Arbitrary-order three-turning-point phase-integral formula for the S matrix in Regge-pole theory

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In this paper we give an arbitrary-order phase-intergral formula for the S matrix in Regge-pole theory. This formula is valid for a particular situation when there are three relevant transition points, two of which may lie at an aribtrary distance from each other, possibly close together, but far away from all other transition points, the pole at the origin inclusive. When the formula is particularized to the first order of the phase-integral approximation, it agrees with formulas given by Connor, Jakubetz, and Sukumar [J. Phys. B **9**, 1783 (1976)] and by Brink and Takigawa [Nucl. Phys. A **279**, 159 (1977)], after these formulas have been rewritten conveniently.

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I. INTRODUCTION

Regge states, which may exist for particular complex values of the angular momentum quantum number, correspond by definition to regular solutions of the radial Schrödinger equation which behave as purely outgoing waves at infinity. Such wave functions correspond to poles (Regge poles) of the one-by-one scattering matrix (S matrix) in the complex angular momentum plane. The quantities that determine the contribution of the Regge states to the scattering cross section are the *positions* l_m and *residues* r_m of the Regge poles, where m is a nonnegative quantum number [1]. Numerical [2–5], semiclassical [6–9], and phase-integral [10–14] methods for calculating l_m and r_m in elastic heavy-particle scattering have been developed during the past twenty years; see Ref. [13], and some further references given there, for short comments on this development. However, the systematic investigation of the accuracy of these methods is incomplete. Until recently, only results obtained by the semiclassical approximation and by the numerical complex coordinate method due to Sukumar and Bardsley [3] had to some extent been compared with each other.

The application of the phase-integral method in Regge-pole theory started with the work by Thylwe and Fröman [10] and by Thylwe [11], in which exact general representations of the scattering matrix were derived within the framework of the rigorous F-matrix method. The recent derivation of arbitrary-order phase-integral formulas [12–14] in Regge-pole theory has provided means for estimating the accuracy of previous analytical as well as numerical results. The investigation in Ref. [14] of the accuracy of the arbitrary-order formula for the positions [11] and the likewise arbitrary-order, uniform formula for the residues [12] shows that these formulas are extremely accurate and that some of the previous numerical results, which have earlier been used for investigating the accuracy of semiclassical results, are not reliable. The investigation in Ref. [13], concerning the application of the uniform residue formula [12] to ion-atom scattering, demonstrates that this formula, already when used in the first-order approximation, considerably improves the previously existing semiclassical Regge-pole description of rainbow scattering.

It has thus been found that the phase-integral method, involving the use of the arbitrary-order phase-integral approximation generated from an *a priori* unspecified base function [15], provides a powerful tool for the development of Regge-pole theory. However, with the use of this tool one has so far treated only nonresonant situations, i.e., situations in which only *two transition zeros* have to be taken into account in the phase-integral treatment. The purpose of the present paper is to derive phase-integral formulas for resonance scattering, when *three transition zeros* are of significant importance.

In Sec. II we quote general, exact formulas for the one-by-one S matrix and the Regge poles. For the two F-matrix elements appearing in these formulas we derive in Sec. III approximate expressions. Section IV presents the final arbitrary-order phase-integral formulas for the Regge-pole positions and residues, which are then, for the first-order approximation, in Sec. V compared with the corresponding semiclassical formulas derived by previous authors [7–9].

II. GENERAL, EXACT FORMULAS FOR THE S MATRIX AND THE REGGE POLES

For the background and the notations we refer to previous papers by Thylwe and Fröman [10] and by Fröman and Fröman [12]. According to Eq. (3.27) in Ref. [10] or Eq. (1.1) in Ref. [12] the elastic one-by-one scattering matrix S_l (the S matrix) is given by the exact formula

$$S_l = i\frac{F_{11}(+\infty,+0)}{F_{21}(+\infty,+0)}\exp(2i\delta_l)\ , \tag{2.1a}$$

