Two-level system in a Gaussian field: An approximate solution

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The problem of the evolution of a two-level quantum system in a radiation field of Gaussian shape with detuning is addressed by seeking an approximate expression for the transformation function governing the interaction. The transformation operator requires the evaluation of two independent matrix elements: the former is obtained by the Rosen-Zener conjecture; the latter is based on an expansion using the main parameters involved in the first one. The expressions obtained, which are supposed to hold in the case of small detunings, have been tested by comparison with a numerical integration. Then the solutions are modified to take into account decay phenomena to other levels. As a final example, the expressions dealing with the transition probability in the case of two separated oscillatory fields are given.

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I. INTRODUCTION

In experiments of high-resolution spectroscopy with atomic or molecular beams, many times the interaction occurs between laser radiation and matter according to the Ramsey or other multizone schemes. In this case for each zone the problem of a two-state quantum-mechanical system coupled by a time-dependent field must be solved. In general the time-dependent Schrödinger equation reduces, under the rotating-wave approximation, to the following well-known differential system for the probability amplitudes C_1 and C_2:

$$\frac{dC_1}{dt}=i\frac{\omega_R}{2}g(t)e^{i(\Omega t+\delta)}C_2\ ,$$
$$\frac{dC_2}{dt}=i\frac{\omega_R}{2}g(t)e^{-i(\Omega t+\delta)}C_1\ , \qquad (1)$$

or to the equivalent second-order differential equation

$$\frac{d^2C_{1,2}}{dt^2}+\frac{dC_{1,2}}{dt}\left[\mp i\Omega-\frac{dg}{dt}\Big/g(t)\right]+C_{1,2}\left[\frac{\omega_R}{2}g(t)\right]^2=0\ , \qquad (2)$$

where the time-dependent part of the Hamiltonian has been written as $H(t)=-\hbar\omega_R g(t)\cos(\omega t+\delta)$, $g(t)$ is the shape function of the field, and $\Omega=\omega-\omega_0$ is the detuning between the radiation frequency and the atomic resonance ω_0. A general solution of (1) is not possible; however, solutions have been found to exist in several cases. Among these it is worthwhile to mention the Rosen and Zener (RZ) solution for $g=\mathrm{sech}(\pi t/\tau)$ [1] with the conjecture of a generalization to a wide class of $g(t)$ (which is unfortunately only an approximation), the solution when $g=\mathrm{const}$ in a time interval, obtained by Rabi [2]; the class of time-asymmetric pulses found by Bambini and Berman [3], where $\mathrm{sech}\pi t/\tau$ is included as a special case; and the studies on the time symmetrization of the above class by Bambini and Lindberg [4]. If Ω too is allowed to be time dependent, different solutions are possible; Hioe and Carroll [5] have discussed this problem and have found solutions which extend some results obtained by different authors and are mentioned in [5]. Moreover, Robiscoe phenomenologically introduced decay constants in the RZ problem [6] and developed a way to add correction terms [7] in the RZ conjecture.

Although the case of a pulse with Gaussian-modulated amplitude and constant detuning, which is essential in studying laser spectroscopy problems, has not been solved exactly, nevertheless, Thomas [8] showed that in this case the RZ conjecture is correct for very small detunings. However, for multizone interaction techniques it is important to find some approximation for the diagonal elements of the matrix connecting the initial and final values of the amplitudes C. This type of solution does not seem to have been examined in the literature. Moreover, the corrections arising from the finite extent of the time interval corresponding to each zone are to be taken into account to an adequate accuracy level.

To find the interaction expressions in the case of a Gaussian field, the solutions holding the well-known case of the hyperbolic secant are used in this work as a starting point. Then expressions are obtained in the case of a Gaussian field and numerical checks are performed to make sure that, at least for small detunings, which is the range of interest in high-resolution spectroscopy, the expressions found are satisfactory. Then some considerations are expressed when decays to other levels are introduced in the problem and the analysis is performed for a Ramsey interaction scheme as an example of multizone interaction.

With the expressions found for a Gaussian field easier evaluations of small effects and uncertainty sources (e.g., the first-order Doppler effect) in multizone interactions can be performed, of particular interest in studying atomic-beam frequency standards. An interesting method used to get approximate expressions is the expansion proposed by Magnus in [9]. For a Gaussian pulse, only the first two iterations can be performed analytically [8] and they give numerical results to a lower level than the present development.

