Quantum kinetic equation of a Bose gas

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(Received 7 March 1991)

The quantum kinetic equation of a Bose gas is derived by means of the Bogoliubov approach and the Wigner distribution function from the Liouville–von Neumann equation. It is an improved Boltzmann-Uehling-Uhlenbeck equation including the correctional binary collision with many-body effects.

PACS number(s): 05.30.Jp, 51.10.+y

I. INTRODUCTION

In previous works, the kinetic equations of Fermi gases were derived in various cases; the nucleon gas and the plasma [1]. It is common knowledge that microscopic particles may be classified into one of two types: fermion and boson. Bose systems exist in many cases. For example, when the nuclear temperature is raised over 100 MeV/u, the nucleon gas returns to meson and nucleon mixed gases [2], and the meson is a boson. In another case, when the plasma is formed from deuterium, the deuteron is a boson. Besides these cases, helium 4 is also a boson in the investigation of superfluidity of liquid ^{4}He. Hence the kinetic equation of Bose is applied in many practical cases.

In this paper we derive the quantum kinetic equation of a Bose gas by means of the Wigner distribution function and the Bogoliubov approach from the Liouville–von Neumann equation. In Sec. II, the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of the Wigner distribution function is derived. The Bogoliubov approach is described in Sec. III. In the final section we derive the kinetic equation of a Bose gas in the quasihomogeneous case.

II. BBGKY HIERARCHY OF THE WIGNER DISTRIBUTION FUNCTION

Let us suppose there are N particles in the Bose system. q^N are all of the coordinates. The time evolution of density matrix ρ_N is determined by the Liouville–von Neumann equation as follows:

$$i\hbar\frac{\partial\rho_N}{\partial t}=H\rho_N-\rho_N H=[H,\rho_N] \ . \tag{1}$$

H is the Hamiltonian of total system as

$$H=\sum_{i=1}^{N}K_i+\sum_{\substack{i,j\\ i<j}}V_{ij} \ . \tag{2}$$

By taking the trace on Eq. (1), the equation of the s particle density matrix ρ_s can be obtained as [3]

$$i\hbar\frac{\partial\rho_s}{\partial t}+\sum_{\substack{i,j\\ j=1}}^{s}[K_j,\rho_s]+\sum_{\substack{i,j\\ i<j}}[V_{ij},\rho_s] + \operatorname*{Tr}_{s+1}\sum_{j=1}^{s}[V_{j,s+1},\rho_{s+1}] \ . \tag{3}$$

Equation (3) is equivalent to Eq. (1). It is convenient to introduce directly the symmetry requirement on the density matrix ρ_s by means of

$$\rho_s=B_sF_s \ , \tag{4}$$

where B_s is a symmetrization operator defined by

$$B_s=\prod_{j=1}^{s}\left[1+\sum_{k=1}^{j-1}P_{jk}\right] \ . \tag{5}$$

Here P_{jk} denotes the permutation operator. Since B_s satisfies the relation

$$B_{s+1}=B_s\left[1+\sum_{j=1}^{s}P_{j,s+1}\right] \ , \tag{6}$$

and commutes with the operators K_j and V_{ij}, one may substitute Eq. (4) into Eq. (3) to obtain the equation

$$i\hbar\frac{\partial F_s}{\partial t}=\sum_{j=1}^{s}[K_j,F_s]+\sum_{\substack{i,j\\ i<j}}[V_{ij},F_s] +\mathrm{Tr}_{s+1}\sum_{j=1}^{s}[V_{j,s+1},F_{s+1}] +\mathrm{Tr}_{s+1}\sum_{j=1}^{s}\left[V_{j,s+1}\sum_{j=1}^{s}P_{j,s+1}F_{s+1}\right] \ . \tag{7}$$

It is convenient to introduce the s-body Wigner distribution function

$$f_s(q^sp^st)=\frac{1}{(2\pi\hbar)^{3s}}\int F_s(q'^sq''^st)\exp\left[\frac{-p^sr^s}{i\hbar}\right]dr^s \tag{8}$$

and

$$F_s(q'^s q''^s t)=\int f_s(q^s p^s t)\exp\left[\frac{p^s r^s}{i\hbar}\right]dp^s , \tag{9}$$

where q^s is brief expression for $\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_s$. The relations between coordinates $\mathbf{q}_i',\mathbf{q}_i''$ and $\mathbf{q}_i,\mathbf{r}_i$ are

$$\mathbf{q}_i'=\mathbf{q}_i-\tfrac{1}{2}\mathbf{r}_i,\quad \mathbf{q}_i''=\mathbf{q}_i+\tfrac{1}{2}\mathbf{r}_i .$$

