Analogies between two optical systems (photon beam splitters and laser beams) and two quantum systems (the two-dimensional oscillator and the two-dimensional hydrogen atom)

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Some striking analogies are exhibited between the lossless quantum-mechanical beam splitter and the following two elementary quantum systems: (1) the two-dimensional isotropic oscillator and (2) the two-dimensional hydrogen atom. For example, we show that the unitary transformation connecting the input and output photon statistics at a lossless 50-50 beam splitter is identical to the transformation connecting the polar and parabolic eigenstates of the two-dimensional (2D) hydrogen atom. A similar connection between (Gaussian) laser beams and the eigenstates of the 2D oscillator is pointed out. These analogies are established by identifying the underlying symmetries of these systems and exploiting them by means of group theory.

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I. INTRODUCTION

The purpose of this paper is to exhibit some striking analogies between the lossless optical beam splitter and the following two elementary quantum systems: (1) the two-dimensional isotropic oscillator and (2) the two-dimensional (2D) hydrogen atom. We will also point out a similar analogy between Gaussian laser beams and the two-dimensional isotropic oscillator. In establishing these analogies we will make use of elementary group-theoretical methods familiar to most physicists from the quantum theory of angular momentum [1].

A great deal is known about the group theory of the hydrogen atom and the harmonic oscillator, not only in two dimensions but in N (>2) dimensions [2]. What little we will say on this subject should therefore come as no surprise to experts in the field. What is interesting about this paper, we hope, are the connections we point out between optical systems (beam splitters, Gaussian beams) and elementary quantum systems (2D oscillator, 2D hydrogen atom). The quantum-mechanical theory of an ideal beam splitter [3–7] appears to have been developed only recently, in connection with such problems as quantum noise reduction [3] and nonclassical effects [7,8] in interferometers. Gaussian beams have, of course, been around for a long time, but the remark we make about them does not seem to have been made earlier.

It is hoped that the present paper will provide some elementary, yet interesting, applications of group-theoretical principles to optical problems that will appeal to a wide cross section of optical physicists.

II. THE LOSSLESS QUANTUM-MECHANICAL BEAM SPLITTER

Consider the lossless, two-port beam splitter shown in Fig. 1. Photons are incident at the input ports 1 and 2 and are partially transmitted to the output ports 1′ and 2′. We will restrict the entire treatment to a single-frequency mode so that all photons, both incident and outgoing ones, are at the same frequency ω. Because the beam splitter is assumed lossless, the average number (or energy) of incident photons is equal to the average number (or energy) of outgoing photons. The quantum-mechanical theory of such a beam splitter has been worked out by several authors in recent years [3–7]. We would like to use this theory to exhibit some analogies between the beam splitter and the two quantum systems mentioned above.

We take as our starting point the quantum-mechanical description of a lossless two-port beam splitter given by Prasad, Scully, and Martienssen [4]. Denote by $|\Psi_{in}\rangle$ the joint input state of the photons at ports 1 and 2 and by $|\Psi_{out}\rangle$ the joint output state at ports 1′ and 2′. Then, as shown by Prasad, Scully, and Martienssen [4], the action of the beam splitter is to perform a unitary transformation on the input state $|\Psi_{in}\rangle$ to yield the output state $|\Psi_{out}\rangle$:

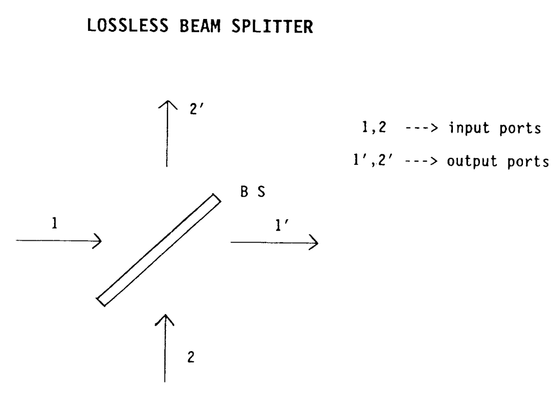

FIG. 1. The lossless quantum-mechanical beam splitter. Photons (of a single frequency ω) are incident at the input ports 1 and 2 and emerge at the output ports 1′ and 2′. The beam splitter need not consist of a single element but could consist of a stack of elements with varying refractive indices. The restriction to lossless elements is, however, essential for the treatment in this paper.

$$|\Psi_{\text{out}}\rangle = B(\psi,\vartheta,\varphi)|\Psi_{\text{in}}\rangle \ . \tag{1}$$