II. AN APPROXIMATE SOLUTION FOR A GAUSSIAN FIELD SHAPE

With reference to the hyperbolic secant field shape, considering that the problem of interest is a multizone interaction, in the kth zone instead of t, it is suitable to introduce the variable $s_k=t-t_k$ in order to get the field distribution $g_k=\mathrm{sech}\,\pi s_k/\tau_k$, whereas in the Hamiltonian the constants are labeled $\omega_{R,k}$ and δ_k.

With the aid of the transformation $z_k=(1+th\,\pi s_k/\tau_k)/2$ [1], the hypergeometric equation is obtained [10]. The two independent solutions of Eq. (2) are

$$p(z_k)=F(a_k,b_k,c,z_k) \tag{3a}$$

and

$$q(z_k)=z_k^{(1-c)}F(a_k+1-c,b_k+1-c,2-c,z_k)\ , \tag{3b}$$

where F is the Gauss hypergeometric series and, dropping for the sake of simplicity the subscript k when confusion is not introduced,

$$a=-b=\omega_R\tau/2\pi\ ,$$

$$c=c_\pm=\tfrac{1}{2}\pm i\Omega\tau/2\pi=c_\mp^*\ ,$$

where plus and minus refer to C_2 and C_1, respectively, in (2).

When the integration is performed from an initial to a final time, exact solutions can be written for C_1 and C_2; however, when $z(s_i)=z_i\simeq 0$ and $z(s_f)=z_f\simeq 1$, it is useful and adequately accurate to express them by a series development up to the first order in $z_i^{1/2}$ and $(1-z_f)^{1/2}$,

$$\begin{aligned}
C_{1f}\simeq{}&\left[W(c_-)+\frac{a}{c_-}\frac{\sin\pi a}{\sin\pi c_+}[(1-z_f)^{c_-}+z_i^{c_-}]\right]C_{1i}\\
&+ie^{i(\delta_k+\Omega t_k)}\left[\frac{\sin\pi a}{\sin\pi c_+}-\frac{a}{c_-}W(c_+)(1-z_f)^{c_-}-\frac{a}{c_+}W(c_-)z_i^{c_+}\right]C_{2i}\ ,\\
C_{2f}\simeq{}&ie^{-i(\delta_k+\Omega t_k)}\left[\frac{\sin\pi a}{\sin\pi c_+}-\frac{a}{c_+}W(c_-)(1-z_f)^{c_+}-\frac{a}{c_-}W(c_+)z_i^{c_-}\right]C_{1i}\\
&+\left[W(c_+)+\frac{a}{c_+}\frac{\sin\pi a}{\sin\pi c_+}[(1-z_f)^{c_+}+z_i^{c_+}]\right]C_{2i}\ ,
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&W(c)=\frac{\Gamma(c)\Gamma(c)}{\Gamma(c+a)\Gamma(c-a)}=\prod_{\kappa=0}^{\infty}\left[1-\left[\frac{a}{\kappa+c}\right]^2\right]\ ,\\
&\frac{1}{\sin\pi c_+}=\frac{1}{\sin\pi c_-}=\mathrm{sech}\frac{\Omega\tau}{2}\ ,\\
&1-z_f\simeq e^{-2\pi s_f/\tau},\quad z_i\simeq e^{2\pi s_i/\tau}\ ,
\end{aligned} \tag{5}$$

and Γ is the gamma function. From (5) the following relation is obtained:

$$W(c_+)W(c_-)=W(c)W^*(c)=1-\sin^2\pi a\ \mathrm{sech}^2\Omega\tau/2\ . \tag{6}$$

This last identity stems from the unitary transformation imposed by Eqs. (1) and can be also obtained directly from expression (5) (see Ref. [10]). Moreover, the expansions used in (4) yield solutions satisfying the unitarity conditions up to the order considered.

When a Gaussian field is considered, namely $g_g(t)=e^{-\pi(t/\tau)^2}$, and τ in this case too has the meaning of pulse area, the RZ approximation is expected to be satisfactory for $\Omega\tau\ll 1$ [8], which is a usual condition in high-resolution spectroscopy techniques. Therefore in (4) the substitution

$$\mathrm{sech}\frac{\Omega\tau}{2}\longrightarrow\exp\left[-\frac{1}{\pi}\left[\frac{\Omega\tau}{2}\right]^2\right]$$

seems appropriate, as has also been checked in [11].

