Nucleation and growth in systems with many stable phases

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(Received 22 July 1991)

We study the kinetics of nucleation and growth in systems with an arbitrary number of distinct stable phases for both homogeneous and heterogeneous nucleation. Exact solutions for the phase transformation kinetics in one dimension are obtained and compared with the mean-field results. We have observed anomalous power-law asymptotics for both homogeneous and heterogeneous nucleation in one dimension, while the mean-field theory predicts exponential asymptotic behavior for large times. Numerical simulations for $2d$ systems show that the mean-field theory is surprisingly accurate. Some properties of the spatial patterns at the final stage of nucleation and growth are elucidated.

PACS number(s): 64.60.Qb, 05.70.Ln, 81.35.+k

I. INTRODUCTION

An interesting class of spatial pattern formation phenomena involves the transformation of one phase into another through a nucleation and growth process. Examples of such phase transformations are numerous and include metallurgical, polymeric, ceramic, crystalline, chemical, and biological systems [1–3]. Consequently, it is of great practical and theoretical interest to possess mathematical predictions concerning the phase transformation kinetics. While many aspects of these transformations have been thoroughly investigated, a number of features are poorly understood, especially those related to inhomogeneities in the material [4].

The Kolmogorov or Johnson-Mehl-Avrami model [5–7] has been widely employed as a model of phase transformation kinetics (see, e.g., [8–12] and references therein). In the model, vanishingly small spherical grains nucleate at a constant rate Γ per unit volume in the metastable phase and grow at constant velocity V_0 once formed. As was shown by Kolmogorov [5], the volume fraction of untransformed material at time t is

$$\Phi(t)=\exp\left[-\frac{\Omega_d}{d+1}\Gamma V_0^d t^{d+1}\right], \tag{1}$$

where d is the dimension of space and Ω_d is the volume of a d-dimensional unit sphere.

The nucleation process in the Kolmogorov model is referred to as homogeneous because it occurs uniformly throughout the metastable phase. In many cases, however, the nucleation is heterogeneous, i.e., the nucleation in the metastable phase is initiated by some "external" reasons such as defects or impurities which are present before the phase transformation began [1]. In biological systems, sites that initiate nucleation may be regions from which some biological species start their expansion. A simple model of heterogeneous nucleation is obtained by randomly placing nuclei with density γ throughout the material. When the phase transformation is initiated at time $t=0$, the volume fraction of untransformed material at time t is

$$\Phi(t)=\exp(-\Omega_d\gamma V_0^d t^d)\ . \tag{2}$$

A very thorough discussion of the Kolmogorov model and analytical expressions for more complex correlation functions may be found in Ref. [9].

There is a single stable phase in both these models. However, in a variety of systems more than one qualitatively different stable (or nearly stable) phase exists. In the simplest case, the metastable system M can transform into one of two different stable phases, A and B [11]. Nucleation rates and growth velocities of the conversion processes $M\rightarrow A$ and $M\rightarrow B$ are generally different.

In the present paper, we focus on a natural generalization of the one-phase model, namely on a model with an arbitrary, possibly infinite, number of different stable phases. We shall specify a particular phase by its growth velocity V. We shall assume that velocities are distributed on some interval (V_1,V_2) with a density $\rho(V)$, $\int_{V_1}^{V_2}\rho(V)dV=1$. Observe that the simplest choice $\rho(V)=\delta(V-V_0)$ corresponds to the Kolmogorov model. To the best of our knowledge, only the two-phase generalization of the Kolmogorov model corresponding to $\rho(V)=\Gamma_A\delta(V-V_A)+\Gamma_B\delta(V-V_B)$ has been studied [11].

An interface between any two phases will fluctuate if both phases are stable, and drift slowly, if the free-energy difference between phases is small. In many systems, e.g., for soap fronts, this defines the long-time behavior [13,14]. Nevertheless, in the present study we will ignore interface fluctuations and drift, i.e., we treat these interfaces as stationary. A more realistic treatment should consider these effects. However, the present model can be solved analytically in one dimension and readily investigated numerically in other dimensions.

A variety of applications of the model can envisioned.

For example, we mention the growth of colonies of sedentary organisms on a nutrient surface. In this problem, the disk-shaped colonies spread over the surface at constant velocities as the organisms reproduce. Many-phase nucleation and growth occurs if colonies of different species compete for the same nutriment. Among other possible applications one can imagine a spread of mutually excluding infections, ethnic groups expansion, etc.