One may substitute Eq. (9) into Eq. (7) to obtain the quantum BBGKY hierarchy of the Wigner distribution functions f_s for the Bose system as

$$\frac{\partial f_s}{\partial t}+\sum_{j=1}^{s}\frac{\mathbf{p}_j}{m}\cdot\frac{\partial f_s}{\partial \mathbf{q}_j}+\sum_{\substack{i,j\\ i<j}}\frac{i}{\hbar}(e^{(i\hbar/2)\theta_{ij}}-e^{-(i\hbar/2)\theta_{ij}})f_s+\sum_{j=1}^{s}\frac{i}{\hbar}\int d\mathbf{x}_{s+1}(e^{(i\hbar/2)\theta_{j,s+1}}-e^{(i\hbar/2)\theta_{j,s+1}})f_{s+1}$$
$$+\sum_{j=1}^{s}\frac{i}{\hbar}\int d\mathbf{x}_{s+1}(e^{(i\hbar/2)\theta_{j,s+1}}-e^{-i(\hbar/2)\theta_{j,s+1}})P_{j,s+1}f_{s+1}=0 , \tag{10}$$

where $\mathbf{x}_{s+1}$ are all of the variables $\mathbf{q}_{s+1}$ and $\mathbf{p}_{s+1}$, and the operators θ_{ij} and $\theta_{j,s+1}$ are represented by

$$\theta_{ij}=\frac{\partial V_{ij}(\mathbf{q}_i-\mathbf{q}_j)}{\partial \mathbf{q}_i}\cdot\left[\frac{\partial}{\partial \mathbf{p}_i}-\frac{\partial}{\partial \mathbf{p}_j}\right] ,$$
$$\theta_{j,s+1}=\frac{\partial V_{j,s+1}(\mathbf{q}_j-\mathbf{q}_{s+1})}{\partial \mathbf{q}_j}\cdot\frac{\partial}{\partial \mathbf{p}_j} . \tag{11}$$

When $s=1$ and 2, one finds

$$\frac{\partial f_1}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1}{\partial \mathbf{q}_1}+\frac{i}{\hbar}\int(e^{(i\hbar/2)\theta_{1,2}}-e^{-(i\hbar/2)\theta_{1,2}})f_2d\mathbf{x}_2+\frac{i}{\hbar}\int(e^{(i\hbar/2)\theta_{1,2}}-e^{-(i\hbar/2)\theta_{1,2}})P_{12}f_2d\mathbf{x}_2=0 , \tag{12}$$

$$\frac{\partial f_2}{\partial t}+\sum_{j=1}^{2}\frac{\mathbf{p}_j}{m}\cdot\frac{\partial f_2}{\partial \mathbf{q}_j}+\frac{i}{\hbar}(e^{(i\hbar/2)\theta_{12}}-e^{-(i\hbar/2)\theta_{12}})f_2+\frac{i}{\hbar}\int d\mathbf{x}_3\sum_{j=1}^{2}(e^{(i\hbar/2)\theta_{j,3}}-e^{-(i\hbar/2)\theta_{j,3}})f_3$$
$$+\frac{i}{\hbar}\int d\mathbf{x}_3\sum_{j=1}^{2}(e^{(i\hbar/2)\theta_{j,3}}-e^{-(i\hbar/2)\theta_{j,3}})P_{j3}f_3=0 . \tag{13}$$

Since f_1 depends on f_2, and f_2 depends on f_3, solving accurately is impossible for the BBGKY hierarchy. It is necessary that the approximated approach will be applied for the solution. The Bogoliubov approach will be applied as follows.

III. BOGOLIUBOV APPROACH

There are two hypotheses in the Bogoliubov approach [4]. With the first hypothesis, provided that the average time between two continuous collisions is much longer than the collision time, it is possible to find a kinetic state for any nonequilibration system. In this state

$$f_s(\mathbf{x}_1,\ldots,\mathbf{x}_s;t)=f_s(\mathbf{x}_1,\ldots,\mathbf{x}_s|f_1) ,$$
$$\frac{\partial f_1}{\partial t}=A(\mathbf{x}f_1) . \tag{14}$$