On the contrary, if we define $W_g(c)$, the analog of $W(c)$ in the Gaussian case, it is not immediate to find a substitution $W_g(c)$ to $W(c)$, even if expression (6) seems to be of some help.

Therefore the way adopted here has been to start from series developments of $W(c)$ for different values of a to yield approximate expressions in terms of the pulse area τ, or the parameter a, and of the Fourier transform of the hyperbolic secant, having as a constraint Eq. (6) to some specified approximation level. Then the appropriate transformation suggested by Rosen and Zener is used.

For instance, by a series expansion of W through the Γ functions around $c=\frac{1}{2}$ and $a=0$ one gets

$$W(c_+,\delta a)\simeq 1-(\psi'+\psi^{(2)}\delta c+\tfrac{1}{2}\psi^{(3)}\delta c^2+\tfrac{1}{6}\psi^{(4)}\delta c^3+\cdots)\delta a^2 +[\tfrac{1}{2}\psi'^2-\tfrac{1}{12}\psi^{(3)}+(\psi'\psi^{(2)}-\tfrac{1}{12}\psi^{(4)})\delta c+\tfrac{1}{2}(\psi'\psi^{(3)}+\psi^{(2)2}-\tfrac{1}{12}\psi^{(5)})\delta c^2+\cdots]\delta a^4+\cdots\ , \tag{7}$$

where ψ is the logarithmic derivative of the Γ function. The different order derivatives are computed for $c=\frac{1}{2}$, and $\delta c = i\Omega\tau/2\pi$.

Rearranging (7), the real and imaginary parts are separated

$$W(c_+,\delta a)\simeq 1-\frac{(\pi\delta a)^2}{2}+\frac{(\pi\delta a)^4}{4!}-\frac{1}{6!}(\pi\delta a)^6+\cdots+2\left[\frac{\Omega\tau}{2}\right]^2\left[\left[\frac{\pi\delta a}{2}\right]^2-\left[\frac{\pi\delta a}{2}\right]^4\left[\frac{4\psi^{(2)^2}}{\pi^6}-\frac{2}{3}\right]+\cdots\right] +i2\frac{\Omega\tau}{2}\left[-2\frac{\psi^{(2)}}{\pi^3}\right]\left[\left[\frac{\pi\delta a}{2}\right]^2-\left[\frac{\pi\delta a}{2}\right]^4\left[2-\frac{1}{3\pi^2}\frac{\psi^{(4)}}{\psi^{(2)}}\right]\cdots\right]. \tag{8}$$

Although the series developments in expression (8) have been reported up to the first correction in $\Omega\tau$ in both the real and in the imaginary part, the evaluations have been performed to higher-order terms both in $\Omega\tau$ and in δa. The constraint $\Omega\tau/2 \ll 1$ is not severe in high-resolution spectroscopy, whereas the extrapolation in the range of a from 0 to $\frac{1}{2}$ is very important.

Considering that, for $\Omega\tau=0$, $W(\frac{1}{2},a)=\cos\pi a$ and that, moreover, $\Omega\tau/2$ can be interpreted as the first term of $\tanh\Omega\tau/2$, some suggestions how to extrapolate in (8) are immediate.

However, two other expansions are useful in checking the extrapolations of the terms in (8). By direct series expansion

$$W(c_+,\tfrac{1}{4})\simeq\frac{1}{\sqrt{2}}\left[1+\left[\frac{\Omega\tau}{2}\right]^2\left[\frac{1}{2}-\frac{2}{\pi^2}(\ln 2)^2\right] +i\frac{\Omega\tau}{2}\frac{2\ln 2}{\pi}\right], \tag{9}$$

and by using the expressions 6.1.28 and 6.1.33 of Ref. [10],

$$W(c_+,\tfrac{1}{2})\simeq\frac{4\ln 2}{\pi}\left[\frac{\Omega\tau}{2}\right]^2+i\frac{\Omega\tau}{2}\ . \tag{10}$$