The rest of this paper is organized as follows. In Sec. II, we develop a mean-field theory for both homogeneous and heterogeneous many-phase nucleation and growth in d dimensions. The models are solved exactly in one dimension in Sec. III. In Sec. IV, we discuss the results of numerical simulation in two dimensions. The conclusions are summarized in Sec. V.

II. MEAN-FIELD APPROACH

First, we develop a mean-field theory (MFT) for heterogeneous many-phase nucleation. Namely, we assume that growth of uniformly and randomly distributed nuclei over the volume begins at time $t=0$. Hereafter, we use the units for which $V_0=1$, $\gamma=1$, and $\Gamma=1$. Let $\psi_v(t)dV$ be the fraction of material which has been transformed at time t by phases having the velocities between V and $V+dV$. If we ignore overlap of growing spheres, we find $\psi_v(t)dV=\Omega_d(Vt)^d\rho(V)dV$. Therefore

$$\frac{d\psi_v}{dt}=\rho(V)\Omega_d dV^d t^{d-1} . \tag{3}$$

This is of course an overestimate of $\psi_v(t)$, since a material which has already been transformed cannot be transformed again. Following the spirit of the MFT we neglect spatial correlations and take into account an overestimate of $\psi_v(t)$ by reducing $d\psi_v/dt$ by a factor $\Phi(t)$, where $\Phi(t)$ is the fraction of untransformed material at time t. Equation (3) then becomes

$$\frac{d\psi_v}{dt}=\rho(V)d\Omega_d V^d t^{d-1}\Phi(t) . \tag{4}$$

Using the obvious relation

$$\Phi(t)=1-\int_0^\infty dV\,\psi_v(t) , \tag{5}$$

we recast (5) into the following equation:

$$\frac{d\Phi}{dt}=-d\Omega_d\langle V^d\rangle t^{d-1}\Phi(t) , \tag{6}$$

where the averaging over the distribution $\rho(V)$ is denoted by angular brackets:

$$\langle V^d\rangle=\int_0^\infty dV\,\rho(V)V^d . \tag{7}$$

Solving (6) yields

$$\Phi(t)=\exp(-\Omega_d\langle V^d\rangle t^d) . \tag{8}$$

Substituting (8) into (4) and performing the integration, one finds

$$\psi_v(t)=\frac{\rho(V)V^d}{\langle V^d\rangle}[1-\Phi(t)] . \tag{9}$$

In particular, when the phase transformation is complete the final density $\psi_v(\infty)$ is given by

$$\psi_v(\infty)=\frac{\rho(V)V^d}{\langle V^d\rangle} . \tag{10}$$

The mean-field equations for homogeneous nucleation are obtained in a manner which is completely analogous to that presented before. Therefore, we only write the final result for the volume fraction of untransformed material at time t:

$$\Phi(t)=\exp\left[-\frac{\Omega_d}{d+1}\langle V^d\rangle t^{d+1}\right] . \tag{11}$$

Note that the function $\psi_v(t)$ is still described by the formula (9) with $\Phi(t)$ given by (11).

As will be discussed below, the MFT for both homogeneous and heterogeneous many-phase nucleation is not exact. It does provide a good approximation in dimensions $d\geq 2$, but it becomes invalid in one dimension. This will be tested in the following sections by comparing analytical and numerical results.

III. EXACT SOLUTIONS IN ONE DIMENSION

It is useful to study phase transformation kinetics in one dimension, since exact solutions are possible to obtain in this case. At first, we consider the simpler case of heterogeneous nucleation. We assume that nucleation centers are distributed uniformly and independently throughout the system. Therefore, the probability that the neighboring nucleus lies in the interval $(x,x+dx)$ is equal to $\exp(-x)dx$.

