Trapping states in a three-level Λ system

J. C. Retamal, L. Roa, and C. Saavedra
Facultad de Física, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile
(Received 12 July 1991)

We study the temporal evolution and the stationary regime of an electromagnetic-field mode interacting with three-level Λ-type atoms in a lossless microwave cavity. We find that under certain conditions the field evolves to the recently defined tangent or cotangent states [J. J. Slosser and P. Meystre, Phys. Rev. A **41**, 3867 (1990)]. In particular, in this system number states can be generated for any value of the atomic-upper-level population. In addition, a strong squeezing in the quadrature fluctuations is found.

PACS number(s): 42.50.Dv, 42.52.+x

I. INTRODUCTION

In recent years the lossless-cavity limit in micromasers has been extensively studied [1–7]. One of the most interesting results in this limit is the generation of number states [1,2]. When the interaction time satisfies the so-called trapping condition, $\sqrt{N+1}\phi=q\pi$ [1], the electromagnetic-field Fock space is separated in disconnected blocks, generating macroscopic superpositions. For example, in the one-photon maser the field evolves to a tangent or cotangent states [3,4] and in a two-photon maser the field evolves to an even or odd state [8]. In addition a strong squeezing in the field-quadrature fluctuations takes place [8,9].

The experimental realization of lossless cavities might be achieved with the improvement of high-Q microwave cavities. Until now, experimental studies in these cavities have been carried out for two- and three-level atomic systems [10–12]. In the latter case, three-level cascade-type atoms were considered. In the present work we study the interaction of a one-mode electromagnetic field with three-level Λ atoms (Fig. 1) in a lossless cavity. The Λ system was recently proposed as an atomic system for lasing without inversion [13]. We show that, under certain conditions, this model generates number states for any value of the atomic-upper-level population. In addition a high quadrature squeezing associated to cotangent states is found. These predictions make the Λ system more general as compared with a two-level system, in which number states are only present for atoms excited to the upper level.

This work is organized as follows: in Sec. II we define the model and we obtain the discrete master equation for the reduced field-density matrix when the atoms are injected in a coherent superposition of the atomic levels. In Sec. III we consider the stationary regime of the master equation and show the generation of the macroscopic superpositions, and its squeezing properties. In Sec. IV we discuss the results.

II. THE MODEL

The general framework to formulate the theory of this master is the same as that for the usual masers. First of all, we have to solve the temporal evolution for the atom-field interaction and find the reduced field-density matrix.

The Hamiltonian which describes the interaction of a one-cavity mode and the Λ three-level atom (Fig. 1) is given by

$$H=\hbar\nu a^\dagger a+\sum_{i=a,b,c}\hbar\omega_i|i\rangle\langle i| +\hbar g[a(|a\rangle\langle b|+|a\rangle\langle c|)+\text{H.c.}]\ , \tag{2.1}$$

where ν is the field frequency and ω_i are the atomic level frequencies. Operators a and $a^\dagger$ are the usual boson field operators and $|i\rangle\langle j|$ are the atomic-transition operators.

Solving the Schrödinger equation for the Hamiltonian in Eq. (2.1) in the detuning condition $(\omega_a-\omega_c)-\nu=\nu-(\omega_a-\omega_b)=\Delta$, we obtain the following expression for the temporal evolution operator of the Λ system:

$$U=\begin{bmatrix} 1-2g^2a\mu^{-1}a^\dagger+2g^2aCa^\dagger & G-igaS & -G-igaS \\ e^{-i\Delta t}(G^\dagger-igSa^\dagger) & e^{-i\Delta t}(D-i\Delta S) & e^{-i\Delta t}E \\ e^{-i\Delta t}(-G^\dagger-igSa^\dagger) & -e^{-i\Delta t}E & e^{-i\Delta t}(D+i\Delta S)\end{bmatrix}, \tag{2.2}$$

