Dynamic quantum-noise reduction in multilevel-laser systems

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Based on the standard laser model of a large number N of model atoms resonantly coupled to a single lasing mode, we show that the nonlinear dynamics of the active atoms of the laser can lead to output-intensity fluctuations significantly reduced below the shot-noise level. We identify the multiple recycling of the active electron from the lower lasing level to the upper level through the pumping as the key process leading to this dynamic-pump-noise reduction. This process has been neglected in most of the standard treatments of the laser so far. We find that the results are closely related to recent calculations based on the assumption of an external regular pump. For the widely used four-level model of the active atoms, the intensity noise can be reduced 50% below the shot-noise level. Generalizing the model to an m-level system, we find a quantum-noise reduction by a factor of $\frac{1}{2}m/(m-1)(m \geq 3)$, leading to perfect output-intensity noise reduction in the limit of a large number of intermediate steps in the recycling process of the active atoms. Finally, we demonstrate that the bandwidth of the noise reduction can be significantly enhanced using a nonlinear absorber in the cavity.

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I. INTRODUCTION

In a number of recent publications it has been shown that quenching the various sources of intrinsic noise in a laser leads to reduction of the output-intensity fluctuations below the shot-noise limit [1–9]. The photon count statistics of the laser then becomes sub-Poissonian, clearly indicating a nonclassical state of the light field. It turns out that one of most promising approaches is the reduction of the pump fluctuations. This can be achieved, e.g., by regular injection of excited atoms into the laser cavity or by pumping the laser active atoms with squeezed light. In the best case one finds a complete suppression of the low-frequency laser output-intensity fluctuations. However, all the schemes to avoid pump noise proposed so far rely on a regularization of the pump process by an externally imposed mechanism. In this work, however, we will demonstrate that under certain operating conditions the desired regularity is generated by the dynamics of the laser atoms. Hence, in contrast to previous calculations, we predict that a laser will emit sub-Poissonian light, without any regularity imposed from the outside, if operated under suitable operating conditions. To some extent it seems surprising that with the exception of very recent work by Khazanov, Koganov, and Gordov [10], Ritsch *et al.* [3], and Ralph and Savage [9] this effect has not been discovered in the vast amount of earlier work on the laser. The main reason is probably that most treatments so far have neglected depletion of the atomic ground state by assuming the number of atoms is so large that only a small portion is excited by the pump. In other treatments [11] fluctuations in the number of laser atoms overwhelming this effect have been explicitly introduced.

From a physical point of view the presented mechanism is closely related to the well-known effect of antibunching in atomic resonance fluorescence [12]. For the fluorescence of a two-level atom driven by an external field, the probability that two photons are emitted within a time interval short compared to the inverse Rabi frequency is very small. This reflects the fact that immediately after the atom has emitted a photon, it is in its ground state and cannot emit a second photon until it is reexcited to the upper state. This behavior is even more strongly pronounced if one introduces additional intermediate levels through which the electron is recycled to the fluorescing level. As we will see later this anticorrelation in the successive emissions of photons leads to a *dynamic self-regularization of the pumping process,* making it completely deterministic in the case of a large number of intermediate levels. The physical mechanism has been recently identified by Ritsch *et al.* [3]. Note that this regularity is generated by the atomic dynamics and does not have to be injected from the outside. For a large number of intermediate levels we find $Q \rightarrow -\frac{1}{2}$, which is the same value as one obtains for a laser with a perfectly regular pump. For a three-level system our results are in agreement with recent predictions of Khazanov, Koganov, and Gordov [10], who find an optimum Mandel Q parameter of $-\frac{1}{4}$. The squeezing spectrum of intensity fluctuations has been discussed by Ralph and Savage [9].

A second approach to a dynamic reduction of the intensity fluctuations in a laser is the insertion of nonlinear elements into the laser cavity. In contrast to regular pumping, nonlinear absorbers tend to reduce the high-frequency components of the laser intensity fluctuations [6]. In this case one finds strong antibunching of the photons emitted by the laser, while fluctuations on a slow time scale are even enhanced. It is thus interesting to investigate the combined effect of such nonlinear elements and a dynamic-pump-noise reduction scheme.

This paper is organized as follows. In Sec. II, following a systematic adiabatic elimination procedure for the atoms as presented in Ref. [13], we first derive a Fokker-Planck-type equation for the Glauber-Sudarshan P representation of the laser mode. Subsequently we calculate physically interesting quantities like the Mandel Q parameter and the output-intensity fluctuation spectrum. In Sec. III we explicitly calculate these quantities for the four-level laser model with pump-electron recycling. A generalization to $m>4$ levels is presented in Sec. IV and the effect of additional nonlinear absorbers placed into the laser cavity is dealt with in Sec. V.

II. FOKKER-PLANCK TREATMENT OF THE LASER

In this section, after introducing some notation and definitions, we will outline a systematic procedure to obtain a Fokker-Planck equation for the Glauber-Sudarshan P representation of the laser field. We use an adiabatic elimination method as presented in Ref. [13(a)], which is closely related to a derivation previously used by Haake and Lewenstein [13(b)]. We then demonstrate how one can extract from this relevant physical quantities like the Mandel Q parameter and the intensity fluctuation spectrum $S(\omega)$.

We consider a single quantized light mode described by creation (annihilation) operators a ($a^\dagger$), with

$$[a,a^\dagger]=1 \tag{1}$$

inside an optical cavity, which is coupled to a continuum of outside modes $b(\omega)$. The corresponding Hamiltonian then can be written as

$$\begin{aligned} H_s &= H_l + H_{\text{free}} + H_{l,\text{free}} , \\ H_l &= \hbar\omega_1 a a^\dagger, \quad H_{\text{free}} = \int_{\Delta\omega} d\omega\, \hbar\omega b(\omega)^\dagger b(\omega) , \\ H_{l,\text{free}} &= i\hbar\sqrt{\kappa}\int_{\Delta\omega} d\omega [a b(\omega)^\dagger + b(\omega) a^\dagger] , \end{aligned} \tag{2}$$

where ω_l is the eigenfrequency of the laser mode and $\Delta\omega$ denotes a suitably chosen frequency interval around ω_l, to which the laser mode is coupled. κ is the corresponding coupling constant which we assume is frequency independent. As is well known, the coupling with the output modes on the one hand leads to a friction force for the internal oscillator describing the losses but on the other hand it introduces fluctuations, so that the commutation relation Eq. (1) is preserved [13(a)].

The laser active medium inside the cavity is described by a large number N of randomly distributed atoms. We assume that their mean distance is large compared to a typical optical wavelength $d \gg \lambda$, so that we can (will) neglect cooperative effects (e.g., superfluorescence, etc.) and atom-atom interactions. In order to get a fairly realistic description, we will treat the lasing atoms as at least four-level systems, coupled to the pump modes $b_p(\omega)$, the laser mode a, and the electromagnetic vacuum. A schematic picture of such an atomic level configuration is presented in Fig. 1. The part of the Hamiltonian for the atoms, the pump field, and the corresponding interaction reads

$$H_a = H_{0a} + H_p + H_v + H_{ap} + H_{av} \tag{3}$$

with

$$\begin{aligned} H_{0a} &= \sum_{i=1}^{N}\sum_{j=0}^{4} \hbar\omega_j \sigma_{jj}^i , \\ H_p &= \sum_{i=1}^{N} \int_{\Delta\omega} d\omega\, \hbar\omega b_p(\omega)_i^\dagger b_p(\omega)_i \end{aligned}$$

representing the Hamiltonians of the uncoupled atoms and pump modes. The pump field is coupled to the pump transition ($|0\rangle - |1\rangle$) via

$$H_{ap} = i\hbar \sum_{i=1}^{N} \int d\omega [g_p^i b_p(\omega)_i^\dagger \sigma_{01}^i + \text{H.c.}] . \tag{4}$$

with coupling constants g_p^i and the atomic operators given by $\sigma_{ij} = |i\rangle\langle j|$. Note that we have assumed that each atom is coupled to its own independent pump modes. In practice this means that we assume that the coherence length of the pump field is much shorter than the atomic distances. In the literature this is known as the "private bath" assumption. Recent calculations and experiments investigating the resonance fluorescence of a sample of atoms exposed to a classical stochastic light field [15] have clearly demonstrated the significance of collective fluctuations of the atomic populations. In view of this, the validity of this approximation in the case of optical pumping by a finite bandwidth "pump" laser is not totally obvious. Fortunately, in the two limits of a white-noise pump and a fully coherent pump, the private bath assumption turns out to be valid. In the following we will restrict ourselves to these limits.