To specify the beam splitter operator $B(\psi,\vartheta,\varphi)$, we need to introduce some auxiliary quantities. Let a_1 and a_2 be the annihilation operators for photons in the input ports 1 and 2, respectively. Then the Schwinger operators [1] J_1, J_2, and J_3 are defined as

$$J_1 = \tfrac{1}{2}(a_1^\dagger a_2 + a_1 a_2^\dagger) \ , \tag{2a}$$

$$J_2 = -\tfrac{1}{2}i(a_1^\dagger a_2 - a_1 a_2^\dagger), \tag{2b}$$

$$J_3 = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \ . \tag{2c}$$

These operators obey the SU(2) or angular momentum commutation rules

$$[J_1,J_2]=iJ_3, \quad [J_2,J_3]=iJ_1, \quad [J_3,J_1]=iJ_2 \ . \tag{3}$$

An explicit expression for the beam splitter operator can now be given as

$$B(\psi,\vartheta,\varphi)=\exp(i\psi J_3)\exp(i\vartheta J_2)\exp(i\varphi J_3) \tag{4}$$

where the angles ψ, ϑ, and φ are related to the properties of the beam splitter as follows: ψ and φ are phase shifts imparted by the beam splitter while $\cos^2(\vartheta/2)$ is the transmittance of the beam splitter. (It should be stressed that for a general beam splitter consisting, say, of a stack of plates with varying refractive indices, the phase shifts ψ and φ cannot be eliminated but must be retained as an essential part of the description.) Equation (4) gives the following geometrical description of the action of a beam splitter: it says that the beam splitter performs a sequence of Euler rotations on the input state, with the generators of the rotations being the Schwinger operators and the angles of rotation being determined by the characteristics of the beam splitter.

A fundamental question one can ask about the beam splitter is the following: if the joint Fock state $|\Psi_{\text{in}}\rangle = |n_1\rangle|n_2\rangle$ (i.e., n_1 photons in port 1 and n_2 photons in port 2) is incident on the beam splitter, what is the probability amplitude for observing n_1' photons in port 1′ and n_2' photons in port 2′? The answer, which we will denote $P(n_1',n_2'|n_1,n_2)$, is obtained simply by projecting the output state (1) onto the final Fock state $|n_1'\rangle|n_2'\rangle$:

$$P(n_1',n_2'|n_1,n_2) = \langle n_1',n_2'|B(\psi,\vartheta,\varphi)|n_1,n_2\rangle \tag{5a}$$

$$= \langle n_1',n_2'|\exp(i\psi J_3)\exp(i\vartheta J_2)\exp(i\varphi J_3)|n_1,n_2\rangle \tag{5b}$$

$$= R^{(1/2)(n_1+n_2)}_{(1/2)(n_1-n_2),(1/2)(n_1'-n_2')}(\psi,\vartheta,\varphi) \ . \tag{5c}$$

Equation (5c) is just a rotation matrix element [1] (or irreducible representation of the rotation group) with the angular momentum parameters $j=\frac{1}{2}(n_1+n_2)$, $m=\frac{1}{2}(n_1-n_2)$, and $m'=\frac{1}{2}(n_1'-n_2')$ and the Euler angles ψ, ϑ, and φ. The above forms for the probability amplitude were noted recently by Campos, Saleh, and Teich [7], who also undertook a detailed investigation of their properties.

A. Analogy with the 2D isotropic oscillator

Before establishing the analogy between the beam splitter and the 2D oscillator, let us recall some facts about the oscillator. Consider a two-dimensional isotropic oscillator of mass μ and frequency ω. The steady-state Schrödinger equation for this system can be separated in both Cartesian and polar coordinates [9]. The Cartesian eigenfunctions are

$$\Psi_{n_1,n_2}(x,y) = [\beta/(n_1!n_2!2^{n_1+n_2}\pi)^{1/2}]H_{n_1}(\beta x)H_{n_2}(\beta y)\exp[-\beta^2(x^2+y^2)/2] \quad (n_1,n_2=0,1,2,\ldots) \tag{6}$$

while the polar eigenfunctions are

$$\Phi_l^n(r,\varphi) = \exp[-\tfrac{1}{2}i(n-l)(\pi/2)]\beta(\pi)^{-1/2}[\Gamma(\tfrac{1}{2}(n-|l|)+1)/\Gamma(\tfrac{1}{2}(n+|l|)+1)]^{1/2}$$
$$\times(\beta r)^{|l|}L^{|l|}_{(1/2)(n-|l|)}(\beta^2 r^2)\exp[-(\beta r)^2/2]e^{il\varphi} \quad [n=0,1,2,\ldots;l=n,n-2,\ldots,-(n-2),-n] \ . \tag{7}$$