Another hypothesis is that there are not correlations in the initial stage of a system. We set the displacement operator

$$\mathcal{S}_t^s\psi(\mathbf{x}_1^0,\ldots,\mathbf{x}_s^0)=\psi(\mathbf{x}_1,\ldots,\mathbf{x}_s) ,$$

where $\mathbf{x}_1^0,\ldots,\mathbf{x}_s^0$ are the values of each $\mathbf{x}$ at $t=0$, and $\mathbf{x}_1,\ldots,\mathbf{x}_s$ are the values at t. The noncorrelative condition at the initial stage indicates

$$\mathcal{S}_{-t}^s\left[f_s(\mathbf{x}_1,\ldots,\mathbf{x}_s)-\prod_{i\le j\le s}f_1(\mathbf{x}_j)\right]\to 0 . \tag{15}$$

Starting from the two Bogoliubov hypotheses, we derive the kinetic equation of a Bose gas. Suppose the interaction radius of the boson is a, and the average separation between two continuous time collisions is d. When the length is measured in d, the mean occupied volume of every boson $v=a^3$ is a small quantity. One can write

$$\frac{\partial f}{\partial t}=A^0(\mathbf{x}f_1)+vA^1(\mathbf{x}f_1)+\cdots \tag{16}$$

and

$$f_s=f_s^0+vf_s^1+v^2f_s^2+\cdots . \tag{17}$$

In a first-order approximation we set $f_2=f_2^0=f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)$ and find from Eq. (12)

$$\frac{\partial f_1(\mathbf{x}_1)}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial \mathbf{q}_1}+\frac{i}{\hbar}\int d\mathbf{x}_2(e^{(i\hbar/2)\theta_{1,2}}-e^{-(i\hbar/2)\theta_{1,2}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)$$
$$+\frac{i}{\hbar}\int d\mathbf{x}_2(e^{(i\hbar/2)\theta_{1,2}}-e^{-(i\hbar/2)\theta_{1,2}})P_{12}f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)=0 . \tag{18}$$

This is a self-consistent equation called the quantum Vlasov equation. In the second-order approximation we write a formal solution

$$f_s^1(\mathbf{x}_1,\ldots,\mathbf{x}_s|f_1)=\sum_{i<j\le s} g(\mathbf{x}_i,\mathbf{x}_j)\prod_{\gamma\neq i\neq j} f_1(\mathbf{x}_i)\ . \quad (19)$$

$g(\mathbf{x}_1,\mathbf{x}_2)$ is the two-body correlative function, whose boundary condition is

$$\lim_{t\to\infty} \mathcal{S}_{-t}^{(2)} g(\mathbf{x}_i,\mathbf{x}_j)\to 0\ . \quad (20)$$

Equation (19) means that the many-body effect is accounted for by the two-body correlation. We may write

$$\frac{\partial f_2}{\partial t}=\frac{\partial f_2}{\partial f_1}\frac{\partial f_1}{\partial t}\approx\left[\frac{\partial f_2^0}{\partial f_1}+v\frac{\partial f_2^1}{\partial f_1}\right]\times[A^0(\mathbf{x}f_1)+vA^1(\mathbf{x}f_1)]\approx\mathcal{D}_0 f_2^0+v[\mathcal{D}_0 g(\mathbf{x}_1,\mathbf{x}_2)+\mathcal{D}_1 f_2^1]\ . \quad (21)$$

We use Eq. (19) and Eq. (13) to obtain

$$\mathcal{D}_0 g(\mathbf{x}_1,\mathbf{x}_2)+\sum_{j=1}^{2}\frac{\mathbf{p}_j}{m}\cdot\frac{\partial}{\partial \mathbf{q}_j}g(\mathbf{x}_1,\mathbf{x}_2)+\frac{i}{\hbar}\sum_{j=1}^{2}(e^{(i\hbar/2)\eta_j}-e^{-(i\hbar/2)\eta_j})g(\mathbf{x}_1,\mathbf{x}_2)=H(\mathbf{x}_1,\mathbf{x}_2)\ , \quad (22)$$

$$\begin{aligned}H(\mathbf{x}_1,\mathbf{x}_2)=&-\frac{i}{\hbar}(e^{(i\hbar/2)\theta'_{12}}-e^{-(i\hbar/2)\theta'_{12}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)-\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{1,3}}-e^{-(i\hbar/2)\theta'_{1,3}})g(\mathbf{x}_2,\mathbf{x}_3)f_1(\mathbf{x}_1)\\&-\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{2,3}}-e^{-(i\hbar/2)\theta'_{2,3}})f_1(\mathbf{x}_2)g(\mathbf{x}_1,\mathbf{x}_3)-\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{1,3}}-e^{-(i\hbar/2)\theta'_{1,3}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)f_1(\mathbf{x}_3)\\&-\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{2,3}}-e^{-(i\hbar/2)\theta'_{2,3}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)f_1(\mathbf{x}_3)\ ,\end{aligned} \quad (23)$$