where we have defined the shorter notation

$$G=-\Delta ga\mu^{-1}+\Delta gaC\ , \tag{2.3}$$

$$D=1-\frac{\bar\mu}{\mu}+\bar\mu C\ , \tag{2.4}$$

$$E=(-g^2a^\dagger a\bar\mu^{-1}+g^2a^\dagger aC)\ , \tag{2.5}$$

$$C=\frac{\cos(g\tau\sqrt{\mu})}{\mu}\ , \tag{2.6}$$

and

$$S=\frac{\sin(g\tau\sqrt{\mu})}{\sqrt{\mu}}\ , \tag{2.7}$$

with μ and $\bar{\mu}$ being the operators

$$\mu=2g^2a^\dagger a+\Delta^2 \tag{2.8}$$

and

$$\bar{\mu}=g^2a^\dagger a+\Delta^2 . \tag{2.9}$$

Using the time evolution operator given by Eq. (2.2) we can find the master equation for the system for an arbitrary detuning. In order to make the analysis simple we consider the on-resonant situation, that is, $\Delta=0$. The evolution operator reduces to

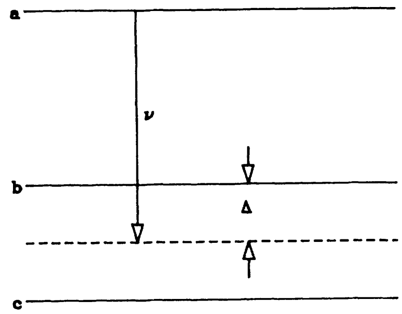

FIG. 1. Atomic three-level Λ system.

$$U=\begin{pmatrix} a\dfrac{\cos(\phi\sqrt{N}\,)}{N}a^\dagger & -ia\dfrac{\sin(\phi\sqrt{N}\,)}{\sqrt{2N}} & -ia\dfrac{\sin(\phi\sqrt{N}\,)}{\sqrt{2N}} \\ -i\dfrac{\sin(\phi\sqrt{N}\,)}{\sqrt{2N}}a^\dagger & \frac{1}{2}[1+\cos(\phi\sqrt{N}\,)] & \frac{1}{2}[-1+\cos(\phi\sqrt{N}\,)] \\ -i\dfrac{\sin(\phi\sqrt{N}\,)}{\sqrt{2N}}a^\dagger & \frac{1}{2}[-1+\cos(\phi\sqrt{N}\,)] & \frac{1}{2}[1+\cos(\phi\sqrt{N}\,)] \end{pmatrix} , \tag{2.10}$$

where $\phi=\sqrt{2}g\tau$ and $N=a^\dagger a$.

Let us consider the evolution of the reduced density matrix for the field. In the absence of cavity loss the change in the field depends only on the number of atoms which passed through the cavity. If only one atom interacts with the field during a finite time τ, the reduced density operator evolves according to the relation

$$\rho^{(k+1)}=\mathrm{tr}_{\mathrm{atom}}[U(\tau)\rho^{(k)}\rho_{\mathrm{atom}}U^\dagger(\tau)] , \tag{2.11}$$

where $\rho^{(k)}$ is the reduced density operator after the interaction with the kth atom and ρ_{atom} corresponds to the initial condition for the injected $(k+1)$th atom. We have assumed that the atom and the field are decoupled before the interaction, so that we can write the expression above the density operator after the $(k+1)$th atom has passed through the cavity. Let us consider an atomic initial condition given by a coherent superposition of the atomic levels

$$|\Psi\rangle_{\mathrm{atom}}=\alpha|a\rangle+\beta|b\rangle+\gamma|c\rangle , \tag{2.12}$$

where the probability amplitudes α, β, and γ are the same for all atoms. Introducing the atomic density matrix into Eq. (2.11) we get the following master equation in the number-state representation:

$$\begin{aligned}\rho_{nm}^{(k+1)}={}&|\alpha|^2\cos(\phi\sqrt{n+1})\cos(\phi\sqrt{m+1})\rho_{nm}^{(k)}+\tfrac{1}{2}[|\beta-\gamma|^2+|\beta+\gamma|^2\cos(\phi\sqrt{n}\,)\cos(\phi\sqrt{m}\,)]\rho_{nm}^{(k)}\\ &+\frac{|\beta+\gamma|^2}{2}\sin(\phi\sqrt{n+1})\sin(\phi\sqrt{m+1})\rho_{n+1m+1}^{(k)}+|\alpha|^2\sin(\phi\sqrt{n}\,)\sin(\phi\sqrt{m}\,)\rho_{n-1m-1}^{(k)}\\ &+\frac{i\alpha(\beta+\gamma)^*}{\sqrt{2}}[\cos(\phi\sqrt{n+1})\sin(\phi\sqrt{m+1})\rho_{nm+1}^{(k)}-\cos(\phi\sqrt{m}\,)\sin(\phi\sqrt{n}\,)\rho_{n-1m}^{(k)}]\\ &+\frac{i\alpha^*(\beta+\gamma)}{\sqrt{2}}[\cos(\phi\sqrt{n}\,)\sin(\phi\sqrt{m}\,)\rho_{nm-1}^{(k)}-\cos(\phi\sqrt{m+1})\sin(\phi\sqrt{n+1})\rho_{n+1m}^{(k)}] .\end{aligned} \tag{2.13}$$

This equation is similar to the master equation for a two-level-atom micromaser when $\beta=\gamma$. In the next section we consider the generation of number states and macroscopic superpositions in this system.

III. NUMBER STATES AND MACROSCOPIC SUPERPOSITIONS

The master equation obtained in the preceding section allows us to study the temporal evolution for the system in terms of the number of atoms, for arbitrary initial atomic and field conditions. In Fig. 2(a) we show the steady-state solution of Eq. (2.13) for the intensity and the photon-number fluctuations, for atoms injected with an atomic-upper-level probability $|\alpha|^2=0.1$ and for $\phi=\pi/\sqrt{11}$. We see that such reduced interaction time imposes to the field to evolve within the Fock-space block [0,10] when the initial conditions of the field are contained in this block. This statement is reminiscent of the existence of the so-called trapping conditions for the elec-

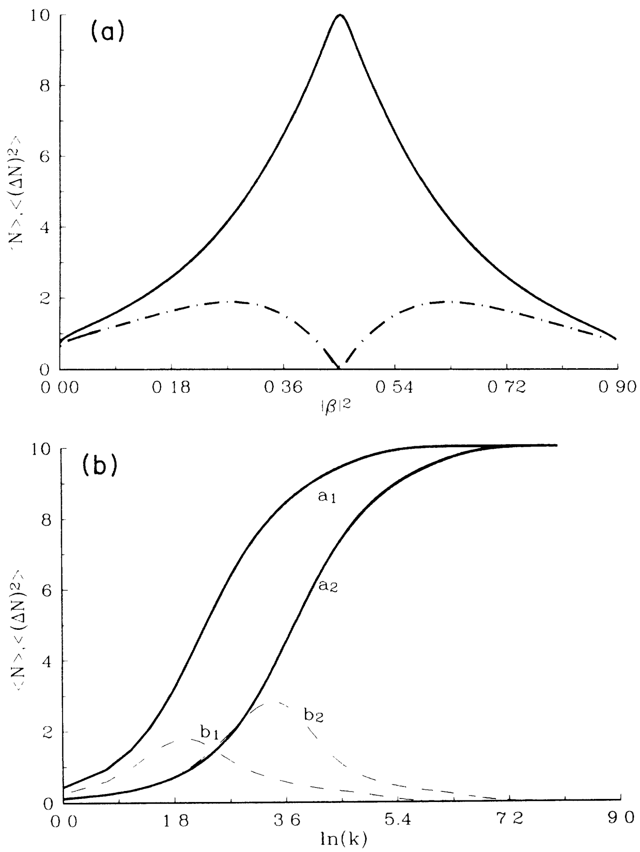

FIG. 2. (a) Steady state of the intensity (solid line) and the photon-number fluctuations (dashed line) as a function of lower-level $|b\rangle$ population, for a fixed $|\alpha|^2=0.1$ and $\phi=\pi/\sqrt{11}$. (b) Temporal evolution of the intensity (a_i, $i=1,2$) and the photon-number fluctuations (b_i, $i=1,2$) as a function of number of atoms passed through the cavity on a ln scale ($i=1$, $\alpha=0.8$, and $i=2$, $\alpha=0.4$), with $\gamma=-\beta$. The initial condition of the field was chosen as the vacuum state.