In the above expressions $\beta=(\mu\omega/\hbar)^{1/2}$, $H_n(x)$ are the Hermite polynomials, $L_m^l(r)$ are the associated Laguerre polynomials, and $\Gamma(n+1)=n!$ is the Γ function. The phase factor $\exp[-\frac{1}{2}i(n-l)(\pi/2)]$ in (7) is not really necessary but has been inserted for later convenience. The eigenfunctions (6) and (7) exhibit degeneracy: all Cartesian eigenfunctions with the same n_1+n_2 are degenerate while all polar eigenfunctions with a definite n (but different l) are degenerate. The degeneracy of the energy level $E_n=(n+1)\hbar\omega$ is $n+1$ (and thus runs through all the natural integers).

From general quantum-mechanical principles we know that any Cartesian eigenfunction can be expressed as a linear combination of the polar eigenfunctions and vice versa. By exploiting the SU(2) degeneracy group of the oscillator one can work out the precise connection between the two sets of eigenfunctions. One finds that [10]

$$\Phi_{2m}^{2j}(r,\varphi) = \sum_{m'=-j}^{j} C^j_{m,m'}\Psi_{j+m',j-m'}(x,y) \tag{8}$$

where the coefficients $C^j_{m,m'}$ are given by

$$C^j_{m,m'}=[(j+m)!(j-m)!(j+m')!(j-m')!]^{1/2}\sum_k i^{2k-m+m'}[\cos(\pi/4)]^{2j-2k+m-m'}[\sin(\pi/4)]^{2k-m+m'} \times[(j+m-k)!k!(j-k-m')!(k-m+m')!]^{-1} \tag{9}$$

with the summation extending over all values of k for which the factorials are nonnegative. The $C^j_{m,m'}$ are particular instances of the rotation matrix elements [1]. The reason for inserting the phase factor in (7) was to ensure that (8) would be valid with the (conventional) choice of phase in (9). Equation (8) demonstrates how any polar eigenfunction can be expressed as a linear combination of the degenerate Cartesian eigenfunctions at the same energy.

Let us proceed to the analogy between the beam splitter and the oscillator. Consider a beam splitter with a transmittance of 50% that also imparts phase shifts of 90° to the outgoing beams so that it is characterized by the Euler angles

$$\vartheta=\pi/2,\quad \psi=-\varphi=\pi/2\ . \tag{10}$$

In this case the expression for $P(n_1',n_2'|n_1,n_2)$, Eq. (5c), reduces precisely to that for $C^j_{m,m'}$, Eq. (9), provided that the quantities $\{n_1,n_2,n_1',n_2'\}$ and $\{j,m,m'\}$ are related in the manner indicated below Eq. (5c). In other words, the fundamental probability amplitudes for the beam splitter (10) are identical to the coefficients relating the Cartesian and polar eigenstates of the 2D oscillator. As an amusing application of this observation, we show how the beam splitter (10) can be used as an analog device to realize the eigenstates of the 2D oscillator. Suppose that the joint Fock state $|\Psi_{\text{in}}\rangle=|n_1\rangle|n_2\rangle$ (i.e., n_1 photons in port 1 and n_2 photons in port 2) is incident on this beam splitter. This input state can be thought of as the Cartesian eigenstate (6) with quantum numbers n_1 and n_2. Then the output state produced by the beam splitter is analogous to the polar eigenstate (7) with quantum numbers $n=n_1+n_2$ and $l=n_1-n_2$. It should be stressed that the output state does not consist of definite numbers of photons in the ports 1′ and 2′ but is rather a superposition of such states, with the coefficients in the superposition being given by (9).

B. Analogy with the 2D hydrogen atom

Consider a hydrogen atom in two spatial dimensions with a $1/r$ potential between the electron and proton. Denote by μ and $-e$ the reduced mass and charge, respectively, of the electron. The Schrödinger equation for this system can be separated both in polar (r,φ) and parabolic (u,v) coordinates [the latter coordinates [11] are related to the former by the equations $u=(2r)^{1/2}\cos(\vartheta/2)$ and $v=(2r)^{1/2}\sin(\vartheta/2)$; by restricting u and v to the ranges $-\infty<u<\infty$ and $0\le v<\infty$ one ensures that the correspondence between the two sets of coordinates is one to one]. The polar eigenfunctions are [12]