where

$$\begin{aligned}&\theta'_{12}=\frac{1}{v}\theta_{12},\quad \theta'_{1,3}=\frac{1}{v}\theta_{1,3},\quad \eta_1=\frac{\partial U(\mathbf{q}_1)}{\partial \mathbf{q}_1}\cdot\frac{\partial}{\partial \mathbf{p}_1}\ ,\\&U_1(\mathbf{q}_1)=\frac{1}{v}\int V_{12}(\mathbf{q}_1-\mathbf{q}_2)f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)d\mathbf{x}_2\ .\end{aligned} \quad (24)$$

$U_1(\mathbf{q}_1)$ is the mean-field potential. Once one knows $g(\mathbf{x}_1,\mathbf{x}_2)$, the second-order approximated equation of $f_1(\mathbf{x})$ will be obtained from Eq. (12). It is very difficult to solve the simultaneous equations (12) and (22) accurately. In the next section it is shown that Eq. (22) may be solved in a quasihomogeneous system.

IV. KINETIC EQUATION OF A BOSE GAS

The condition of a quasihomogeneous system is

$$g(\mathbf{x}_1,\mathbf{x}_2)=g(\mathbf{q}_1-\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2)\ .$$

This states that the correlative function depends only on the relative coordinate. In this case one may obtain the formal solution of $g(\mathbf{q},\mathbf{p}_1,\mathbf{p}_2)$ by means of the displacement technique as follows:

$$\begin{aligned}g(\mathbf{q},\mathbf{p}_1,\mathbf{p}_2)=\int_0^\infty dt\Bigg[&\frac{i}{\hbar}(e^{(i\hbar/2)\theta'_{12}}-e^{-(i\hbar/2)\theta'_{12}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)\\&+\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{1,3}}-e^{-(i\hbar/2)\theta'_{1,3}})[g(\mathbf{x}_2,\mathbf{x}_3)f_1(\mathbf{x}_1)+f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)f_1(\mathbf{x}_3)]\\&+\frac{i}{\hbar}\int d\mathbf{x}_3(e^{(i\hbar/2)\theta'_{2,3}}-e^{-(i\hbar/2)\theta'_{2,3}})[g(\mathbf{x}_1,\mathbf{x}_3)f_1(\mathbf{x}_2)+f_1(\mathbf{x}_1)f_1(\mathbf{x}_3)f_1(\mathbf{x}_3)]\Bigg]\ .\end{aligned} \quad (25)$$

It is convenient to introduce the Fourier transform in order to solve Eqs. (12) and (25). We set

$$\begin{aligned}&\tilde{g}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2)\equiv\int d\mathbf{q}\,g(\mathbf{q},\mathbf{p}_1,\mathbf{p}_2)e^{-i\mathbf{k}\cdot\mathbf{q}}\ ,\\&\tilde{V}(\mathbf{k})\equiv\int d\mathbf{q}V(\mathbf{q})e^{-i\mathbf{k}\cdot\mathbf{q}}\ .\end{aligned} \quad (26)$$

We may substitute Eq. (26) into Eq. (12) to obtain

$$\frac{\partial f_1}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1}{\partial \mathbf{q}_1}+\mathcal{F}\cdot\frac{\partial f_1}{\partial \mathbf{p}_1}=\frac{-i}{(2\pi)^3\hbar}\int d\mathbf{k}\left[\exp\left[\frac{\mathbf{k}\hbar}{2}\cdot\frac{\partial}{\partial \mathbf{p}_1}\right]-\exp\left[-\frac{\hbar\mathbf{k}}{2}\cdot\frac{\partial}{\partial \mathbf{p}_1}\right]\right]\tilde{V}_{12}h(\mathbf{k},\mathbf{p}_1)\ , \quad (27)$$

where

$$h(\mathbf{k},\mathbf{p}_1)=\int d\mathbf{p}g(\mathbf{k},\mathbf{p}_1\mathbf{p}_2)\ ,$$
$$\mathcal{F}\cdot\frac{\partial f_1}{\partial \mathbf{p}_1}=\frac{i}{\hbar}(e^{(i\hbar/2)\eta_1}-e^{-(i\hbar/2)\eta_1})f_1(\mathbf{x}_1)+\frac{i}{\hbar}\int d\mathbf{x}_2(e^{(i\hbar/2)\theta'_{1,2}}-e^{-(i\hbar/2)\theta'_{12}})f_1(\mathbf{x}_1)f_1(\mathbf{x}_2)\ . \tag{28}$$