In Eq. (3) H_v accounts for all the unexcited modes of the electromagnetic vacuum not contained in H_p, which interact with the atom via H_{av} like a zero-temperature heat bath. Effectively they lead to various additional spontaneous decay terms in the atomic equations. The coupling between the laser mode and the laser active transition from level $|2\rangle$ to level $|3\rangle$, which we assume is resonant, is given in the standard rotating-wave approximation (RWA) by

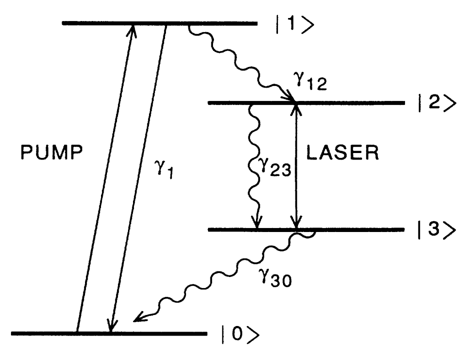

FIG. 1. Schematic representation of the four-level atom.

$$H_{al}=i\hbar\sum_{i=1}^{N} g_l^i(a^\dagger\sigma_{32}^i-\text{H.c.})\ , \tag{5}$$

where g_l^i in general depends on the position of the ith atom. The total laser Hamiltonian is then given by the sum $H=H_s+H_a+H_{al}$. Using standard techniques we can trace over the output H_{out} and vacuum H_v modes to derive a master equation for the combined density matrix W of the intracavity field, the atoms, and the pump field, which in the interaction representation reads [13(a)]

$$\frac{d}{dt}W=(L_{av}+L_{l0}+L_{al}+L_{ap})W \tag{6}$$

with

$$L_{av}W=\sum_{n=1}^{N}\sum_{i>j}\gamma_{ij}(2\sigma_{ij}^n W\sigma_{ji}^n-\sigma_{ji}^n\sigma_{ij}^n W-W\sigma_{ij}^n\sigma_{ji}^n)$$

with

$$L_{l0}W=\kappa(2aWa^\dagger-a^\dagger aW-Wa^\dagger a)\ . \tag{7}$$

The other Liouville operators in Eq. (6) are formally defined by $L_iX=(i/\hbar)[H_i,X]$, where X stands for any operator. γ_{ij} are the various atomic decay constants, stemming from H_{av}. We now assume the limit of a good cavity, i.e., the time scales of the atomic motion $1/\gamma_{ij}$ and the pump fluctuations τ_p are much faster than the cavity decay time $1/\kappa$. This enables us to adiabatically eliminate the atomic and pump degrees of freedom from the density matrix Eq. (6), so that we are left with an equation for the reduced density matrix of the internal mode amplitude alone. In order to do this systematically we will use a projection-operator technique such as, e.g., is presented in Ref. [13(a)]. As a first step we represent the combined atom-field density operator W by a Glauber P representation for the laser field

$$W=\int d^2\alpha|\alpha\rangle\langle\alpha|W(\alpha)\ , \tag{8}$$

where $W(\alpha)$ is still an operator in the many-atom and pump Hilbert space. Note that for notational convenience we use a Glauber P representation for the laser field density matrix in the following derivation; nevertheless it can be easily generalized to a generalized P representation, which we will need to correctly describe the nonclassical properties of the laser field. Inserting this expansion into Eq. (6) and using the well-known operator–c-number correspondences [16], we find

$$\begin{aligned} &aW\to\alpha W(\alpha)\ ,\\ &a^\dagger W\to(\alpha^*-\partial_\alpha)W(\alpha)\ , \end{aligned} \tag{9}$$

$$\begin{aligned}\frac{d}{dt}W(\alpha)=&[\kappa(\partial_\alpha\alpha+\partial_{\alpha^*}\alpha^*)+2\kappa n_{th}\partial^2_{\alpha\alpha^*}]W(\alpha)\\ &+\sum_{k=1}^{N}g_l^k[\partial_\alpha\sigma_{32}^k W(\alpha)+\partial_{\alpha^*}W(\alpha)\sigma_{23}^k]\\ &+\partial_t W(\alpha)|_r\ ,\end{aligned} \tag{10}$$

with

$$\partial_\alpha=\frac{\partial}{\partial\alpha},\ \dots\ . \tag{11}$$

Here $\partial_t W(\alpha)|_r$ denotes the remaining terms in Eq. (6), which do not explicitly contain $a,a^\dagger$. We will now regroup the various terms in Eq. (10) into three parts:

$$\frac{d}{dt}W(\alpha)=(L_1+L_2+L_3)W(\alpha)\ . \tag{12}$$

The first term

$$\begin{aligned}L_1W(\alpha)=&-\sum_{k=1}^{N}g_l^k\{\alpha^*[\sigma_{32}^k,W(\alpha)]-\alpha[\sigma_{23}^k,W(\alpha)]\}\\ &+L_{av}W(\alpha)+L_{ap}W(\alpha)\end{aligned} \tag{13}$$

describes the "fast" dynamics of the atoms coupled to the pump and laser fields with a characteristic time scale of $1/\gamma_{ij}$. The second part,

$$L_2W(\alpha)=\sum_{k=1}^{N}g_l^k[\partial_\alpha(\sigma_{32}^k-\langle\sigma_{32}^k\rangle)+\text{H.c.}]\ , \tag{14}$$

which is proportional to the atom-field coupling g_l^i, contains the fluctuations of the atomic polarization around the steady-state expectation value. Here the atomic expectation values are to be taken with respect to the stationary atomic density matrix ρ_a^s satisfying

$$L_1\rho_a^s=0\ . \tag{15}$$

Note that ρ_a^s depends parametrically on α. Physically this is motivated by the fact that the dynamics of the laser field is slow on the time scale of atomic relaxation. Hence ρ_a^s can be considered as a quasistationary atomic density matrix, which adiabatically "follows" the laser field α.

Finally

$$L_3W(\alpha)=\partial_\alpha\left[\kappa\alpha+\sum_{k=1}^{N}g_l^k\langle\sigma_{32}^k\rangle\right]+\text{H.c.} \tag{16}$$

describes the "slow" dynamics of the laser mode driven by the mean atomic polarization and damped by the cavity losses, with a typical time scale $1/\kappa$. Defining a projection operator

$$PW(\alpha)=\text{Tr}_{a+p}[W(\alpha)]\otimes\rho_a^s \tag{17}$$

we can easily show that the above choice of $L_{(1,2,3)}$ implies the identities

$$PL_1=L_1P=0\ , \tag{18}$$

$$PL_2P=0\ , \tag{19}$$

$$PL_3=L_3P\ . \tag{20}$$

Thus as demonstrated in Refs. [13] and [14] in the case of fast atomic dynamics $\gamma_{ij}\gg\kappa$, one finds the following equation of motion for the P-distribution function of the laser field:

$$\begin{aligned}\partial_tP(\alpha)=L_3P(\alpha)-\text{Tr}_{a+p}&\left[L_2L_1^{-1}\left[=\int_0^\infty dt\,e^{L_1t}\right]L_2\rho_a^s\right]\\ &\times P(\alpha)\end{aligned} \tag{21}$$

with $P(\alpha)=\mathrm{Tr}_{a+p}[W(\alpha)]$.