Let us consider the process of phase transformation of the interval of length x by the neighboring nuclei growing with velocities in the intervals $(V,V+dV)$ and $(W,W+dW)$, respectively. Averaging over lengths and velocities yields the volume fraction of untransformed material at time t:

$$\begin{aligned}\Phi(t)&=\int_0^\infty dV\,\rho(V)\\ &\quad\times\int_0^\infty dW\,\rho(W)\\ &\quad\quad\times\int_{Vt+Wt}^\infty dx\,(x-Vt-Wt)\exp(-x)\\ &=[\langle\exp(-Vt)\rangle]^2 .\end{aligned} \tag{12}$$

Using the well-known relation

$$\langle\exp(-Vt)\rangle\geq\exp[-(\langle V\rangle t)] , \tag{13}$$

which is valid for an arbitrary positively distributed random variable, we see that true asymptotic decay (12) is slower than predicted by the MFT, except the special case of one-phase nucleation, $\rho(V)=\delta(V-V_0)$.

Now we turn to systems with a continuum number of stable phases. First, we consider a flat distribution

$$\rho(V)=\begin{cases}(2\epsilon)^{-1} & \text{at } 1-\epsilon<V<1+\epsilon\\ 0 & \text{otherwise} .\end{cases} \tag{14}$$

We find

$$\Phi(t)=\exp[-2(1-\epsilon)t]\left[\frac{1-\exp(-2\epsilon t)}{2\epsilon t}\right]^2 \tag{15}$$

while the MFT predicts $\Phi(t)=\exp(-2t)$. We see that the leading-order asymptotic behavior of $\Phi(t)$ in the model (14) is the same as the one-phase (or MFT) result with the smallest velocity, $V=1-\epsilon$ in our model. The power-law correction to the leading exponential decay is a feature that does not appear in the MFT. Moreover, this feature is an intrinsic property relevant to a continuum distribution and does not occur in systems with a finite number of distinct phases.

In the limit $\epsilon\to 1$, we arrive at the quite unexpected result

$$\Phi(t)=\left[\frac{1-\exp(-2t)}{2t}\right]^2 \propto t^{-2} \quad \text{at } t\to\infty\ , \tag{16}$$

i.e., the material is transformed much more slowly than in the usual one-phase model. Anomalous kinetics of this kind are not expected in higher dimensions, since a slowly growing nucleus stops the growth of usual grains only in one dimension due to the "trolleybus effect" (no trolleybus can pass the front one).

A slowing down of phase transformation kinetics may become more significant if $\rho(V)$ tends to infinity as V tends to zero. For a model with power-law behavior $\rho(V)\propto 1/V^{\alpha}$ as $V\to 0$ [$\alpha<1$ due to the normalization condition $\int\rho(V)dV=1$], one can derive that the fraction of untransformed material decays as a power law,

$$\Phi(t)\propto t^{-2(1-\alpha)} \quad \text{at } t\to\infty\ . \tag{17a}$$

One can also consider the limiting case $\alpha=1$ by treating somewhat more special distribution, namely $\rho(V)\propto V^{-1}[\ln(1/V)]^{-\beta}$ ($\beta>1$ due to the normalization condition). Evaluating (12), we obtain

$$\Phi(t)\propto(\ln t)^{1-2\beta} \quad \text{at } t\to\infty\ . \tag{17b}$$

Thus we see that the true asymptotic decay of $\Phi(t)$ may be significantly slower than that predicted by the MFT.

Let us now turn to homogeneous many-phase nucleation and growth in one dimension. Proceeding with a solution we first study an auxiliary "one-sided" problem in which no nuclei are placed to the left of the origin. Nuclei are scattered to the right of the origin as in the original problem. We define $p(t,V)dV$ as the probability that the origin is transformed before time t by a grain with velocity in the interval $(V,V+dV)$. The probability that the origin is in the metastable phase at time t is

$$f(t)=1-\int_0^{\infty}dV\,p(t,V)\ . \tag{18}$$

By definition, $\dot p(t,V)dt\,dV$ is the probability that the origin is transformed between time t and $t+dt$ by a grain with velocity between V and $V+dV$. Such a grain could be nucleated at any point x in the interval $0\le x\le Vt$ between times $t-x/V$ and $t-x/V+dt$. Hence

$$\dot p(t,V)dt\,dV=\int_0^{Vt}dx\,dt\,dV\,\rho(V)f(t-x/V)\times\exp\left[-x\left[t-\frac{x}{2V}\right]\right]\ . \tag{19}$$