tromagnetic field in zero-loss cavities. It is observed that the field distribution is sub-Poissonian for all values of the atomic-lower-level populations. In addition, for a particular value of the lower-level populations ($\gamma=-\beta$) the photon-number fluctuations tend to zero, i.e., there is a pure number state, which corresponds precisely to the number state $|10\rangle$. Figure 2(b) illustrates the temporal evolution for $\langle N\rangle$ and $(\Delta N)^2$ considering the same reduced interaction time of Fig. 2(a) and the condition $\gamma=-\beta$. It is observed that for different atomic-upper-level probabilities the system evolves to the same pure number state $|10\rangle$. The difference between these two cases is in the time it takes the system to reach a steady state.

The previous numerical results give us a feeling of the possibilities of this system, that is, the existence of trapping states and the generation of number states, in a broad range of the input parameters. The existence of number states can be established from Eq. (2.13). When we consider the situation $\gamma=-\beta$ the diagonal elements of the master equation reduce to

$$\rho_{nn}^{k+1}=[1-|\alpha|^2\sin^2(\phi\sqrt{n+1})]\rho_{nn}^{k} + |\alpha|^2\sin^2(\phi\sqrt{n})\rho_{n-1n-1}^{k} . \tag{3.1}$$

In the stationary regime of the above equation, $\rho_{nn}^{k+1}=\rho_{nn}^{k}$, we get

$$|\alpha|^2\sin^2(\phi\sqrt{n+1})\rho_{nn}=|\alpha|^2\sin^2(\phi\sqrt{n})\rho_{n-1n-1} . \tag{3.2}$$

We consider also that the reduced interaction times ϕ satisfy the condition

$$\phi(N_\mu+1)^{1/2}=q\pi , \tag{3.3}$$

with N_μ and q integers. In this situation Eq. (3.2) has the solution

$$\rho_{nn}=\delta_{n,N_\mu} , \tag{3.4}$$

that is, when $\gamma=-\beta$ and ϕ satisfies Eq. (3.3), the field mode evolves to number state $|N_\mu\rangle$ for any nonzero value of the atomic-upper population $|\alpha|^2$. This result makes the difference between this system and the two-level-atom system in which the number states can be generated only when the injected atom is excited to the upper level [1,2].

From here on we will consider the more general conditions under which the system evolves to pure states in the stationary regime, for arbitrary atomic-level populations α, β, and γ. When the field mode reaches the steady state, before and after a new interaction, the atom-field state is given by the tensor product of the field and atomic state. Explicitly

$$|\Phi\rangle \xrightarrow{\tau} U|\Phi\rangle ,$$

where

$$|\Phi\rangle=\sum_{n=N_d}^{N_\mu} S_n(\alpha|a\rangle+\beta|b\rangle+\gamma|c\rangle)|n\rangle , \tag{3.5}$$

and

$$U|\Phi\rangle=\sum_{n=N_d}^{N_\mu} S_n(\alpha'|a\rangle+\beta'|b\rangle+\gamma'|c\rangle)|n\rangle , \tag{3.6}$$

where N_d and N_μ are the lower and upper limits of the Fock-space block, and U the time evolution operator given by Eq. (2.10). Writing explicitly Eq. (3.6) we get

$$\begin{aligned}U|\Phi\rangle=\sum_{n=N_d}^{N_\mu} & S_n\left[\alpha\cos(\phi\sqrt{n+1})|n\rangle-\frac{i}{\sqrt{2}}(\beta+\gamma)\sin(\phi\sqrt{n})|n-1\rangle\right]|a\rangle \\ & +S_n\left[\frac{-i}{\sqrt{2}}\alpha\sin(\phi\sqrt{n+1})|n+1\rangle+\tfrac{1}{2}[\beta-\gamma+(\beta+\gamma)\cos(\phi\sqrt{n})]|n\rangle\right]|b\rangle \\ & +S_n\left[\frac{-i}{\sqrt{2}}\alpha\sin(\phi\sqrt{n+1})|n+1\rangle+\tfrac{1}{2}[\gamma-\beta+(\beta+\gamma)\cos(\phi\sqrt{n})]|n\rangle\right]|c\rangle .\end{aligned} \tag{3.7}$$