Taking into account developments (8)–(10) a satisfactory approximation to $W(c_+,a)$ is

$$W(c_+,a)\simeq\cos\pi a+2\sin^2\frac{\pi a}{2}\tanh^2\frac{\Omega\tau}{2}\left[1-\left[\frac{2}{\pi}\right]^4\sin^2\frac{\pi a}{2}\right] +i2\sin^2\frac{\pi a}{2}\left[-2\frac{\psi^{(2)}}{\pi^3}\right]\tanh\frac{\Omega\tau}{2}\left[1-\left[\frac{5}{3}-\frac{\psi^{(4)}}{3\pi^2\psi^{(2)}}\right]\sin^2\frac{\pi a}{2}\left[1+\tfrac{1}{2}\sin^2\frac{\pi a}{2}\right]\right], \tag{11}$$

where the dependence on $\tanh\Omega\tau/2$ has been suggested by expression (6). The comparison between (11) and expansions (8)–(10) shows in general a reasonable agreement. Moreover the real part of $W(c_+,a)$ in (11) appears approximated to better than 10^{-3} for $a=\frac{1}{2}$ and $\Omega\tau\le 0.25$, and for $a<\frac{3}{8}$ and $\Omega\tau\le 0.25$ the approximation is better than the 10^{-4} level, whereas in the imaginary part for the same couples of values the agreements are 1.5×10^{-3} and $<5\times10^{-4}$. In general, expression (11) does not compare unfavorably with series developments (8) and (9).

Expression (11) has been written in such a way that the RZ conjecture can be directly applied. In fact, $a=\omega_R\tau/2\pi$ has a very general meaning, and through the relation $\tanh\Omega\tau/2=\pm(1-\mathrm{sech}^2\Omega\tau/2)^{1/2}$, with the $\pm$ sign according to $\Omega\tau \gtrless 0$, the substitutions in the Gaussian case are straightforward and the following expression for $W_g(c_+,a)$ is obtained:

$$W_g(c_+,a)\simeq\cos\pi a+2\sin^2\frac{\pi a}{2}\left\{1-\exp\left[-\frac{2}{\pi}\left[\frac{\Omega\tau}{2}\right]^2\right]\right\}\left[1-\left[\frac{2}{\pi}\right]^4\sin^2\frac{\pi a}{2}\right] +i2\left[-\frac{2\psi^{(2)}}{\pi^3}\right]\left\{1-\exp\left[-\frac{2}{\pi}\left[\frac{\Omega\tau}{2}\right]^2\right]\right\}^{1/2}\sin^2\frac{\pi a}{2}\left[1-\frac{1}{3}\left[5-\frac{\psi^{(4)}}{\pi^2\psi^{(2)}}\right]\sin^2\frac{\pi a}{2}\left[1+\tfrac{1}{2}\sin^2\frac{\pi a}{2}\right]\right]. \tag{11'}$$

Then the solutions of Eqs. (1) and (2) for a Gaussian field, obtained through the RZ conjecture and its application to (11) to get $W_g(c,a)$, are compared to numerical integration results obtained with an estimated accuracy of 10^{-4}. Moreover, expressions similar to (8)–(10) have been obtained taking into account that series developments of the Fourier transforms suggest the substitutions $(\Omega\tau)_s^2\rightarrow(2/\pi)(\Omega\tau)_g^2$. As a result of these comparisons, the real part of expression (11′) agrees with numerical integration better than 1×10^{-3} for $a\le\frac{1}{2}$ and $\Omega\tau\le 1$, whereas the imaginary part agreement is better than 4×10^{-3} for $\Omega\tau\le 0.25$. Also for the Gaussian shape the situation improves as detuning and field amplitude decrease.

III. MODIFICATIONS INTRODUCED BY DECAY

Phenomenological decay constants γ_1 and γ_2 to other levels can be easily introduced through the relation (see Ref. [13], p. 84)

$$C_i = \mathcal{C}_i e^{-(1/2)\gamma_i t} , \tag{12}$$

and the functions $\mathcal{C}_i$ satisfy the same differential equations (1) and (2) provided Ω is substituted with

$$\Omega' = \Omega + i\gamma/2, \quad \gamma = \gamma_2 - \gamma_1 . \tag{13}$$

In this case, in solutions (3a) and (3b) of the hyperbolic secant field, the parameters