Strictly this is only correct if all eigenvalues of the Liouville operator L_1 are negative, which is obviously not the case as we see from Eq. (15). However, in our case L_1 is applied only on $L_2\rho_a^s$, which gives zero if projected onto the eigenspace connected to the eigenvalue zero, as can be seen from the second identity in Eq. (18). Hence we have to evaluate

$$L_2 e^{L_1 t} L_2 \rho_a^s = \sum_{n,m=1}^{N} [g_l^n \partial_{\alpha^*} \delta\sigma_{32}^n \times (\partial_\alpha \delta\sigma_{32}^m \rho_a^s|_t + \partial_{\alpha^*} \rho_a^s \delta\sigma_{23}^m|_t) + (\partial_\alpha \delta\sigma_{32}^m \rho_a^s|_t + \partial_{\alpha^*}\rho_a^s \delta\sigma_{23}^m|_t) \times \partial_{\alpha^*} \delta\sigma_{23}^n] ,$$

with

$$\delta\sigma_{32}^i = \sigma_{32} - \langle \sigma_{32} \rangle \quad \text{and} \quad \delta\sigma_{23}^i = \sigma_{23} - \langle \sigma_{23} \rangle .$$

As a next step, we will move all derivatives with respect to α and α^* to the left-hand side. The small corrections to the drift proportional to g_l^2 stemming from this reordering can be neglected [13(a)]. We thus get the following Fokker-Planck-type equation for the Glauber P distribution for the laser mode:

$$\partial_t P(\alpha) = (\partial_\alpha A_\alpha + \partial_{\alpha^*} A_\alpha^* + 2\partial^2_{\alpha\alpha^*} D_{\alpha\alpha^*} + \partial^2_{\alpha^2} D_{\alpha\alpha} + \partial^2_{\alpha^{*2}} D_{\alpha^*\alpha^*}) P(\alpha) \tag{22}$$

with

$$A_\alpha = \kappa\alpha + \sum_{k=1}^{N} g_l^k \langle \sigma_{32}^k \rangle ,$$

$$D_{\alpha^*\alpha} = \sum_{k,m=1}^{N} g_l^k g_l^m \int d\tau \langle \delta\sigma_{23}^k(\tau) \delta\sigma_{32}^m(0) \rangle ,$$

$$D_{\alpha\alpha} = D^*_{\alpha^*\alpha^*} = \sum_{k,m=1}^{N} g_l^k g_l^m \int d\tau \langle \delta\sigma_{32}^k(\tau)\delta\sigma_{32}^m(0)\rangle .$$

Again the atomic averages have to be taken with respect to the stationary atomic density matrix as defined in Eq. (15). As expected, the drift terms A_α that determine the mean amplitude are given by the cavity losses and the mean induced polarization, providing for the laser gain. Apart from the contributions stemming from the vacuum and thermal fluctuations entering through the laser-output coupling, the diffusion is given by the time integral over atomic polarization autocorrelation functions. These look very similar to correlation functions, which enter the calculation of resonance fluorescence. In fact $D_{\alpha^*\alpha}$ is just proportional to the incoherent part of the resonance fluorescence intensity emitted at the atomic transition frequency [17]. In order to evaluate this correlation function, we first have to solve Eq. (15) for the stationary atomic density matrix ρ_a^s, where we can treat the laser field amplitude α as a c number. Using the quantum regression theorem we can then find the necessary integrals over the correlation functions by a Laplace transform of the corresponding equations.

To gain more insight into the physical contents of Eq. (22) it proves advantageous to convert the amplitude variables α,α^* to intensity and phase variables:

$$I = \alpha^*\alpha \quad \text{and} \quad \phi = \frac{i}{2}\ln\left[\frac{\alpha}{\alpha^*}\right] . \tag{23}$$

As is well known the phase diffusion coefficient $D_{\phi\phi}$ determines the laser linewidth, whereas the intensity diffusion coefficient D_{II} is connected to the intensity noise and thus the photon statistics of the laser output. In the new variables the drift and diffusion coefficients read [16]

$$A_I = A_\alpha \alpha^* + A_{\alpha^*}\alpha + 2D_{\alpha^*\alpha}, \quad A_\phi = \mathrm{Im}\left[\frac{D_{\alpha\alpha}}{\alpha}\right] , \tag{24}$$

and

$$D_{II} = 2\alpha^*(\alpha D_{\alpha^*\alpha} + \alpha^* D_{\alpha\alpha}) ,$$
$$D_{\phi\phi} = \frac{2\alpha^*}{I^2}(\alpha D_{\alpha^*\alpha} - \alpha^* D_{\alpha\alpha}) , \tag{25}$$
$$D_{I\phi} = \mathrm{Im}\left[\frac{\alpha D_{\alpha\alpha}}{\alpha^*}\right] .$$

In terms of atomic correlations D_{II} is then

$$D_{II} = \sum_{k,l=1}^{N} \int d\tau [2 g_l^k g_l^l \times \langle \alpha\delta\sigma_{23}^k(\tau) + \alpha^*\delta\sigma_{32}^k(\tau), \alpha^*\delta\sigma_{32}^l\rangle] = \sum_{k,l=1}^{N} \int d\tau [g_l^k g_l^l \langle \delta\sigma_I^k(\tau), \delta\sigma_I^l\rangle] , \tag{26}$$

with $\delta\sigma_I^i = \alpha\delta\sigma_{23}^i + \alpha^*\delta\sigma_{32}^l$. Hence the diffusion coefficients $D_{II}(D_{\phi\phi})$ are given by the autocorrelation function of the atomic polarization in phase (in quadrature) with the laser field amplitude.

Note that the private bath assumption has not yet been made in Eq. (26). We will do this now and thus keep only the terms with $k=l$. The diffusion coefficients hence are proportional to the number of atoms N and not N^2, as one might expect from Eq. (26).

Due to the nonlinearity in the drift and diffusion coefficients an exact analytic solution of the Fokker-Planck equation is in general not possible. However, well above threshold we can linearize the intensity around the steady-state value $\bar{I}$. The resulting linearized Fokker-Planck equation has constant drift and diffusion terms and can be solved analytically. In this case one derives the following expression for the Mandel Q parameter of the laser:

$$Q = \frac{\langle n(n-1)\rangle - \langle n\rangle^2}{\langle n\rangle} = \frac{D_{\bar{I}I}}{\bar{I}a_{\bar{I}}}, \quad a_I = \frac{dA_I}{dI} , \tag{27}$$

with $n = a^\dagger a$.

Q is a well-known measure of the deviation of the photon statistics from a Poissonian distribution ($Q=0$), which one finds for a coherent state. A negative Q (equivalent to sub-Poissonian statistics) is a signature of a nonclassical state of the field. Another physically interesting quantity is the spectrum of the output-intensity correlation function:

$$S(\omega)=\int d\omega \cos\omega\tau \langle i(t+\tau)i(t)\rangle_s = \langle \bar{i}\rangle \left[1+2\kappa \frac{D_{\bar{I}I}}{\bar{I}(a_I^2+\omega^2)} \right] , \qquad (28)$$

where $\langle \bar{i}\rangle = \kappa\langle \bar{I}\rangle$ denotes mean output intensity. In the following sections we will evaluate these quantities for various physically interesting level schemes of the laser atoms.

III. LASER WITH FOUR-LEVEL ATOMS

In order to establish a fairly realistic model of the active atoms of a laser, one usually describes them by four-level systems. Here one pair of levels is coupled to the pump light, whereas a second independent pair forms the actual lasing transition. Both pairs are assumed to be coupled by various radiative and nonradiative decay channels, which we will simply describe by transition rates. In Fig. 1 we sketch the level scheme of such an atom, where wiggly arrows stand for incoherent transitions. The pump transition is indicated by an upward arrow and a double arrow denotes the laser active transition. In most treatments of the laser so far it has been assumed that the pumping is weak, so that there is no depletion of the atomic ground state. In this case the ground state can be eliminated from the atomic equations and we are effectively left with three levels. Additionally the transitions from the lower levels to the ground state of the atom are often ignored.

In our approach we will now abandon the weak pump approximation and include the dynamics of the ground-state level of the atom. Consequently we will introduce recycling terms $(\gamma_{30},\gamma_{20})$ in the four-level system from the lasing transition back to the pump transition. This allows us to study saturation effects in the pump transition, as well as the effect of one atom undergoing several cycles of absorbing a pump photon and emitting a laser photon.