Performing the Fourier transform of Eq. (25) and making some manipulations one may find

$$\mathrm{Im}h(\mathbf{k},\mathbf{p}_1)=\int \frac{\pi\widetilde{V}_{12}(\mathbf{k})}{\hbar k|1-(1/\hbar)\widetilde{V}_{13}\psi|^2}[f_1^+(\mathbf{x}_1)f_1^-(\mathbf{x}_2)-f_1^-(\mathbf{x}_1)f_1^+(\mathbf{x}_2)]\delta\left[\mathbf{k}\cdot\left[\frac{\mathbf{p}_1}{m}-\frac{\mathbf{p}_2}{m}\right]\right]d\mathbf{p}_2\ , \tag{29}$$

where

$$f^{\pm}=f\left[\mathbf{p}\pm\frac{\hbar\mathbf{k}}{2}\right]\left[1+f\left[\mathbf{p}\mp\frac{\hbar\mathbf{k}}{2}\right]\right]\ ,$$
$$f\left[\mathbf{p}\pm\frac{\hbar\mathbf{k}}{2}\right]=\exp\left[\pm\frac{\hbar\mathbf{k}}{2}\cdot\frac{\partial}{\partial\mathbf{p}}\right]f(\mathbf{p})\ , \tag{30}$$
$$\psi=\int_{-\infty}^{\infty}\frac{d\mathbf{p}_3}{\mathbf{k}\cdot\left[\frac{\mathbf{p}_2}{m}-\frac{\mathbf{p}_1}{m}\right]-i\epsilon}[f_1^+(\mathbf{x}_3)-f_1^-(\mathbf{x}_3)]\ . \tag{31}$$

Substituting Eq. (29) into Eq. (27), one finds

$$\frac{\partial f_1(\mathbf{x}_1)}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial \mathbf{q}_1}+\mathcal{F}\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial \mathbf{p}_1}=\frac{\pi}{(2\pi)^3\hbar}\int d\mathbf{k}\left[\exp\left[\frac{\hbar\mathbf{k}}{2}\cdot\frac{\partial}{\partial\mathbf{p}_1}\right]-\exp\left[-\frac{\hbar\mathbf{k}}{2}\cdot\frac{\partial}{\partial\mathbf{p}_1}\right]\right],$$
$$\times\int d\mathbf{p}_2\delta\left[\mathbf{k}\cdot\left[\frac{\mathbf{p}_1}{m}-\frac{\mathbf{p}_2}{m}\right]\right]\frac{\widetilde{V}_{12}^2(k)}{|1-(1/\hbar)\widetilde{V}_{23}\psi|^2}$$
$$\times[f_1^+(\mathbf{x}_1)f_1^-(\mathbf{x}_2)-f_1^+(\mathbf{x}_2)f_1^-(\mathbf{x}_1)]\ . \tag{32}$$

Equation (32) is the kinetic equation of a Bose gas in the quasihomogeneous case. This is an improved BUU equation. It is reduced to the usual BUU equation provided that many-body effects are neglected and that the first approximation for the term $\mathcal{F}$ [5] is taken:

$$\frac{\partial f_1(\mathbf{x}_1)}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial \mathbf{q}_1}-\nabla\mathbf{U}_1\cdot\frac{\partial f_1(\mathbf{x}_1)}{\partial \mathbf{p}_1}=\frac{\pi}{\hbar(2\pi)^9}\int d\mathbf{p}_2d\mathbf{p}'_1d\mathbf{p}'_2|\langle\mathbf{p}_1\mathbf{p}_2|V_{12}|\mathbf{p}_1\mathbf{p}_2\rangle|^2\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}'_1-\mathbf{p}'_2)$$
$$\times\{f_1(\mathbf{x}'_1)f_1(\mathbf{x}'_2)[1+f_1(\mathbf{x}_1)][1+f_1(x_2)]$$
$$-f_1(\mathbf{x}_1)f_2(\mathbf{x}_2)[1+f_1(\mathbf{x}'_1)][1+f_1(\mathbf{x}'_2)]\}\ .$$

Since Eq. (32) includes the influence of many-body effects, a greater improvement will be produced for the systems that have a higher particle density or a larger force range of particle interaction.

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