where

$$\delta_l = \lim_{r\to+\infty}[w(r)-kr]+(l+\tfrac{1}{2})\pi/2\ . \tag{2.1b}$$

By definition the Regge poles are the poles of the S matrix in the complex l plane. From (2.1a) and (2.1b) one therefore finds that the condition determining a Regge pole l_m is

$$F_{21}(+\infty,+0)=0 \quad \text{for } l=l_m \tag{2.2}$$

and that the residue r_m of this Regge pole is

$$r_m=\left[i\frac{F_{11}(+\infty,+0)}{\partial F_{21}(+\infty,+0)/\partial l}\exp(2i\delta_l)\right]_{l=l_m} . \tag{2.3}$$

In Ref. [10] the exact formula (2.1a) with (2.1b) was used for the derivation of an approximate formula for S_l on the assumption that one needs to take into account only *one simple complex transition zero.* In such situations there exist, however, no Regge poles. In Ref. [12] the exact formulas (2.2) and (2.3) were applied to a situation when there exist Regge poles and were used for the derivation of approximate formulas for l_m and r_m on the assumption that there are *two relevant transition zeros,* lying at an arbitrary distance from each other, possibly close together, but far away from the origin and from other transition points. The resulting formulas were used in particular applications by Thylwe and Amaha [13] and by Amaha and Thylwe [14]. Starting from the situation treated in Ref. [12], we now assume that the two transition zeros t_1 and t_2 do not lie close together, which means that the quantum number m is not too small. As the energy decreases, a third transition zero t_3 may approach the cluster consisting of the transition zeros t_1 and t_2 such that all three transition zeros t_1, t_2, and t_3 finally form a cluster. We shall assume that the distance between t_1 and t_3 (like the distance between t_1 and t_2) remains sufficiently large, while there is no restriction on the distance between t_2 and t_3. In the present paper we shall use the exact formulas (2.2) and (2.3) for the treatment of such a cluster of *three relevant transition zeros* t_1, t_2, and t_3; see Fig. 1.

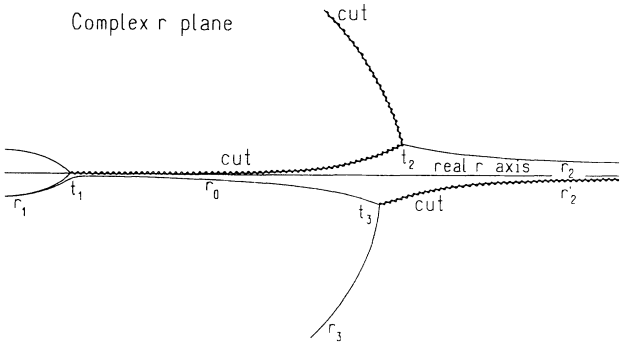

FIG. 1. Three transition zeros t_1, t_2, and t_3 in the complex r plane and the first-order anti-Stokes lines emerging from these points. The point t_1 is assumed to lie far away from both t_2 and t_3, but t_2 and t_3 may lie at an arbitrary distance from each other, possibly close together. In order not to have too many lines in the figure, we have drawn it for the case that the quantity $\overline{\gamma}_{12}$, defined by (3.3), is real and positive, which implies that there is an anti-Stokes line joining t_1 and t_2. To make $q(z)$ single valued we have introduced a cut (wavy line) between t_1 and t_2 and another cut (wavy line) along the first-order anti-Stokes line emerging from t_3 and proceeding toward $+\infty$. To make also $w(z)$ single valued we have introduced a further cut (wavy line) along the first-order anti-Stokes line emerging from t_2 and proceeding upward. By r_0, r_1, r_2, r_3, and r_2' we denote points on some of the first-order anti-Stokes lines emerging from t_1, t_2, and t_3. In the derivation in the present paper there is no assumption of an anti-Stokes line joining t_1 and t_2, although the figure has been drawn for such a case.