In order to evaluate the corresponding atomic correlation functions entering D_{II}, Q, etc. [Eqs. (27) and (28)], we now have to solve the full system of equations for the $(4\times4)=16$ density matrix elements. Fortunately in our case only eight are nonzero. Explicitly the dynamic equation $(d/dt)\mathbf{r}=A\mathbf{r}$ for the vector of atomic density matrix elements reads

$$\frac{d}{dt}\begin{bmatrix}\rho_{00}\\ \rho_{11}\\ \rho_{22}\\ \rho_{33}\\ \rho_{10}\\ \rho_{01}\\ \rho_{23}\\ \rho_{32}\end{bmatrix}=\begin{bmatrix} 0 & \gamma_1 & 0 & \gamma_{30} & -i\frac{1}{2}\Omega^* & i\frac{1}{2}\Omega & 0 & 0\\ 0 & -(\gamma_1+\gamma_{12}) & 0 & 0 & i\frac{1}{2}\Omega^* & -i\frac{1}{2}\Omega & 0 & 0\\ 0 & \gamma_{12} & -\gamma_2 & 0 & 0 & 0 & g\alpha^* & -g\alpha\\ 0 & 0 & \gamma_{23} & -\gamma_{30} & 0 & 0 & -g\alpha^* & g\alpha\\ i\frac{1}{2}\Omega^* & -i\frac{1}{2}\Omega^* & 0 & 0 & -\gamma_p & 0 & 0 & 0\\ i\frac{1}{2}\Omega & -i\frac{1}{2}\Omega & 0 & 0 & 0 & -\gamma_p & 0 & 0\\ 0 & 0 & -g\alpha^* & g\alpha^* & 0 & 0 & -\gamma_s & 0\\ 0 & 0 & -g\alpha & g\alpha & 0 & 0 & 0 & -\gamma_s \end{bmatrix}\begin{bmatrix}\rho_{00}\\ \rho_{11}\\ \rho_{22}\\ \rho_{33}\\ \rho_{10}\\ \rho_{01}\\ \rho_{23}\\ \rho_{32}\end{bmatrix} . \qquad (29)$$

Here Ω stands for the (fluctuating) amplitude of the pump field and Eq. (29) still has to be averaged over the pump degrees of freedom (for a coherent pump field we have $\Omega=\text{const}$, while in the white-noise case Ω is proportional to a complex Wiener process). In Eq. (29) $\gamma_2=\gamma_{23}+\gamma_{20}$ denotes the sum of the decay rates out of level $|2\rangle$ into levels $|3\rangle$ and $|0\rangle$; γ_s and γ_p are the transverse decay rates of the atomic polarization. The correlation functions needed for the diffusion coefficients then can be obtained by the quantum regression theorem. The Laplace-transformed differential equations for the atomic correlation functions read

$$(s+A)\mathcal{F}[\mathbf{V}](s)=\mathbf{V}(0) . \qquad (30)$$

Here $\mathcal{F}[f](s)$ denotes the Laplace transform of f and $\mathbf{V}$ stands for the vector of atomic correlation functions

$$\mathbf{V}(\tau)=(V_{00},V_{11},V_{22},V_{33},V_{10},V_{01},V_{23},V_{32})(\tau) ,$$
$$V_{ij}(\tau)=\langle (\sigma_{ij})(\tau),(\sigma_{32})(0)\rangle . \qquad (31)$$

From the definition of the Laplace transform it is trivial to find the identity

$$\int_0^\infty d\tau\, V_{ij}(\tau)=\lim_{s\to0}\mathcal{F}[V_{ij}(\tau)](s)=\lim_{s\to0}(s+A)^{-1}\mathbf{V}(0) . \qquad (32)$$

In order to reduce the computational effort, it proves advantageous to introduce the in-phase atomic polarization correlation function $V_I=\alpha V_{23}+\alpha^* V_{32}$, with $D_{II}=2\alpha^*\mathcal{F}[V_I](s)|_{s=0}$, which reduces the number of equations by one. Similarly we define $V_\phi=\alpha V_{23}-\alpha^* V_{32}$, which can be shown to decouple from the other equations. Hence the phase diffusion coefficient $D_{\phi\phi}=(2\alpha^*/I^2)\mathcal{F}[V_\phi](s)|_{s=0}$ can be obtained very easily from

$$D_{\phi\phi}=\frac{2g^2N\rho_{22}^s}{I\gamma_s}\ , \tag{33}$$

which is of course related to the Shawlow-Townes linewidth.

As the most general expressions for quantities like D_{II}, Q, etc., get rather lengthy and tedious to evaluate, we will restrict ourselves for the moment to the special case of a broadband pump. In this case the pumping is described simply by transition rates between the pump levels. Using standard master-equation techniques [18,19] we can average Eq. (29) over the pump field fluctuations to obtain the following equation for the atomic density matrix elements:

$$\begin{bmatrix} -\gamma n_b & \gamma(n_b+1) & 0 & \gamma_{30} & 0 & 0 \\ \gamma(n_b+1) & -[\gamma_1+\gamma(n_b+1)] & 0 & 0 & - & 0 \\ 0 & \gamma_1 & -\gamma_2 & 0 & g\alpha^* & -g\alpha \\ 0 & 0 & \gamma_{23} & -\gamma_{30} & -g\alpha^* & g\alpha \\ 0 & 0 & -g\alpha^* & g\alpha^* & -\gamma_s & 0 \\ 0 & 0 & -g\alpha & g\alpha & 0 & -\gamma_s \end{bmatrix} \begin{bmatrix} \rho_{00} \\ \rho_{11} \\ \rho_{22} \\ \rho_{33} \\ \rho_{23} \\ \rho_{32} \end{bmatrix} = 0\ , \tag{34}$$

where $n_b=\text{Tr}_p(2\Omega^*\Omega/\gamma_p)/\gamma$ is a measure of the spectral intensity of the pump field. Note that the matrix elements connected to the atomic polarization on the pump transition are decoupled from this equation, so that we are left with a system of six coupled equations. These can be reduced further by introducing proper linear combinations of the density matrix elements as mentioned in Sec. II:

$$\begin{aligned} \frac{d}{dt}\rho_I(t) &= \frac{d}{dt}(\alpha\rho_{23}+\alpha^*\rho_{32}) \\ &= -\gamma_s\rho_I+2|g\alpha|^2(\rho_{22}-\rho_{33})\ , \end{aligned} \tag{35}$$

$$\frac{d}{dt}\rho_\phi(t)=\frac{d}{dt}(\alpha\rho_{23}-\alpha^*\rho_{32})=-\gamma_s\rho_\phi\ . \tag{36}$$

Inverting Eqs. (34) and (30) we can then evaluate the intensity and phase diffusion coefficients. With some effort we find

$$D_{\phi\phi}=g^2N\frac{4R\gamma_1}{\gamma_s^2 n}\frac{\gamma_3+n}{R\gamma_1(\gamma_3-\gamma_{23})+(\gamma_{23}+n)[2R(\gamma_1+\gamma_3)+\gamma_1\gamma_3]}\ , \tag{37}$$

$$D_{II}=g^2N\frac{R\gamma_1 n}{\gamma_s}\frac{\gamma_3\gamma_s[\gamma_1R(\gamma_{23}+\gamma_3)+\gamma_3\gamma_{23}(2R+\gamma_1)]+nz_1+n^2\gamma_s z_2}{\{\gamma_1R(\gamma_2 3+\gamma_3+2n)+n[\gamma_3(2R+\gamma_1)(\gamma_{23})+n]\}^3}\ , \tag{38}$$

with

$$\begin{aligned} z_1 = {} & \gamma_1^2\gamma_{23}\gamma_3\gamma_s(\gamma_3^2+\gamma_2 3^2)-R\gamma_1\gamma_{23}\gamma_3\{\gamma_1(\gamma_3-\gamma_{23})^2-\gamma_s[(\gamma_{23}+\gamma_3)(\gamma_{23}+\gamma_3+4\gamma_1)+2(\gamma_3^2+\gamma_{23}^2)]\} \\ & -R^2\{(\gamma_3-\gamma_{23})^2[\gamma_1(\gamma_{23}+\gamma_3)+2\gamma_3\gamma_{23}]-4\gamma_3\gamma_{23}(\gamma_1^2+\gamma_3^2+\gamma_{23}^2)-\gamma_1(\gamma_{23}+\gamma_3)[\gamma_1(\gamma_{23}+\gamma_3)+8\gamma_3\gamma_{23}]\} \end{aligned}$$

and

$$z_2=\gamma_1^2\gamma_{23}^2\gamma_3-R\gamma_1\gamma_3(\gamma_3-3\gamma_{23})(\gamma_3-\gamma_{23}+2\gamma_1)+4R^2(\gamma_1^2\gamma_{23}+3\gamma_1\gamma_3\gamma_{23}+\gamma_3\gamma_{23}^2-\gamma_1\gamma_3)\ . \tag{39}$$