Here $dx\,dt\,dV\,\rho(V)$ is the probability of nucleation of a grain in the space interval $(x,x+dx)$, the time interval $(t-x/V,t-x/V+dt)$, and velocity interval $(V,V+dV)$. This can occur only if the interval $(x,x+dx)$ was not previously transformed—hence the factor $(f(t-x/V)$. The exponential factor in (19) ensures that other grains do not prevent our grain [in $(x,x+dx)$ with velocity between V and $V+dV$] from reaching the origin. This exponential factor is easily derived if we observe that it is equal to

$$\exp\left[-x\left[t-\frac{x}{V}\right]\right]\exp\left[-\int_0^{x/V}(x-V\tau)d\tau\right]\ ,$$

where the former factor is the probability that no nucleation has occurred in the interval $(0,x)$ during the time interval $(0,t-x/V)$ while the latter factor is the probability of the same during the time interval $(t-x/V,t)$ in the untransformed part of the interval $(0,x)$.

Changing variables from x to $s=t-x/V$ in Eq. (19) and using (18) we find

$$\dot f(t)=-\int_0^t ds\,f(s)\int_0^{\infty}dV\,\rho(V)V\exp\left[-\frac{V}{2}(t^2-s^2)\right]\ , \tag{20}$$

which is a closed integrodifferential equation for $f(t)$. For the particular case of a two-phase system this equation reduces to the simpler one, derived in Ref. [11].

We now return to the original problem, which may be regarded as a two-sided analog of the previous one. The origin may be transformed by the growth of a nucleus that has been nucleated to the right provided that the origin has not been previously transformed by the grains nucleated to the left; the opposite situation is also possible. Introducing the probability $P(t,V)$ that the origin is transformed before time t by a grain with velocity in the interval $(V,V+dV)$, we therefore obtain

$$\dot P(t,V)=2f(t)\dot p(t,V)\ . \tag{21}$$

Combining (18), (21), and a two-sided analog of (18),

$$\Phi(t)=1-\int_0^{\infty}dV\,P(t,V)\ , \tag{22}$$

we arrive at the final result

$$\Phi(t)=f^2(t)\ . \tag{23}$$

For the one-phase system $\rho(V)=\delta(V-1)$ and Eq. (20)

is readily solved to give $f(t)=\exp(-t^2/2)$. Using (23) we then arrive at the expected result, Eq. (1) for $d=1$. We now discuss a system with the flat distribution of velocities. As for heterogeneous nucleation the leading-order asymptotic behavior of the model (14) coincides with the mean-field exponential behavior corresponding to velocity $1-\epsilon$, the smallest velocity in the distribution (14). For a correction $\chi(t)$ to the leading exponential decay,

$$f(t)=\exp\left[-\frac{1-\epsilon}{2}t^2\right]\chi(t)\ , \tag{24}$$

we derive from (20)

$$\dot{\chi}(t)-(1-\epsilon)t\chi(t)=-\int_0^t ds\,\chi(s)\frac{1+(1-\epsilon)\lambda-[1+(1+\epsilon)\lambda]\exp(-2\epsilon\lambda)}{2\epsilon\lambda^2}\ , \tag{25}$$

where $\lambda=(t^2-s^2)/2$. An analysis of this equation shows that at $t-\infty$, the main contribution to the integral on the right-hand side of (25) is accumulated near the lower limit, i.e., at $s\ll t$. Hence the integral in (25) tends to $(1-\epsilon)(\epsilon t^2)^{-1}\int_0^\infty ds\,\chi(s)$ and we finally obtain the power-law correction to the leading exponential decay.

$$\chi(t)\to t^{-3}\epsilon^{-1}C,\quad C=\int_0^\infty ds\,\chi(s)\ \text{ at } t\to\infty\ . \tag{26}$$

As it follows from Eqs. (23), (24), and (26), we get a power-law asymptotic for $\epsilon=1$, i.e., $\Phi(t)\propto t^{-6}$. It is interesting to note that power-law decay is again exhibited when nucleation sites with zero-growth velocity are present.