$$c'_{\pm} = c_{\pm} \mp \gamma^{\tau/4\pi} = \tfrac{1}{2} \pm i\frac{\Omega\tau}{2\pi} \mp \frac{\gamma\tau}{4\pi} \tag{14}$$

are introduced. Provided $\mathrm{Re}(c') > 0$ which means $|\gamma|\tau < 2\pi$, the hypergeometric Gauss series holds. According to (14)

$$c'_{+} = 1 - c'_{-} , \tag{15}$$

but

$$c'_{+} \neq c'^{*}_{-}$$

As it was proved in the special case considered in [6], under the above indicated condition, the amplitudes $C_i(t)$ behave regularly; the same regular behavior is found also in the most general case when both the initial amplitudes are different from zero. Defining in the kth zone with s_i and s_f the initial and final values of the variable s, respectively, under the hypothesis $|\mathrm{Re}(c' - \frac{1}{2})| \ll 1$, that is $|\gamma|\tau \ll 4\pi$, which is a safe condition in multizone interactions, the final values of the amplitudes C_i can be written by means of a truncated Taylor expansion:

$$\begin{aligned} C_1(s_f) &\simeq C_1(s_i) e^{-(1/2)\gamma_1(s_f - s_i)} \left[W(c'_{-}) + \frac{a}{c'_{-}} \frac{\sin\pi a}{\sin\pi c'_{+}} [(1 - z(s_f))^{c'_{-}} + z(s_i)^{c'_{-}}] \right] \\ &\quad + iC_2(s_i) e^{-(1/2)(\gamma_1 s_f - \gamma_2 s_i)} e^{i(\delta_k + \Omega t_k)} \left[\frac{\sin\pi a}{\sin\pi c'_{+}} - \frac{a}{c'_{-}} W(c'_{+})[1 - z(s_f)]^{c'_{-}} - \frac{a}{c'_{+}} W(c'_{-}) z(s_i)^{c'_{+}} \right] , \\ C_2(s_f) &\simeq iC_1(s_i) e^{-(1/2)(\gamma_2 s_f - \gamma_1 s_i)} e^{-i(\delta_k + \Omega t_k)} \left[\frac{\sin\pi a}{\sin\pi c'_{+}} - \frac{a}{c'_{+}} W(c'_{-})[1 - z(s_f)]^{c'_{+}} - \frac{a}{c'_{-}} W(c'_{+}) z(s_i)^{c'_{-}} \right] \\ &\quad + C_2(s_i) e^{-(1/2)\gamma_2(s_f - s_i)} \left[W(c'_{+}) + \frac{a}{c'_{+}} \frac{\sin\pi a}{\sin\pi c'_{+}} \{[1 - z(s_f)]^{c'_{+}} + z(s_i)^{c'_{+}}\} \right] . \end{aligned} \tag{16}$$

These analytical evaluations are straightforward but rather cumbersome. Detailed developments are found in [12]. There is a rather high symmetry in (16) because $\sin\pi c'_{+} = \sin\pi c'_{-} = \cosh\Omega'\tau/2$.

According to the hypothesis above, the terms in $1 - z(s_f)$ and $z(s_i)$ can be disregarded. Moreover,

$$\frac{1}{\sin\pi c'_{\pm}} = \mathrm{sech}\frac{\Omega'\tau}{2} = \mathrm{sech}\frac{\Omega\tau}{2} \exp\left[-i \arctan\left[\tan\frac{\gamma\tau}{4} \tanh\frac{\Omega\tau}{2} \right] \right] \Big/ \left[1 - \sin^2\frac{\gamma\tau}{4} \mathrm{sech}^2\frac{\Omega\tau}{2} \right]^{1/2} \tag{17a}$$

which expressed in amplitude and phase to the first order in $\gamma\tau$ gives

$$\mathrm{sech}\frac{\Omega'\tau}{2} \simeq \mathrm{sech}\frac{\Omega\tau}{2} \exp\left[-i\frac{\gamma\tau}{4} \tanh\frac{\Omega\tau}{2} \right] , \tag{17b}$$

where the amplitude is independent of $\gamma\tau$, and the phase shows a linear dependence.