Here we have introduced the laser-induced stimulated transition rate $n=2|\alpha|^2/\gamma_s$ and the pump transition rate $R=\gamma n_b$. As our interest in this work is mainly focused on intensity noise, we will now look at the expression for D_{II} in more detail. Note that D_{II} contains positive as well as negative terms. In order to compare our results with previous calculations we first consider the limit of a weak pump. To lowest order in R we find

$$D_{II}=g^2N\frac{R\gamma_{23}n}{\gamma_3^2}\frac{\gamma_3^2+\gamma_{23}n}{(\gamma_{23}+n)^2}\ . \tag{40}$$

This is precisely what Lax and Louisell obtained in their well-known series of papers on quantum noise in a laser [14]. Obviously in this limit D_{II} is always positive and goes to zero if we neglect the spontaneous emission rate γ_{23}. However, this is not true in the general case, where higher-order terms in the pump rate R become important. These are connected to depletion and recycling processes in the atomic ground state. As we will show below these terms can cause D_{II} to become negative, leading to a nonclassical steady state for the laser. Neglecting spontaneous emission for notational convenience, we see that Eq. (37) reduces to

$$D_{II}=g^2N\frac{(R\gamma_1\gamma_3)^2}{\gamma_s}\frac{R\gamma_1\gamma_3\gamma_s-nR\gamma_1(\gamma_3-\gamma_s)-n^2\gamma_s(4R+2\gamma_1+\gamma_3)}{\{R\gamma_1\gamma_3+n[2R(\gamma_1+\gamma_3)+\gamma_1\gamma_3]\}^2}\ , \tag{41}$$

which obviously turns negative if n gets sufficiently large, (i.e., the laser is operated far above threshold). Fortunately in this limit one can linearize the Fokker-Planck equation for the laser intensity around its steady value $\bar{n}$ and obtain an explicit analytical expression for the Mandel Q parameter using Eq. (27):

$$Q=\frac{R\gamma_1\gamma_3}{\gamma_s n}\frac{R\gamma_1[\gamma_3\gamma_s-\bar{n}(\gamma_3-\gamma_s)]-\bar{n}^2\gamma_s(4R+2\gamma_1+ga_3)}{[2R(\gamma_1+\gamma_3)+\gamma_1\gamma_3][\gamma_3(2R+\gamma_1)(\gamma_3+\bar{n})+R\gamma_1(\gamma_3+2\bar{n})]}\ , \tag{42}$$

with mean laser-induced atomic transition rate $\bar{n}$ given by

$$\bar{n}=(c-1)\frac{R\gamma_1(\gamma_3-\gamma_{23})}{2R(\gamma_1+\gamma_3)+\gamma_1\gamma_3}-\gamma_{23}\ . \tag{43}$$

Here $c=g^2N/(\gamma_s\kappa)$ is closely related to the laser cooperativity parameter, with the laser threshold given by

$$c_{\rm thr}=\frac{R\gamma_1\gamma_3+\gamma_{23}[R(\gamma_1+2\gamma_3)+\gamma_1\gamma_3]}{R\gamma_1(\gamma_3-\gamma_{23})}\ . \tag{44}$$

In the limit of large n ($c\gg1$), Q reduces to

$$Q=-R\gamma_1\gamma_3\frac{4R+2\gamma_1+\gamma_3}{[2R(\gamma_1+\gamma_3)+\gamma_1\gamma_3]^2}\ . \tag{45}$$

As is obvious from Eq. (45) Q in this limit is negative and thus we find sub-Poissonian behavior of the laser. A short calculation yields an optimum Q value of $-\frac{2}{7}$ in the case of $\gamma_3=\frac{4}{3}R$ and $\gamma_1=2R$. Hence we can expect more than 60% intensity noise reduction in the laser-output field. In order to estimate the influence of spontaneous emission we calculate the correction to Eq. (45) to lowest order in the spontaneous emission rate γ_{23},

$$Q=Q|_{\gamma_{23}=0}+\frac{\gamma_{23}}{\gamma_3}\frac{4R\gamma_1[R\gamma_1+2R\gamma_3+\gamma_1\gamma_3+(\gamma_3/2)^2]}{[2R(\gamma_1+\gamma_3)+\gamma_1\gamma_3]^2}$$

$$\Longrightarrow Q|_{\gamma_{23}=0}+\frac{5}{7}\frac{\gamma_{23}}{\gamma_3}\ . \tag{46}$$

Here in the second line we have inserted the optimum value for the pump rate R. As the spontaneous emission rate in a laser is typically much smaller than the decay rate of the lower level $\gamma_{23}\ll\gamma_3$, the correction to Q stays small. This behavior is demonstrated in Fig. 2, where we plot the Mandel Q parameter as a function of the pump rate R in units of the decay rate γ_{30} out of the lower level $|3\rangle$. In curve a we have assumed optimally matched rates $\gamma_{12}=1.5\gamma_{30}$ and negligible spontaneous emission $\gamma_{10}=\gamma_{23}=0$, whereas in curve b we have chosen a fast decay out of the upper pump level $\gamma_{12}=10\gamma_{30}$. The influence of spontaneous emission is demonstrated in curves c (d), where we have set $\gamma_{10}=\gamma_{30}$, $\gamma_{12}=2.5\gamma_{30}$, $\gamma_{23}=0$ ($\gamma_{10}=0$, $\gamma_{23}=0.1\gamma_{30}$).

Note that the results are very similar to the case of a laser with sub-Poissonian pump. However, we do not need any regular pump mechanism from the outside, but the noise reduction is dynamically produced by the laser itself. Hence under suitably chosen operating conditions the laser can become a source of strongly amplitude-squeezed light. The physical origin of this effect is very closely related to the effect of antibunching of photons in resonance fluorescence. Once an atom has emitted a photon, it is projected into the lower state and cannot emit a second photon until it is reexcited into the upper level. This introduces a sub-Poissonian behavior into the laser pumping. We will discuss this in more detail in Sec. IV, where we will treat the m-level atom, for which this effect becomes even more pronounced and its origin can be traced more precisely.

Let us now turn to the output-intensity power spectrum of the laser as measured by a photodetector. As can be seen from Eq. (28), for a negative D_{II} the spectrum shows a Lorentzian dip of area $|Q|$ below the shot-noise level. It is centered around zero frequency, with a width given by

$$a_I=2\kappa\frac{(c-1)R\gamma_1\gamma_3+\gamma_{23}[(c+1)R\gamma_1+\gamma_3(2R+\gamma_1)]}{cR\gamma_1(\gamma_3-\gamma_{23})}\ . \tag{47}$$

In the limit of a negligible spontaneous emission $\gamma_{23}\ll(\gamma_3,R,\gamma_1)$ this reduces to a very simple form:

$$a_I=2\kappa\frac{c-1}{c}\ . \tag{48}$$

As is easily seen from Eq. (47), sufficiently above threshold ($c\gg1$) the width of the dip is 2κ, which is the

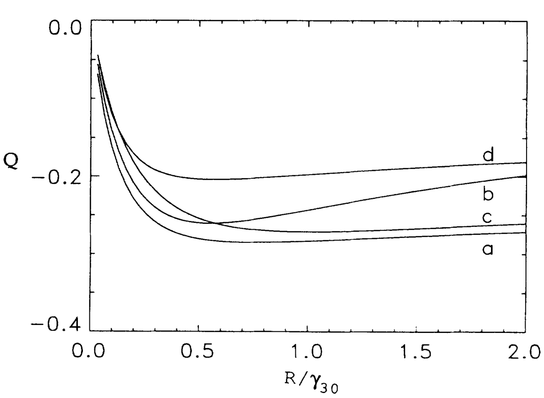

FIG. 2. Mandel Q parameter as function of the pump rate R in units of γ_{30}. The parameters for the various curves are $\gamma_{12}=3/2\gamma_{30}$, $\gamma_{10}=0=\gamma_{23}$ (curve a); $\gamma_{12}=10\gamma_{30}$, $\gamma_{10}=0=\gamma_{23}$ (curve b); $\gamma_{12}=5/2\gamma_{30}$, $\gamma_{10}=\gamma_{30}$, $\gamma_{23}=0$ (curve c); and $\gamma_{12}=3/2\gamma_{30}$, $\gamma_{10}=0$, $\gamma_{23}=0.1\gamma_{30}$ (curve d).

same as one finds for the laser with a regular pump. Hence the maximum noise reduction, which in this case occurs at $\omega=0$, amounts to

$$S(0)=\frac{4\kappa Q}{a_I}\approx 2Q\approx -\tfrac{4}{7} \, . \tag{49}$$

In the best limit ($c \gg 1$, $\gamma_{23} \ll \gamma_3$) we thus find about 60% noise reduction below shot-noise level in the standard four-level laser model.