The pure states are apparent from Eq. (3.7). Trapping states take place when the downward, $(N_d)^{1/2}\phi=q\pi$, and upward, $(N_\mu+1)^{1/2}\phi=p\pi$, conditions are satisfied, with q and p integers. From Eqs. (3.6) and (3.7) we get the following recursion relations for the coefficients S_n in the steady state of the field:

$$\frac{i}{\sqrt{2}}(\beta+\gamma)\sin(\phi\sqrt{n})S_n=[\alpha\cos(\phi\sqrt{n})-\alpha']S_{n-1}\ , \tag{3.8}$$

$$\{\beta'-\tfrac{1}{2}[\beta-\gamma+(\beta+\gamma)\cos(\phi\sqrt{n})]\}S_n =\frac{-i}{\sqrt{2}}\alpha\sin(\phi\sqrt{n})S_{n-1}\ , \tag{3.9}$$

and

$$\{\gamma'-\tfrac{1}{2}[\gamma-\beta+(\beta+\gamma)\cos(\phi\sqrt{n})]\}S_n =\frac{-i}{\sqrt{2}}\alpha\sin(\phi\sqrt{n})S_{n-1}\ . \tag{3.10}$$

These equations must be satisfied simultaneously, so that we get a system of equations which allows us to find the values for the atomic probability amplitudes after the interaction in terms of the initial values. From Eqs. (3.9) and (3.10) we get

$$\beta'-\gamma'=\beta-\gamma\ . \tag{3.11}$$

Similarly, from Eqs. (3.8) and (3.9) we find

$$\alpha'\beta'=\tfrac{1}{2}[\alpha'(\beta-\gamma)-\alpha(\beta+\gamma)] \tag{3.12}$$

and

$$\alpha\beta'=\tfrac{1}{2}[-\alpha'(\beta+\gamma)+\alpha(\beta-\gamma)]\ . \tag{3.13}$$

The solution to these equations is given by

$$\alpha'=\begin{bmatrix}\alpha\\-\alpha\end{bmatrix}\ ,\quad \beta'=\begin{bmatrix}-\gamma\\ \beta\end{bmatrix}\ ,\quad \gamma'=\begin{bmatrix}-\beta\\ \gamma\end{bmatrix}\ . \tag{3.14}$$

Replacing these solutions into the recursive equations found previously we obtain

$$S_n=i\frac{\sqrt{2}\alpha}{\beta+\gamma}\tan\left[\frac{\phi\sqrt{n}}{2}\right]S_{n-1} \tag{3.15}$$

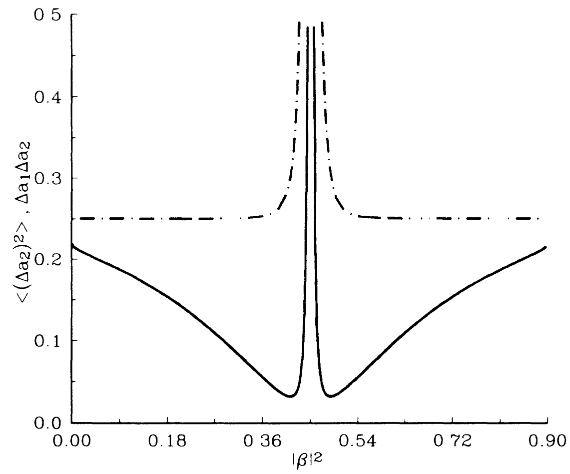

FIG. 3. $(\Delta a_2)^2$ (solid line) and $(\Delta a_1\Delta a_2)$ (dashed line) as a function of the population of the level $|b\rangle$, for an upper-level population $|\alpha|^2=0.1$ and $N_\mu=400$ under a π-trapping condition.

and

$$S_n=-i\frac{\sqrt{2}\alpha}{\beta+\gamma}\cot\left[\frac{\phi\sqrt{n}}{2}\right]S_{n-1}\ . \tag{3.16}$$

From these expressions we get a family of tangent and cotangent states. When we consider a particular value of $|\alpha|\neq 0$ there is a family of states corresponding to different populations of the lower levels.