However, from (5), to the same approximation order

$$W(c'_{\pm}, a) \simeq W(c_{\pm}, a) \left[1 \pm \frac{\gamma\tau}{4\pi} [\psi(c_{\pm} + a) + \psi(c_{\pm} - a) - 2\psi(c_{\pm})] \right] , \tag{18}$$

with the choice of the upper or lower sign according to the plus or minus subscript in c' and c, respectively. This expression shows that both amplitude and phase of $W(c', a)$ are linearly dependent on $\gamma\tau$.

When considering a Gaussian-shaped field, the following approximated expression is obtained:

$$\exp\left[-\frac{1}{\pi} \left[\frac{\Omega'\tau}{2} \right]^2 \right] \simeq \exp\left[-\frac{1}{\pi} \left[\frac{\Omega\tau}{2} \right]^2 - i\frac{\Omega\tau}{2} \frac{\gamma\tau}{2\pi} \right] \tag{19}$$

with the same properties as (17b), therefore after substitution into the expression of $W_g(c', a)$ that is, formula (11′), with the appropriate modifications, the same linear dependences in amplitude and phase are found.

Then a suitable approximation of the matrix M_I linking the amplitudes $C_j(s_f)$ to $C_j(s_i)$ in the zone k is obtained from (16)

$$M_I \simeq \begin{vmatrix} W(c'_-)e^{-(1/2)\gamma_1(s_f-s_i)} & ie^{-(1/2)(\gamma_1 s_f-\gamma_2 s_i)}e^{i(\delta_k+\Omega t_k)} \times \sin\pi a \, \mathrm{sech}\frac{\Omega\tau}{2}\exp\left[-i\frac{\gamma\tau}{4}\tanh\frac{\Omega\tau}{2}\right] \\ ie^{-(1/2)(\gamma_2 s_f-\gamma_1 s_i)-i(\delta_k+\Omega t_k)} \times \sin\pi a \, \mathrm{sech}\frac{\Omega\tau}{2}\exp\left[-i\frac{\gamma\tau}{4}\tanh\frac{\Omega\tau}{2}\right] & W(c'_+)e^{-(1/2)\gamma_2(s_f-s_i)} \end{vmatrix} . \tag{20}$$

The elements of M_I in (20) can be transformed with the same rules for the Gaussian-field case obtaining the following matrix M_{Ig}:

$$M_{Ig} \simeq \begin{vmatrix} W_g(c'_-)e^{-(1/2)\gamma_1(s_f-s_i)} & ie^{-(1/2)(\gamma_1 s_f-\gamma_a s_i)}e^{i(\delta_k+\Omega t_k)} \times \sin\pi a \exp\left[-\frac{1}{\pi}\left[\frac{\Omega\tau}{2}\right]^2 - i\frac{\Omega\tau}{2}\frac{\gamma\tau}{2\pi}\right] \\ ie^{-(1/2)(\gamma_2 s_f-\gamma_1 s_i)}e^{-i(\delta_k+\Omega t_k)} \times \sin\pi a \exp\left[-\frac{1}{\pi}\left[\frac{\Omega\tau}{2}\right]^2 - i\frac{\Omega\tau}{2}\frac{\gamma\tau}{2\pi}\right] & W_g(c'_+)e^{-(1/2)\gamma_2(s_f-s_i)} \end{vmatrix} . \tag{20'}$$

IV. AN APPLICATION TO RAMSEY'S METHOD

As an example of high resolution spectroscopy let us consider a Ramsey interaction scheme with a field shaped as the exactly solvable hyperbolic secant, namely

$$g(t) = \begin{cases} g_1 = \mathrm{sech}[\pi(t+t_0)/\tau], & -\infty < t_i < t \le 0 \quad \text{(first zone)} \\ g_2 = \mathrm{sech}[\pi(t-t_0)/\tau], & 0 \le t < t_f < \infty \quad \text{(second zone)} \end{cases} \tag{21}$$

and, as usual, $2t_0 \gg \tau$. Moreover, let us suppose $\omega_{R,1}=\omega_{R,2}=\omega_R$, but $\delta_1 \neq \delta_2$ to account for phase shift between the two zones.