It is also possible to describe an asymptotic behavior of one-dimensional homogeneous nucleation with an arbitrary velocity distribution $\rho(v)$. First, we consider systems with no zero-growth velocity nucleation sites. This means that growth velocity distribution has a lower cutoff $V_{\min}$. We obtain for such distributions

$$f(t)=\exp(-\tfrac{1}{2}V_{\min}t^2)\chi(t)\ , \tag{27a}$$

where at long times $\chi(t)$ behaves as

$$\chi(t)\to\frac{1}{t}\int_0^\infty du\,\rho(V_{\min}+u)\exp(-\tfrac{1}{2}t^2u)\left[\int_0^\infty ds\,\chi(s)\right]\ . \tag{27b}$$

When the zero-growth velocity nucleation sites are present in the system, i.e., when $V_{\min}=0$, we obtain much slower asymptotic decay,

$$f(t)\to\frac{1}{t}\left[\int_0^\infty du\,\rho(u)\exp(-\tfrac{1}{2}t^2u)\right]\left[\int_0^\infty ds\,\chi(s)\right]\ . \tag{28}$$

If the velocity distribution has a power-law form at $V\to 0$, $\rho(V)\to CV^{-\alpha}$ $(\alpha<1)$, we obtain a power-law asymptotic decay

$$\Phi(t)=f^2(t)\propto t^{-(6-4\alpha)}\ \text{ at } t\to\infty\ . \tag{29}$$

As expected, the fraction of untransformed material for homogeneous nucleation decays always faster than for heterogeneous nucleation.

In closing this section, we give the exact results for the fraction of material $\psi_v(t)$ which has been transformed by phases with velocities in the interval $(V,V+dV)$. It is not difficult to show that for heterogeneous nucleation $\psi_v(t)$ may be expressed in terms of $f(t)$,

$$\psi_v(t)=2V\rho(V)\int_0^t d\tau f(\tau)\exp(-V\tau)\ , \tag{30}$$

where $f(t)$ is again the probability that the origin is in the metastable phase at time t in an auxiliary "one-sided" problem. Combining (30) and the general relation (23), which is also valid for the heterogeneous nucleation and the exact solution (12), we finally obtain

$$\psi_v(t)=2V\rho(V)\int_0^\infty dW\,\rho(W)\frac{1-\exp(-Vt-Wt)}{V+W}\ . \tag{31}$$

For the flat distribution (14) one can compute the in-

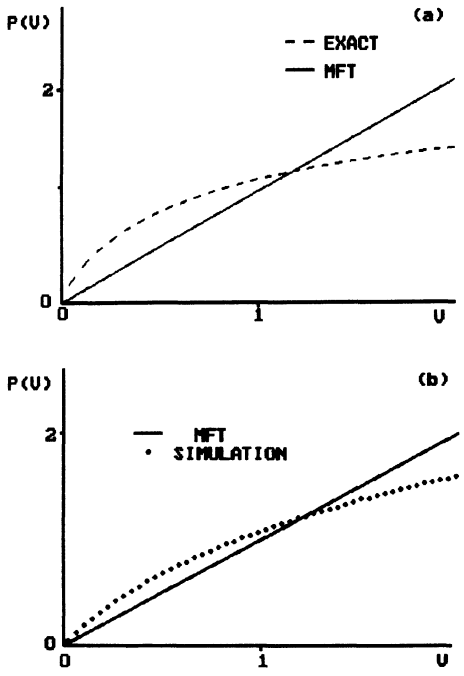

FIG. 1. The function $P(V)=2\epsilon\psi_v$ $(t=\infty)$, where $\psi_v(\infty)$ is a fraction of material at the final state, transformed by phases with growth velocity V, vs velocity for a 1D system for (a) heterogeneous nucleation and (b) homogeneous nucleation. The results of the mean-field approximation (——), exact result (— — —), and simulational results (○) for the flat growth velocity distribution (14) at $\epsilon=1$ are presented.

tegral (31) and find

$$\psi_v(t)=\frac{V}{2\epsilon^2}\left[\ln\left[\frac{1+\epsilon+V}{1-\epsilon+V}\right]-\int_{1-\epsilon+v}^{1+\epsilon+v}du\frac{e^{-ut}}{u}\right]. \quad (32)$$

This exact result for the heterogeneous nucleation should be compared with the mean-field predictions (8) and (9). Figure 1(a) plots the final density $\psi_v(\infty)$.

It is not difficult to define $\psi_v(t)$ for the homogeneous nucleation:

$$\psi_v(t)=2V\rho(V)\int_0^t ds_1 f(s_1)\times\int_0^{s_1}ds_2 f(s_2)\exp\left[-V\frac{s_1^2-s_2^2}{2}\right]. \quad (33)$$

We have not been able to define $\psi_v(t)$ analytically. Instead of the numerical evaluation of the double integral in the right-hand side of Eq. (33), we study the homogeneous nucleation by a direct computer simulation.