$$\Phi^n_l(r,\varphi)=\beta[(2\pi)(2n-1)]^{-1/2}[(n-1-|l|)!/(n-1+|l|)!]^{1/2}(\beta r)^{|l|}L^{2|l|}_{n-1-|l|}(\beta r)\exp(-\tfrac{1}{2}\beta r)e^{il\varphi}$$
$$[n=1,2,3\ldots;l=(n-1),(n-2),\ldots,-(n-2),-(n-1)] \tag{11}$$

where $\beta=(8\mu|E|/\hbar^2)^{1/2}$ with E, the energy eigenvalue, being given by Eq. (13) below. The parabolic eigenfunctions are [10]

$$\Psi_{n_1,n_2}(u,v)=\exp(-\tfrac{1}{2}i\pi n_2)\beta[n_1!n_2!2^{n_1+n_2+1}(n_1+n_2+1)\pi]^{-1/2} \times H_{n_1}((\beta/2)^{1/2}u)H_{n_2}((\beta/2)^{1/2}v)\exp[-\beta(u^2+v^2)/4]\quad (n_1,n_2=0,1,2,\ldots\text{but with } n_1+n_2 \text{ even}) \tag{12}$$

where an extra phase factor $\exp(-\frac{1}{2}i\pi n_2)$ has been inserted on the right-hand side for later convenience. The energy eigenvalues corresponding to the polar (n,l) and parabolic (n_1,n_2) eigenfunctions are

$$E_{n,l}=-\mu e^4/[2\hbar^2(n-\tfrac{1}{2})^2]\quad\text{and}\quad E_{n_1,n_2}=-(2\mu e^4/\hbar^2)/(n_1+n_2+1)^2\ . \tag{13}$$

The energies depend only on the polar quantum number n, or only on the sum of the parabolic quantum numbers n_1 and n_2, and this gives rise to the well-known degeneracies of this system.

From general quantum-mechanical principles we know that the polar eigenfunctions must be expressible as linear combinations of the parabolic eigenfunctions and vice versa. The precise connection between the two sets of eigenfunctions can be worked out with the aid of the degeneracy group of this system, which is SO(3) [13]; the three generators of this group are the z component of angular momentum and the x and y components of the three-dimensional Runge-Lenz vector [14] (the plane of the hydrogen atom is taken to be the x-y plane). On carrying out the requisite calculations [10] we find that the parabolic eigenfunctions are given in terms of the polar eigenfunctions by

$$\Psi_{l-m,l+m}(u,v)=\sum_{m'=-l}^{l}D^l_{m,m'}\Phi^{l+1}_{m'}(r,\varphi) \tag{14}$$

where

$$D^{l}_{m,m'}=[(l+m)!(l-m)!(l+m')!(l-m')!]^{1/2}\times\sum_{k}(-1)^{k+m-m'}[\cos(\pi/4)]^{2l-2k-m+m'}[\sin(\pi/4)]^{2k+m-m'}[(l-m-k)!k!(l-k+m')!(k+m-m')!] \quad (15)$$

with the summation extending over all values of k for which the factorials are non-negative. As in the oscillator problem, the above coefficients are particular instances of the rotation matrix elements. Equation (14) shows how any parabolic eigenfunction can be expressed as a linear combination of the degenerate polar eigenfunctions at the same energy. The additional phase factor was inserted in (12) so that (14) would hold without the necessity of modifying the (conventional choice of) phase in (15).

To indicate the connection between the beam splitter and the hydrogen atom, consider a beam splitter with a transmittance of 50% that introduces no phase shifts so that it is characterized by the Euler angles

$$\vartheta=\pi/2,\quad \psi=\varphi=0\ . \quad (16)$$

For such a beam splitter the matrix of probability amplitudes [with elements $P(n_1',n_2'|n_1,n_2)$ given by (5c)] becomes identical to the inverse rotation matrix D^{-1} that transforms the parabolic eigenstates at any energy into the polar eigenstates at the same energy. Let us rephrase the preceding statement more transparently. For this purpose, suppose that the joint Fock state $|\Psi_{\text{in}}\rangle=|n_1\rangle|n_2\rangle$ (with n_1+n_2 even) is incident on the beam splitter (16). This input state can be thought of as the parabolic eigenstate (12) with n_1 and n_2 representing the parabolic quantum numbers. Then the output state produced by the beam splitter is analogous to the polar eigenstate (11) with quantum numbers $n=\frac{1}{2}(n_1+n_2)+1$ and $l=\frac{1}{2}(n_2-n_1)$. It should be stressed that the output state does not consist of definite numbers of photons in the ports 1′ and 2′ but is rather a superposition of such states, with the coefficients in the superposition being determined by (15).