Detailed analytical developments are reported in [12]. For the particular initial conditions $C_1(t_i)=1$ and $C_2(t_i)=0$, the following final values are obtained:

$$\begin{aligned} C_1(t_f) &\simeq e^{-(1/2)\gamma_1(t_f-t_i)}e^{-\gamma t_0}e^{i\Omega t_0} \\ &\times \left[W^2(c'_-)e^{\gamma t_0}e^{-i\Omega t_0} - \sin^2\pi a \, \mathrm{sech}^2\frac{\Omega'\tau}{2}e^{-\gamma t_0}e^{i(\delta+\Omega t_0)} \right] , \\ C_2(t_f) &\simeq i\sin\pi a \, \mathrm{sech}\frac{\Omega' t}{2}e^{-(1/2)(\gamma_2 t_f-\gamma_1 t_i)} \\ &\times [e^{\gamma t_0}e^{-i(\delta_2+\Omega t_0)}W(c'_-) \\ &+ e^{-\gamma t_0}e^{-i(\delta_1-\Omega t_0)}W(c'_+)] . \end{aligned} \tag{22}$$

The transition probability is then

$$\begin{aligned} C_2C_2^* &= \sin^2\pi a \, |\mathrm{sech}\Omega'\tau/2|^2 e^{-(\gamma_2 t_f-\gamma_1 t_i)} \\ &\times e^{2\gamma t_0}W(c'_-)W(c'^*_-) + e^{-2\gamma t_0}W(c'_+)W(c'^*_+) \\ &+ e^{-i(\delta_2-\delta_1+2\Omega t_0)}W(c'_-)W(c'^*_+) \\ &+ e^{i(\delta_2-\delta_1+2\Omega t_0)}W(c'_+)W(c'^*_-) , \end{aligned} \tag{23}$$

but

$$\left|\mathrm{sech}\frac{\Omega'\tau}{2}\right|^2 = \frac{\mathrm{sech}^2\frac{\Omega\tau}{2}}{1-\sin^2\frac{\gamma\tau}{4}\mathrm{sech}^2\frac{\Omega\tau}{2}} \simeq \mathrm{sech}^2\frac{\Omega\tau}{2} . \tag{24}$$

According to (18), recalling that $c_+=c_-^*$, and introducing $\psi(c_\pm+a)+\psi(c_\pm-a)-2\psi(c_\pm)=A\pm iB$, one gets

$$\begin{aligned} W(c'_+)W(c'^*_+) &\simeq W(c_+)W(c^*_+)[1+(A+iB)\gamma\tau/4\pi] \\ &\times[1+(A-iB)\gamma\tau/4\pi] \\ &\simeq (1-\sin^2\pi a \, \mathrm{sech}^2\Omega\tau/2)(1+2A\gamma\tau/4\pi) , \end{aligned} \tag{25}$$

$$\begin{aligned} W(c'_-)W(c'^*_-) &\simeq W(c_-)W(c^*_-)[1-(A-iB)\gamma\tau/4\pi] \\ &\times[1-(A+iB)\gamma\tau/4\pi] \\ &\simeq (1-\sin^2\pi a \, \mathrm{sech}^2\Omega\tau/2)(1-2A\gamma\tau/4\pi) , \end{aligned} \tag{26}$$

$$\begin{aligned} W(c'_-)W(c'^*_+) &\simeq W(c_-)W(c^*_+)[1-(A-iB)\gamma\tau/4\pi] \\ &\times[1+(A-iB)\gamma\tau/4\pi] \\ &\simeq (1-\sin^2\pi a \, \mathrm{sech}^2\Omega\tau/2)e^{-i2\chi} , \end{aligned} \tag{27}$$

where χ is the phase of $W(c_+)$. Moreover,

$$
\begin{aligned}
e^{2\gamma t_0}|W(c'_-)|^2+e^{-2\gamma t_0}|W(c'_+)|^2 &\simeq (1-\sin^2\pi a\,\mathrm{sech}^2\Omega\tau/2)[e^{2\gamma t_0}(1-2A\gamma\tau/4\pi)+e^{-2\gamma t_0}(1+2A\gamma\tau/4\pi)] \\
&\simeq (1-\sin^2\pi a\,\mathrm{sech}^2\Omega\tau/2)2[\cosh 2\gamma t_0-2(A\gamma\tau/4\pi)\sinh 2\gamma t_0] \\
&\simeq (1-\sin^2\pi a\,\mathrm{sech}^2\Omega\tau/2)2\cosh 2\gamma t_0 \; . \qquad (28)
\end{aligned}
$$

This last approximation can be accepted if attention is paid to the presence of $\sinh 2\gamma t_0$, which compares with $\cosh 2\gamma t_0$ and in many cases $\gamma t_0 \ll 1$. Moreover, by ignoring A in (28) the approximation is introduced in an amplitude term and this should be of minor importance in interference problems, whereas expressions (25)–(28) are independent of B, which appears as a part of the phase factor in $W(c',a)$.