IV. LASER WITH MULTISTEP RECYCLING: MULTILEVEL SYSTEM

As we have demonstrated in Sec. III the inclusion of dynamical recycling of the active laser atoms to the upper laser level after having emitted a laser photon can lead to a substantial reduction of intensity fluctuations below shot-noise level. In close relation to the well-known antibunching in atomic resonance fluorescence, we have suggested that the regularity of successive emissions of laser photons by each active atom plays a key role in this effect. To further substantiate this suggestion we will now generalize the above model to the case of an m-level system, $|1\rangle,\ldots,|m\rangle$. From the lower of the two lasing levels $|3\rangle$ the active electron is now recycled via $(m-2)$ steps $|i\rangle\leftarrow|i-1\rangle$, which are simply described by unidirectional rates r, to the upper lasing level $|2\rangle$. In Fig. 3 we schematically depict such an "atom." The conditional probability for an electron, being prepared in state $|i\rangle$ at time $t=t_0$, to jump to level $|i-1\rangle$ in the time interval $[t,t+dt)$ is given by $\bar{c}(t)=re^{-r(t-t_0)}$. For $m-1$ consecutive independent steps we find

$$\bar{c}(t)=r[r(t-t_0)]^{m-1}e^{-r(t-t_0)}/m! \, . \tag{50}$$

In our m-level system this probability can be identified with the conditional probability for the atom to be repumped into the upper lasing state at time t after having emitted a laser photon at $t=t_0$. We see that for $m\geq 2$ we find an anticorrelation $\bar{c}(t=0)=0$, which gets stronger the more levels we include. Of course this is just

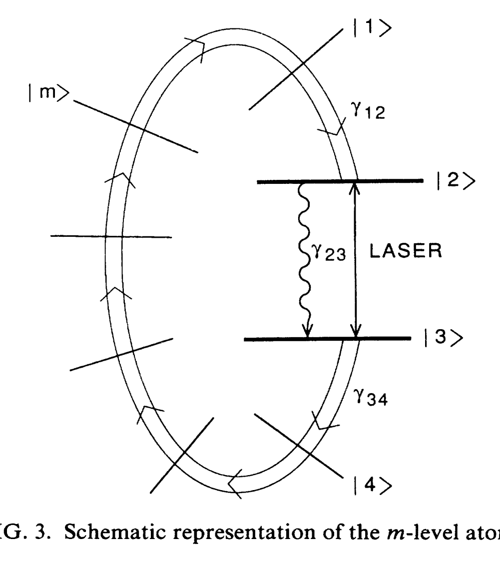

FIG. 3. Schematic representation of the m-level atom.

a reformulation of the well-known fact that a Poisson process as a sequence of consecutive unidirectional random steps gets deterministic in the limit of a large number of fast individual steps (keeping the ratio of the individual transition rate r and the number of steps m constant) [16]. For our case, where we consider a Poisson process on a circle, this limit implies a deterministic number of cycles for the active electron in a given time interval. Consequently if the number of atoms is constant, then the total number of photons added to the cavity mode in this time interval is fixed. Hence we have no pump noise, although the number of atoms in the upper lasing level exhibits random fluctuations in time. Obviously these results can be related to the calculations for a laser with an external regular pump, where $c(t)$ is the conditional probability for the next pump event [2]. Indeed in a recent paper Marte and Zoller have calculated the Mandel Q parameter for exactly the same conditional pump probabilities $\bar{c}(t)$ as given in Eq. (50).

Let us now generalize the four-level equations Eq. (34) to the case of m levels as depicted in Fig. 3. We will do this by introducing $m-4$ intermediate levels $|i\rangle$ between levels $|3\rangle$ and $|0\rangle$. These are assumed to be coupled only by spontaneous transitions. For the sake of simplicity the pump transition from level $|0\rangle$ to level $|1\rangle$ is replaced by a unidirectional rate as well. Hence we have no nonzero off-diagonal matrix elements on the nonlasing transitions and the corresponding equations for the diagonal density matrix elements read

$$\frac{d}{dt}\rho_{ii}=-\gamma_{i,i+1}\rho_{ii}+\gamma_{i-1,i}\rho_{i-1,i-1} \, , \tag{51}$$

where $\gamma_{i,i+1}$ is the (spontaneous) transition rate from level $|i\rangle$ to level $|i+1\rangle$. In the stationary limit we then have

$$\gamma_{m0}\rho_{mm}=\gamma_{m-1,m}\rho_{m-1,m-1}=\gamma_{i-1,i}\rho_{i-1,i-1}=\gamma_{34}\rho_{33} \, . \tag{52}$$

Using the above identities, we see that Eq. (34) for the stationary average of the atomic density matrix is unchanged with the exception of the first line, where we have to replace γ_{30} by γ_{34}. Of course the additional condition of having a closed atomic system now reads

$$\sum_{i=1}^{m}\rho_{ii}=1 \, . \tag{53}$$

Similarly, in the augmented set of Eqs. (30) for the atomic correlation functions, which we must solve to find the diffusion coefficients, the equations containing the additional levels can be eliminated, using the relation

$$(s+\gamma_{i,i+1})\mathcal{F}[V_{ii}](s)=\gamma_{i-1,i}\mathcal{F}[v_{i-1,i-1}](s) \, . \tag{54}$$

The resulting equations for the m-level system then can be solved analytically as for the four-level case. As one might expect from the considerations at the beginning of this section, the best noise reduction occurs for the laser operated well above threshold with all the various decay rates matched. The Mandel Q parameter obtained by linearization of the resulting Fokker-Planck equation then reads

$$Q=\frac{\bar{\gamma}(2\bar{\gamma}-n)-n^2(m-1)(m-2)}{2n(m-1)[\bar{\gamma}+(l-1)n]}\ , \tag{55}$$

where we have set $\gamma_{i,i+1}=\bar{\gamma}$ for $i\in[4,\ldots,m]$, $\gamma_{34}=2\bar{\gamma}$, and neglected spontaneous emission ($\gamma_{23}=0$). As before, n here denotes the laser-induced stimulated transition rate given by

$$n=(c-1)\frac{\gamma_{34}-\gamma_{23}}{(m-2)\gamma_3+2\bar{\gamma}}-\gamma_{23}\ . \tag{56}$$

In the limit of operation far above threshold Q reduces to

$$Q=-\frac{1}{2}\frac{m-2}{m-1}\ , \tag{57}$$

which is exactly what one would expect from Eq. (50). For a large number m of levels we find $Q\approx-\frac{1}{2}$, which is the same result as obtained for the laser with a completely regular pump. Hence, as suggested by Eq. (50), the atomic multistep recycling process leads to a complete suppression of pump noise.

So far we have neglected spontaneous emission on the laser transition. Including the first-order correction to Eq. (57) from a nonzero spontaneous emission rate γ_{23} on the lasing transition leads to

$$Q=-\frac{1}{2}\frac{m-2}{m-1}+\frac{\gamma_{23}}{2\gamma_{34}}\frac{m}{m-1}\ , \tag{58}$$

which fortunately, in a laser, is normally quite small. In Fig. 4 we plot the Mandel Q parameter as a function of the laser cooperativity parameter, for various numbers of atomic levels m. As expected, the maximum noise reduction occurs for large cooperativity parameter c, which has to be of order $c\approx 10$ to get a significant effect. Fortunately this is independent of m.