Performing a computer simulation of one-dimensional homogeneous nucleation is straightforward. First we generate the time until the next possible nucleation event. This quantity follows a simple exponentially decreasing probability distribution. The space coordinate for the event is taken from a uniform distribution over the size of the system. This coordinate is then checked to see whether it falls into an untransformed region. If so, a new grain is formed. The growth velocity of the grain is chosen randomly, based on the distribution $\rho(V)$.

The size of the system was varied to provide about 10^5 nucleation events. This was found to make the finite-size effects negligible relative to the statistical errors. Numerical simulations, performed up to the minimum value $\Phi(t)=10^{-6}$, confirm the power-law dependence $\Phi(t)\propto t^{-6}$ for $\epsilon=1$. The results for $\psi_v(\infty)$ for the flat distribution (14) are presented in Fig. 1(b). Predictions of the MFT are also shown in this figure. As expected, the numerical results differ significantly from the values predicted by the mean-field theory.

IV. NUMERICAL SIMULATION IN TWO DIMENSIONS

Numerical simulations of the growth kinetics for the 2D systems with many stable phases were performed for both heterogeneous and homogeneous nucleation. We have used the flat distribution of velocities (14) with various values of parameter ϵ, specifying the width of the distribution. The influence of the distribution width on the growth kinetics as well as on the properties of the final patterns were investigated. The simulations were performed for the following set: $\epsilon=0$ (Kolomogorov model), 0.1, 0.25, 0.5, 0.875, and 1.0.

To simulate heterogeneous nucleation in 2D, about 10^2 nucleation sites were distributed randomly over the plane and the growth velocities were taken from the flat distribution. Periodic boundary conditions were imposed. The simulations were then repeated 10–15 times for each value of ϵ.

We evaluated the function $\Phi(t)$, which is the fraction of untransformed area at time t, and compared it with predictions of the MFT. The later gives for $\Phi(t)$ in the case of flat distribution

$$\Phi_\epsilon^{\mathrm{MFT}}(t)=\exp[-(t/\tau_\epsilon)^2], \quad (34)$$

where

$$\tau_\epsilon^{-2}=\pi(1+\epsilon^2/3). \quad (35)$$

The numerical and MFT results for $\Phi(t)$ are given in Fig. 2(a). We also calculate the function $\psi_v(\infty)$, which gives the final (at $t=\infty$) fraction of material transformed by the phase growing with velocity V. The numerical results for $\psi_v(\infty)$ are compared with the MFT prediction

$$\psi_v^{\mathrm{MFT}}(\infty)=\begin{cases}V^2[2\epsilon(1+\epsilon^2/3)]^{-1} & \text{at } 1-\epsilon<V<1+\epsilon\\ 0 & \text{otherwise}\end{cases} \quad (36)$$

in Fig. 3. One can see that the discrepancy between numerical data and MFT is very slight. The deviations are not more than the statistical errors in our simulations. This contrasts with the 1D system, for which the multiphase growth model gives qualitatively different behaviors in MFT and exact considerations.

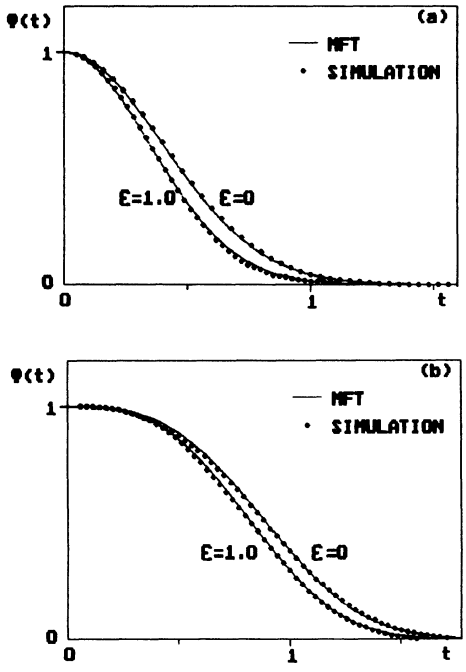

FIG. 2. The fraction of untransformed material $\Phi(t)$ vs t for a 2D system for (a) heterogeneous nucleation and (b) homogeneous nucleation. The results of the mean-field approximation (——) and simulational results (○) for the flat growth velocity distribution (14) at $\epsilon=0$ (the Kolmogorov model) and $\epsilon=1$ are shown.