III. GAUSSIAN BEAMS

Gaussian beams are solutions to the paraxial wave equation and are useful in describing the fields of lasers [15,16]. There are essentially two different types of Gaussian beams—the so-called Hermite-Gaussian and Laguerre-Gaussian beams. If the beam has rectangular symmetry about its axis of propagation, then the appropriate solutions of the paraxial wave equation are the Hermite-Gaussian modes

$$\Psi_{n,m}(x,y,z)=(2/\pi)^{1/2}[w(z)]^{-1}(n!m!2^{n+m})^{-1/2}\times\exp\{i(n+m+1)[\psi(z)-\psi_0]\}\times H_n(\sqrt{2}x/w(z))H_m(\sqrt{2}y/w(z))\times\exp[-ik(x^2+y^2)/2q(z)]\ . \quad (17)$$

If, on the other hand, the beam has cylindrical symmetry about its axis of propagation, the appropriate solutions are the Laguerre-Gaussian modes

$$\Phi^{p}_{l}(r,\vartheta,z)=(2/\pi)^{1/2}[w(z)]^{-1}[p!/(|l|+p)!]^{1/2}\times\exp\{i(2p+l+1)[\psi(z)-\psi_0]\}\times[\sqrt{2}r/w(z)]^{|l|}L_p^{|l|}(2r^2/w(z)^2)\times\exp[-ikr^2/2q(z)+il\vartheta]\ . \quad (18)$$

Our notation in the above equations is similar to Siegman's [15] [see especially his Eq. (64) in Chap. 16 and Eq. (41) in Chap. 17]. The rectangular and cylindrical solutions each constitute a complete and orthonormal set, so either may be used as a basis for the expansion of an arbitrary paraxial beam. The question naturally arises as to what the connection between the two sets is, i.e., how can one expand the cylindrical beams in terms of the rectangular ones, and vice versa? For this purpose we replace the radial index p in (18) by the new index $p'=2p+|l|$. Then, on noting the close similarity of (17) to (6) and (18) to (7), we readily find that

$$\Phi^{p'}_{l}(r,\vartheta,z)=\exp\left[\frac{i\pi}{4}(p'-l)\right]\sum_{m'=-p'/2}^{p'/2}C^{p'/2}_{l/2,m'}\Psi_{\frac{p'}{2}+m',\frac{p'}{2}-m'}(x,y,z) \quad (19)$$

where the quantities $C^{p'/2}_{l/2,m'}$ have been defined earlier in (9). On comparing (19) with (8) we see that (apart from a phase factor) the connection between the rectangular and cylindrical Gaussian beams is precisely the same as that between the Cartesian and polar eigenfunctions of the 2D oscillator. Thus, in the process of solving the latter problem, we have automatically solved the former one.

We have not been able to find Eq. (19) in the literature (although its existence is hinted at by several authors [15–17]). It is well known that the paraxial wave equation is isomorphic to the free particle Schrödinger equation in two dimensions, with the time variable in the latter being replaced by the spatial variable along the axis of propagation in the former. In view of this, and the fact that a free particle is a limiting case of a harmonic oscillator, one might have guessed (19) on the basis of (8). [This argument is, of course, not rigorous and merely advanced in an effort to rationalize (19) after the fact.] If one considers a beam propagating in a square-law medium [with refractive index $n(r)=n_0-n_2r^2$, where r is the radial distance from a central axis], the analogy between optics and quantum mechanics becomes even closer. In this case [17] the optical wave equation resembles the quantum-mechanical equation for a harmonic oscillator and the rectangular and cylindrical Gaussian beam solutions are again related by (19).

IV. CONCLUSION

We have exhibited some analogies between the lossless quantum-mechanical beam splitter and two simple quantum systems, namely, the two-dimensional oscillator and the two-dimensional hydrogen atom. It may be asked whether this analogy has any useful practical applications in optics or elsewhere, or whether it can be extended any further (say, to the three-dimensional oscillator or the three-dimensional hydrogen atom). The answer to both these questions is (regrettably) no, at present. However, we feel that the existence of the analogy itself is interesting and worth publicizing, inasmuch as the beam splitter is one of the commonest pieces of optical apparatus and the hydrogen atom and the harmonic oscillator are basic paradigms for many kinds of more complex systems in nature. Perhaps both optical physicists and quantum mechanicians will derive some pleasure from this simple example that bridges their two disciplines.

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