In the output-density fluctuation spectrum we again find a dip below the shot-noise level with a width of approximately 2κ and a minimum around zero frequency $S(0)=2Q$. Hence in principle one could achieve zero noise in the slow fluctuations of the intensity. In other words, the number of photons emitted in a sufficiently long time interval ($\delta t\gg 1/\kappa$) becomes deterministic.

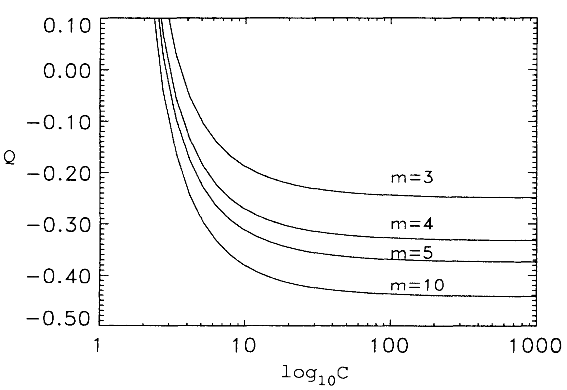

FIG. 4. Mandel Q parameter and mean intensity of laser as a function of pump rate R for various number of intermediate levels m.

V. NONLINEAR ABSORBERS INSIDE THE LASER CAVITY

As has been recently shown by Ritsch [6], as well as by Walls, Collett, and Lane [8], a nonlinear absorbing medium inside the laser cavity can lead to a significant reduction of photon number fluctuations in the laser output. Physically, in contrast to the situation described in Sec. IV instead of the gain we are now modifying the losses of the laser mode. We can easily include such nonlinear absorbers in our laser model. In contrast to a description based on an abstract nonlinear medium with a macroscopic nonlinear susceptibility, we will add a further independent species of atoms inside the laser cavity. These are assumed to have an n-photon transition resonant with a multiple cavity frequency ω_l. For notational simplicity we will restrict ourselves here to the case of a two-photon absorber. The methods, however, can be easily generalized to the case of more complicated nonlinear transitions as treated in Ref. [6]. For instance, the case of a nonlinear feedback can be treated by introducing a four-level system as in Fig. 3 and replacing the coupling to the pump transition by a nonlinear coupling to the lasing mode.

For the standard two-photon absorber the additional terms in the Hamiltonian Eq. (2) read

$$H_{nl}=\sum_{k=1}^{M}\left[\sum_{l=0}^{1}\hbar\omega_l(|l\rangle\langle l|)_k+[\hbar g_{nl}^k(a^\dagger)^2(|0\rangle\langle 1|)_k+\text{H.c.}]\right]+H_{nl,v}\ , \tag{59}$$

where $H_{nl,v}$ describes the decay out of the upper level $|1\rangle$ and we have assumed $\omega_{nl}=\omega_1-\omega_0\approx 2\omega_l$. Using the operator replacement rules

$$\begin{aligned} &a^2W\to\alpha^2W(\alpha)\ ,\\ &(a^\dagger)^2W\to(\alpha^{*2}-2\alpha^*\partial_\alpha+\partial_{\alpha^2}^2)W(\alpha)\ , \end{aligned} \tag{60}$$

we find the following additional terms for the Liouville operators L_i [Eq. (12)]:

$$\begin{aligned} &L_1\to L_1-\sum_{k=1}^{M}\{g_{nl}^k\alpha^{*2}[\sigma_{32}^k,W(\alpha)]-\alpha^2[\sigma_{23}^k,W(\alpha)]\}\ ,\\ &L_2\to L_2-\sum_{k=1}^{M}g_{nl}^k[(\partial_\alpha 2\alpha^*+\partial_{\alpha^2}^2)(\delta\sigma_{32}^k)+\text{H.c.}]\ ,\\ &L_3\to L_3+\sum_{k=1}^{M}g_{nl}^k[(\partial_\alpha 2\alpha^*+\partial_{\alpha^2}^2)(\langle\sigma_{32}^k\rangle)+\text{H.c.}]\ . \end{aligned} \tag{61}$$

It is easy to see that conditions of Eqs. (18) are still fulfilled, and we can use the same adiabatic elimination procedure as in Sec. III. Note that we find additional second-order derivatives already in the slow part L_3, which later will turn out to be the origin of the noise reduction in the laser intensity. In the differential equation for $W(\alpha)$ we will neglect derivatives of order 3 and higher [13] and transform the resulting Fokker-Planck equation in α and α^* to intensity I and phase ϕ [see Eqs. (24)]. Approximating the absorber atoms by simple two-level systems, which are independent of the laser active atoms, we find the following rather simple equations for the corresponding in- and out-of-phase polarization correlation functions of the absorber atoms:

$$\frac{d}{d\tau}V_I=-\gamma V_I-2|g\alpha^2|^2V_z ,$$
$$\frac{d}{d\tau}V_z=-2\gamma V_z+2V_I-2\gamma\langle\sigma_{32}\rangle , \qquad (62)$$
$$\frac{d}{d\tau}V_\phi=-\gamma V_\phi .$$

After some algebra we find two additional contributions to the intensity diffusion stemming from nonlinear absorbers.

$$D_{II}^{nl}=D_{II,L_3}^{nl}+D_{II,L_{1,2}}^{nl}$$

with

$$D_{II,L_3}^{nl}=2M\bar{g}(\alpha^*)^2\langle\sigma_{32}\rangle=-\gamma M\frac{I_p}{\gamma+I_p} \qquad (63)$$

and

$$D_{II,L_{1,2}}^{nl}=4M\gamma\frac{I_p^2}{(\gamma+I_p)^2} .$$

Here

$$I_p=\bar{g}^{\,2}\frac{2|\alpha|^4}{\gamma} \qquad (64)$$

denotes the rate of nonlinear absorptions per atom. For a weakly coupled nonsaturated absorber transition $\gamma\gg I_p$ the upper-state population ρ_{11}^{nl} can be neglected and only the first term contributes significantly to

$$D_{II}^{nl}=-\gamma MI_p\frac{\gamma-3I_p}{(\gamma+I_p)^2}\rightarrow -MI_p , \qquad (65)$$

which thus becomes negative. If this "nonlinear" contribution to D_{II} dominates over the positive part stemming from the usual laser terms, we get nonclassical sub-Poissonian statistics of the laser light, with Q given by

$$Q=\frac{D_{II}^0-\bar{g}^{\,2}MI^2/\gamma}{2I(d_I^0+4\bar{g}^{\,2}MI^2/\gamma)} . \qquad (66)$$

Here D_{II}^0 denotes the diffusion term stemming from the conventional laser gain terms. It tends to zero far above threshold as we have noted above [Eq. (40)]. In the limit that the laser losses are mainly due to the nonlinear absorption rate I_p, we find

$$Q=-\frac{\gamma^2-6(\bar{g}I)^2}{4\gamma^2}\xrightarrow[\gamma\gg I_p]{}-\frac{1}{4} . \qquad (67)$$

Thus the laser intensity fluctuations are reduced by 25% compared to the Poissonian limit of the standard laser. Here the magnitude of the noise reduction is limited by the fact that the nonlinear absorption indeed reduces the intensity diffusion on the one hand, but on the other hand also diminishes the mean intensity. In agreement with previous calculations [6,8,20] the minimum Mandel Q is bounded by $Q=-\frac{1}{4}$.

If we start to saturate the absorbing transition, the upper-state population cannot be neglected any more. In this limit the first contribution stemming from L_3 tends to $-\gamma M$, while the second contribution gives $4\gamma M$, so that the net effect of the absorber is an increased intensity noise. Hence a saturated nonlinear absorber acts as a loss introducing additional fluctuations. From Eq. (65) it is obvious that this transition from noise reduction to additional noise occurs at $\gamma=3I_p$. The best noise reduction due to the nonlinear absorber hence is achieved in the weak coupling limit, where no saturation of the nonlinear absorbing atoms occurs. This is probably the situation easiest to achieve in an experiment anyway. In order to obtain a significant effect in this limit one of course needs a large number of atoms.