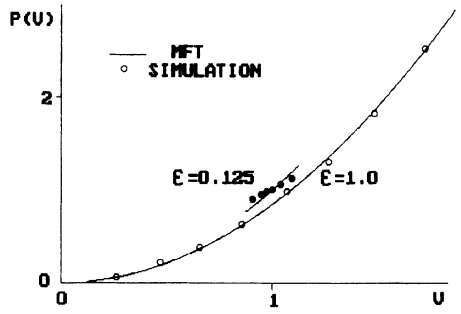

FIG. 3. The function $P(V)=2\epsilon\psi_v$ $(t=\infty)$, where $\psi_v(\infty)$ is a fraction of material at the final state, transformed by phases with growth velocity V, vs velocity for a 2D system for heterogeneous nucleation. The results of the mean-field approximation (——) and simulational results for the flat growth velocity distribution (14) at $\epsilon=1$ (○) and $\epsilon=0.125$ (●) are presented.

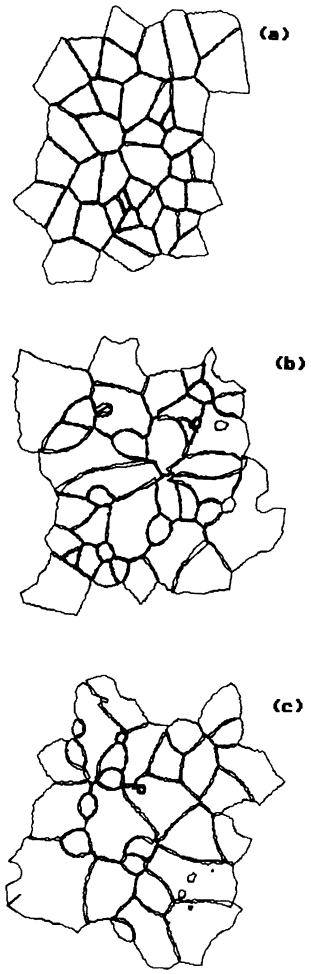

FIG. 4. The typical grain geometry of the final patterns for heterogeneous nucleation and growth model for the flat growth velocity distribution (14) at (a) $\epsilon=0$, (b) $\epsilon=0.5$, and (c) $\epsilon=1$. Slight overlaps of the grains and fluctuations of border lines are artifacts of the numerical technique.

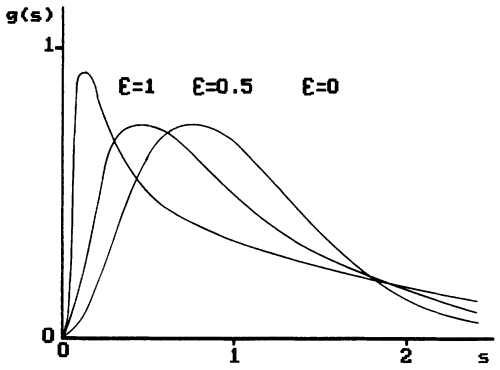

FIG. 5. Smoothed grains area distribution function $g(s)=dn(s)/ds$, vs grains area s, for heterogeneous nucleation for the flat growth velocity distribution (14) at various ϵ.

The analogous simulations were performed for homogeneous nucleation. The function $\Phi(t)$ and the MFT prediction

$$\Phi_\epsilon^{MFT}(t)=\exp[-(t/\tau_\epsilon)^3] , \tag{37}$$

where

$$\tau_\epsilon^{-3}=\pi(1+\epsilon^2/3)/3 , \tag{38}$$

are shown in Fig. 2(b). Once again we see that MFT yields quite satisfactory results.

We believe that our findings are qualitatively general and do not depend on the particular form of the growth velocities distribution. Therefore, one may suppose that the upper critical dimension for the many-phase-growth model is $d_c=2$. In the limiting case of $\epsilon=0$ (one-phase-growth model), however, the critical dimension is $d_c=1$.