To the level of approximation discussed the transition probability in the hyperbolic secant case is

$$C_2C_2^* \simeq 2\sin^2\pi a\,\mathrm{sech}^2\Omega\tau/2\,(1-\sin^2\pi a\,\mathrm{sech}^2\Omega\tau/2)e^{-(\gamma_2 t_f-\gamma_1 t_i)}\times[\cosh 2\gamma t_0+\cos(2\Omega t_0+2\chi+\delta_2-\delta_1)] \; , \qquad (29)$$

and through the usual transformation for a Gaussian field

$$
\begin{aligned}
(C_2C_2^*)_G \simeq\; & 2\sin^2\pi a \exp\left[-\frac{2}{\pi}\left[\frac{\Omega\tau}{2}\right]^2\right]\left\{1-\sin^2\pi a\exp\left[-\frac{2}{\pi}\left[\frac{\Omega\tau}{2}\right]^2\right]\right\}e^{-(\gamma_2 t_f-\gamma_1 t_i)} \\
& \times[\cosh 2\gamma t_0+\cos(2\Omega t_0+2\chi_g+\delta_2-\delta_1)] \qquad (29')
\end{aligned}
$$

where χ and χ_g are the phase factors of $W(c_+)$ and $W_g(c_+)$, respectively, and go to zero if $\Omega\tau\to 0$, giving the well-known dependence of the transition probability on the phase shift $\delta_2-\delta_1$.

An evaluation of the linewidth in the Ramsey scheme, taking into account also the field shape, can be obtained from (25) and (29′) assuming $\tau \ll t_0$ and keeping only linear terms in $\Omega\tau$. With $\delta_1=\delta_2$ and $\gamma=0$ the half width at half maximum for a hyperbolic secant field occurs at Ω_1, satisfying the following condition:

$$\Omega_1 t_0+\chi(\Omega_1\tau,a)=\pi/4 \; , \qquad (30)$$

where the dependence of χ on a is given by (11) retaining only the linear terms in $\Omega\tau$.

As an example in a monokinetic beam the field intensity for a complete population inversion at $\Omega=0$ requires $a=\frac{1}{4}$, and from (9) [see also expression (34) of Ref. [12] where $s(\frac{1}{4})=2^4\ln 2$]

$$\chi(\Omega_1\tau,\tfrac{1}{4})\simeq\frac{\Omega_1\tau}{2}\frac{2\ln 2}{\pi} \qquad (31)$$

and therefore

$$\Omega_1=\frac{1}{t_0}\frac{\pi}{4}\left[1-\frac{2\ln 2}{\pi}\frac{\tau}{2t_0}\right] . \qquad (32)$$

In the same way for a Gaussian field

$$\chi_g(\Omega_1\tau)\simeq\left[\frac{2}{\pi}\right]^{1/2}\frac{\Omega_1\tau}{2}\frac{2\ln 2}{\pi} \qquad (31')$$

with a halfwidth at half maximum given by

$$\Omega_{1g}\simeq\frac{1}{t_0}\frac{\pi}{4}\left[1-\frac{2}{\pi}\left[\frac{2}{\pi}\right]^{1/2}\ln 2\frac{\tau}{2t_0}\right] . \qquad (32')$$

V. CONCLUSION

Approximate expressions for the final values of the elements of the transformation matrix governing the interaction between a two-level quantum system and an electromagnetic field with a Gaussian shape and a small detuning have been obtained. The approximation has been tested by comparison with numerical integration. The introduction of decay phenomena has been discussed and the analysis of a Ramsey interaction scheme has been performed. The solutions given appear suitable for evaluations of high-resolution spectroscopy such as in multizone interaction schemes as well as a base for the evaluation of uncertainty sources, for example, the first-order Doppler effect in infrared and optical frequency standards.

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