So far our considerations on nonlinear absorbers have been based on the standard approach to the laser, where one neglects dynamic-pump-noise reduction. In the following we will investigate how a nonlinear absorber modifies the results obtained in Secs. III and IV [i.e., we replace D_{II}^0 and a_I^0 in Eq. (66) with the expressions obtained in Sec. III]. As the general expression for the Mandel Q parameter in this case is rather complicated and not very instructive, we will demonstrate the key effect in the following plots. In Fig. 5 we plot Q as a function of the nonlinear absorption rate $r_{nl}=MI_p$ for $m=10$, fixed laser intensity, and ideally matched atomic transition rates. We see that increasing the nonlinear absorption rate gradually decreases the total amount of intensity noise reduction from $Q\approx-\frac{1}{2}$ to $Q=-\frac{1}{4}$. Hence at first glance a nonlinear absorber seems to have a detrimental effect on the noise reduction. However, as before, the intensity fluctuation spectrum shows a Lorentzian dip around $\omega=0$ below the shot-noise level. Its width, however, is not given simply by the cavity linewidth κ, but in-

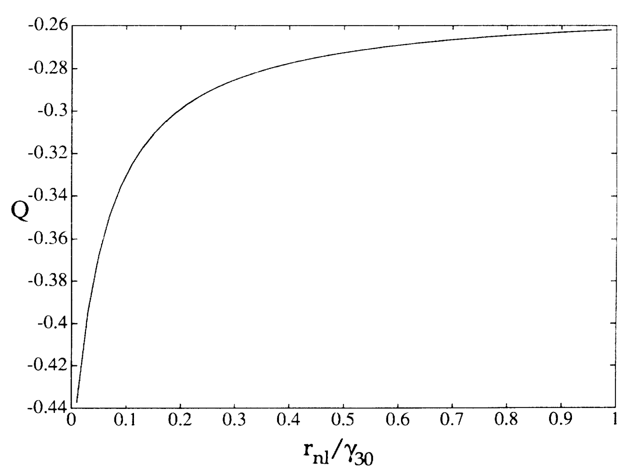

FIG. 5. Mandel Q as a function of nonlinear absorption rate r_a in units of γ_{30} for optimally matched rates and fixed laser intensity.

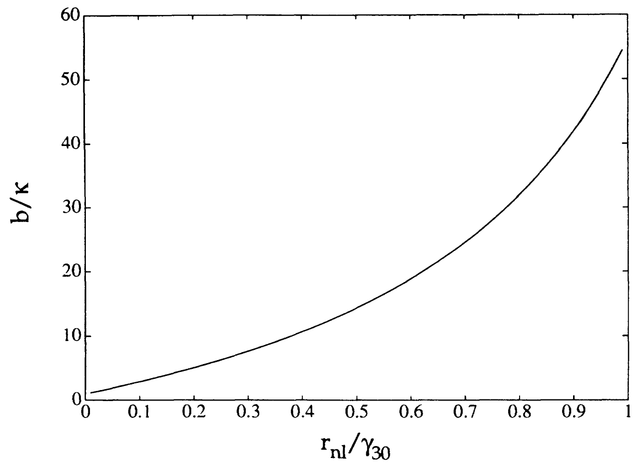

FIG. 6. Bandwidth b of intensity noise reduction (in units of the cavity width κ) for the same parameters as in Fig. 5.

creases with the nonlinear absorption rate. This is demonstrated in Fig. 6, where we show the linewidth a_I as a function of r_{nl} for the same parameters as in Fig. 5. We see that although a nonlinear absorber tends to reduce the total amount of noise reduction characterized by the Mandel Q parameter, the frequency range over which a significant noise reduction can be achieved can be greatly enhanced. Thus we get a large frequency range over which the intensity noise reduction is significantly enhanced. This effect could be very important for applications involving high-frequency modulations or very fast measurements, as the time needed to measure the output intensity with a certain precision can be significantly reduced.

As a last point, let us now look at the explicit expression

$$D_{\phi\phi}^{nl} = -2M\bar{g}(\alpha^*)^2\langle\sigma_{32}\rangle + \frac{\rho_{11}^{nl}}{2\gamma} = M\frac{\gamma I_p}{\gamma + I_p} \tag{68}$$

for the additional term in the phase diffusion rate. Obviously $D_{\phi\phi}^{nl}$ is positive. Hence we always find an enlarged spectral width of the laser.

VI. CONCLUSIONS

For quite a long time, apart from some small effects found for the two-level laser model [21], it has been believed that the laser sufficiently above threshold emits light, which is closely related to a coherent state, but with a slowly time-varying phase. In this work we have demonstrated that under certain conditions, such as keeping the number of laser active atoms fixed and allowing a depletion of the atomic ground state, the laser will emit highly nonclassical light, with strong amplitude squeezing. For a three-level atom our results agree with recent predictions by Khazanov *et al.*, who obtained a Mandel parameter of $Q = -\frac{1}{4}$.

In this paper all the explicit calculations were done assuming a broadband pump. As has been pointed out to us by T. Ralph in a recent discussion, for a coherent pump [set Ω=const in Eq. (29)] one finds even better noise reduction [9]. Indeed an explicit calculation based on the full density matrix Eq. (29) instead of Eq. (34) yields an optimum Mandel Q parameter of $Q = -m/[2(m+1)]$; i.e., the result is similar to Eq. (57) with m replaced by $m+2$. Hence replacing one of the rate transitions in the recycling circle by a coherent transition has an effect similar to adding two further levels.

ACKNOWLEDGMENTS

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[1] Y. Yamamoto, Phys. Rev. Lett. **66**, 2867 (1991).
[2] M. A. M. Marte and P. Zoller, Phys. Rev. A **40**, 5774 (1989); A. W. Troshin, A. I. Trubilko, and R. Ibarra, Opt. Spektrosk. **65**, 1195 (1988) [Opt. Spectrosc. (USSR) **65**, 674 (1988); C. Benkert, M. O. Scully, J. Bergou, L. Davidovic, M. Hillary, and M. Orzag, Phys. Rev. A **41**, 2756 (1990).
[3] H. Ritsch, P. Zoller, C. W. Gardiner, and D. F. Walls, Phys. Rev. A **44**, 3361 (1991).
[4] M. A. M. Marte and P. Zoller, Quantum Opt. **3**, 229 (1990).
[5] F. Haake, S. M. Tan, and D. F. Walls, Phys. Rev. A **40**, 7121 (1989).
[6] H. Ritsch, J. Quantum Opt. **2**, 189 (1990).
[7] T. A. B. Kennedy and D. F. Walls, Phys. Rev. A **40**, 6366 (1989); A. Sizmann, R. Schak, and A. Schenzle, Europhys. Lett. **13**, 109 (1990).
[8] D. F. Walls, M. J. Collett, and A. S. Lane, Phys. Rev. A **42**, 4366 (1990).
[9] T. Ralph and C. M. Savage, Opt. Lett. **16**, 1113 (1991).
[10] A. M. Khazanov, G. A. Koganov, and E. P. Gordov, Phys. Rev. A **42**, 3065 (1990).
[11] A. Smith and C. W. Gardiner, Phys. Rev. A **41**, 2730 (1990).
[12] H. J. Carmichael and D. F. Walls, J. Phys. B **9**, 1199 (1976); H. J. Kimble, M. Dagenais, and L. Mandel, Phys. Rev. A **18**, 201 (1978).
[13] (a) A. W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991); (b) F. Haake and M. Lewenstein, Phys. Rev. A **27**, 1031 (1983); **28**, 3606 (1983).
[14] M. Lax and W. H. Louisell, Phys. Rev. **185**, 586 (1969).
[15] H. Ritsch, P. Zoller, and J. Cooper, Phys. Rev. A **41**, 2653 (1990); M. H. Anderson *et al.*, Phys. Rev. Lett. **64**, 1346 (1990).
[16] C. W. Gardiner, *Handbook of Stochastic Methods*

(Springer-Verlag, Berlin, 1983).
[17] B. R. Mollow, Phys. Rev. A **5**, 1522 (1972).
[18] H. Ritsch and P. Zoller, Phys. Rev. A **38**, 4657 (1988).
[19] C. W. Gardiner, IBM J. Res. Dev. **32**, 1 (1988).
[20] Y. M. Golubev and I. V. Sokolov, Zh. Eksp. Teor. Fiz. **87**, 408 (1984) [Sov. Phys.—JETP **60**, 234 (1984)].
[21] L. A. Lugiato, F. Casagrande, and L. Pizzuto, Phys. Rev. A **26**, 3438 (1982).