Now let us discuss the properties of the final patterns for the heterogeneous nucleation (for homogeneous nucleation our results do not differ significantly). Figures 4(a)–4(c) plot typical grain geometry at $t\to\infty$ for $\epsilon=0$, 0.5, and 1.0. Observe that the increase in the parameter ϵ, specifying the velocity distribution width, leads to the appearance of "freakish," nonconvex grains and of small grains fully incorporated into another. It is of interest that for large ϵ the final patterns remind one of geographical maps. For large ϵ one can also notice "growth suppression" effect: The grains with very high velocities suppress the growth of their neighbors. As a result, the final patterns consist of a few big grains, surrounded by a number of smaller (suppressed) grains.

The growth suppression effect may be illustrated by some properties of the distribution functions $g(s)=dn(s)/ds$, with $n(s)$ being the number (fraction) of grains with area smaller than s. An increase in ϵ from 0 to 1 leads to the following transformation of the distribution $g(s)$ (see Fig. 5): its peak narrows (at the level 0.5 of its maximum) and drifts to the smaller s, while the standard deviation of this distribution increases more than twice. One can expect that if the growth velocity distribution has a high-velocity "tail" [e.g., $\rho(V)\propto V^{-\beta}$, $V\gg 1$, $\beta>1$], the effect of growth suppression should be more pronounced.

V. CONCLUSIONS AND DISCUSSION

The growth kinetics for the many-phase-growth model was investigated analytically for 1D systems and numerically for 2D systems. In this model the growth of nucleating sites of different phases occurs with different growth velocities. Both homogeneous and heterogeneous nucleation have been considered.

For one-dimensional systems we have found an anomalous power-law correction to the leading-order exponential decay of the untransformed fraction of the material. We have established that for both types of nucleation the untransformed fraction decays as a power law, if the system has a continuum number of stable phases with nonvanishing density of the phases with vanishingly small velocities. Noting of this kind appears in the mean-field theory.

We have also carried out numerical simulations for 2D systems. We observed that the mean-field theory described surprisingly well the nucleation kinetics for growth velocities distributions of a "flat" type. Some properties of the final patterns for models with various widths of the growth velocities distribution were also investigated.

ACKNOWLEDGMENTS

It is a pleasure to thank R. M. Bradley for providing us with a copy of Ref. [11].

[1] J. W. Christian, *The Theory of Transformations in Metals and Alloys* (Pergamon, Oxford, 1975).

[2] *Nucleation and Crystallization in Glasses–Advances in Ceramics,* edited by J. H. Simmons, D. R. Uhlmann, and E. H. Beall (American Ceramic Society, Columbus, OH, 1982).

[3] *Oscillations and Travelling Waves in Chemical Systems,* edited by R. J. Field and M. Burger (Wiley, New York, 1985).

[4] A. Kolb-Telieps and Tan Shu-Song, J. Non-Cryst. Solids **107**, 122 (1988).

[5] A. N. Kolmogorov, Izv. Akad. Nauk SSSR, Ser. Fiz. **3**, 355 (1937).

[6] W. A. Johnson and R. F. Mehl, Trans. Am. Inst. Min. Metall. Pet. Eng. **135**, 416 (1939).

[7] M. Avrami, J. Chem. Phys. **7**, 1103 (1939); **8**, 212 (1940); **9**, 177 (1941).

[8] J. D. Axe and Y. Yamada, Phys. Rev. B **34**, 1599 (1986).

[9] K. Sekimoto, Physica A **135**, 328 (1986).

[10] S. Ohta, T. Ohta, and K. Kawasaki, Physica A **140**, 478 (1987).

[11] R. M. Bradley and P. N. Strenski, Phys. Rev. B **40**, 8967 (1989).

[12] M. Weinberg and R. Kapral, J. Chem. Phys. **91**, 7146 (1989).

[13] J. A. Glazier, S. P. Gross, and J. Stavans, Phys. Rev. A **36**, 306 (1987); M. Marder, *ibid.* **36**, 438 (1987); C. W. J. Beenaker, *ibid.* **37**, 1697 (1988); J. Stavans, *ibid.* **42**, 5049 (1990).

[14] K. Kawasaki, T. Nagai, and K. Nakashima, Philos. Mag. B **60**, 399 (1989); K. Nakashima, T. Nagai, and K. Kawasaki, J. Stat. Phys. **57**, 759 (1989).