The last point motivates the study of the field-quadrature fluctuations, which will have different features as compared with a two-level system. The field quadratures are defined by

$$a_1=\tfrac{1}{2}(a+a^\dagger) \tag{3.17}$$

and

$$a_2=\frac{1}{2i}(a-a^\dagger)\ . \tag{3.18}$$

Using the states defined in Eq. (3.16), we readily get the following expressions for the quadrature fluctuations:

$$(\Delta a_1)^2=\tfrac{1}{4}\left[1+2\langle N\rangle-2\left|\frac{\beta+\gamma}{\sqrt{2}\alpha}\right|^2\langle\sqrt{N(N-1)}\tan(\phi\sqrt{N})\tan(\phi\sqrt{N-1})\rangle\right] \tag{3.19a}$$

and

$$(\Delta a_2)^2=\tfrac{1}{4}\left[1+2\langle N\rangle+2\left|\frac{\beta+\gamma}{\sqrt{2}\alpha}\right|^2[\langle\sqrt{N(N-1)}\tan(\phi\sqrt{N})\tan(\phi\sqrt{N-1})\rangle-2\langle\sqrt{N}\tan(\phi\sqrt{N})\rangle^2]\right]\ . \tag{3.19b}$$

Figure 3 shows the quadrature fluctuations $(\Delta a_2)^2$ and the product $(\Delta a_1\Delta a_2)$ as a function of the population of the level $|b\rangle$, for an upper-level population $|\alpha|^2=0.1$. We observe the existence of squeezing for a broad range of $|\beta|^2$, with the maximum reduction around the symmetric point $\gamma=-\beta$. As we pointed out before, at this point the field is in a number state $|N_\mu\rangle$ (Fig. 3 corresponds to $N_\mu=400$). In this case we have a maximum squeezing of approximately 88%. This reduction is stronger than the one in a two-level system, for the same upper-level population [14].

The minimum value of the quadrature fluctuations depends on the dimension of the Fock-space block and for increasing dimension a best squeezing is obtained. From numerical analysis the minimum has a tendency to zero, but analytically it is not clear that perfect squeezing takes place for a given set of parameters. Finally, we see that the states are of minimum uncertainty for a broad range of $|\beta|^2$.

IV. DISCUSSION

We have analyzed the three-level Λ-type maser system in the lossless cavity limit. This system exhibits interesting properties which make it more general than a two-level system. As a first conclusion we can say that when $\gamma=-\beta$ and trapping conditions are satisfied, the system evolves to a number state for any nonzero value of the atomic-upper-level population. We recall this last point as compared with a two-level system in which number states can be generated only when the injected atom is excited to the upper level. On the other hand, we know that squeezing states are generated around the region of a number state, so we can say that a family of squeezed states is generated in addition for each value of the upper-level population. Finally, in a broad range of given atomic-upper-level population, when the Fock-space dimension is increased and adequate atomic-lower-level populations in the Λ system are chosen, the reduction is unbounded, on the contrary, the reduction in the field quadratures in the two-level system is bounded.

The previous conclusions have to be reconsidered when the effects of finite cavity loss and atomic decay rates are included in the analysis. The authors in Ref. [15] have analyzed these effects for a two-level atom maser. They found that these effects will play a critical role in the experimental realization of trapping states. We envisage that a possible realization of trapping states in a Λ three-level system will be restricted by similar conditions to those found in that work. A detailed analysis of these conditions in this system is an open question, and is planned to be the subject of future work.

ACKNOWLEDGMENTS

The authors acknowledge financial support from CONICYT. J.C.R. acknowledges support from FONDECYT under Grant No. 0053/90.

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