III. APPROXIMATE EXPRESSIONS FOR $F_{11}(+\infty,+0)$ AND $F_{21}(+\infty,+0)$

We assume that the three relevant transition zeros t_1, t_2, and t_3 (simple zeros of the square of the base function [15]) lie far away from the origin and from other transition points, and we consider the situation that t_1 is well separated from both t_2 and t_3, while t_2 and t_3 may lie at an arbitrary distance from each other, i.e., even close together; see Fig. 1. To make $q(r)$ single valued we introduce in the complex r plane a cut between t_1 and t_2 and another cut from t_3 along an anti-Stokes line proceeding toward $+\infty$, and to make also $w(r)$, to be defined in (3.1), single valued, we continue the former cut from t_2 toward infinity; see Fig. 1. We now define

$$w(r)=\tfrac{1}{2}\int_{\Gamma_{t_2}(r)}q(r)dr=\int_{(t_2)}^{r}q(r)dr , \tag{3.1}$$

where the contour of integration $\Gamma_{t_2}(r)$, which starts from the point corresponding to r, but lying on a Riemann sheet adjacent to the complex r plane under consideration, encircles the transition zero t_2, and ends at the point r in the complex r plane, is chosen such that the function $w(r)$ can in the first order of the phase-integral approximation be written as an integral starting at t_2 on the *right-hand lip* of the cut in Fig. 1 (that starts at t_1 and via t_2 continues to infinity) and ending at r. The last member of (3.1) is by definition a simplified notation for the contour integral in the second member. This simplified notation, reminding of an ordinary integration from t_2 to r, which also makes it convenient to use a corresponding language, was introduced by Fröman, Fröman, and Lundborg [16], pp. 160 and 161, and will be used in the present paper. It makes it possible to use, for an arbitrary order of the phase-integral approximation, a similar simple notation and almost the same simple language (although in a generalized sense) as for the first order of the phase-integral approximation. This really simplifies the treatment of concrete problems.

It is assumed that $q(r)$ tends to a positive constant k as r tends to $+\infty$, and therefore, with $w(r)$ defined by (3.1) and r_2 lying on the anti-Stokes line emerging from t_2 toward $+\infty$, $w(r_2)$ is positive and increases as r_2 moves away from t_2, while $\mathrm{Re}\,w(r_1)$ decreases as r_1 moves away from t_1; see Fig. 1. The origin is assumed to lie in (or on the border of) the region limited by the two anti-Stokes lines emerging from t_1 toward the left.

According to Eqs. (3.3a) and (3.3b) in Ref. [12] we have

$$F_{11}(r_1,+0)=1+o(1) , \tag{3.2a}$$

$$F_{21}(r_1,+0)=o(1)\exp(-2\pi i\overline{\gamma}_{12}) , \tag{3.2b}$$

where

$$\overline{\gamma}_{12}=\frac{1}{\pi}\int_{(t_1)}^{(t_2)}q(r)dr , \tag{3.3}$$

with the integration performed below the cut from t_1 to t_2 in Fig. 1, and hence on the same side of the cut as the point r_0 lies; see Fig. 1. This explanation of the path of integration is strictly valid for the first order of the phase-integral approximation, and (as explained above) it can also be used in a generalized sense when a higher-order approximation is used. The integral in (3.3) is (according to its definition) actually equal to half of the integral along a closed contour encircling in the positive sense t_1 and t_2 by disregarding the cut from t_2 toward infinity. We also recall that the symbols $o(1)$ in (3.2a) and (3.2b) and in the following denote quantities which to their absolute values are small compared to unity.

Assuming $\mathrm{Re}\,\bar{\gamma}_{12}$ to be positive and sufficiently large, we have, according to Eq. (2.4b) in Ref. [12], the approximate formula

$$\underline{F}(r_0,r_1)=\begin{bmatrix} 1 & o(1)\exp(+2\pi i\bar{\gamma}_{12}) \\ i\exp(-2\pi i\bar{\gamma}_{12}) & 1 \end{bmatrix}. \tag{3.4}$$

As a result of arguments analogous to those leading to Eqs. (2.3), (2.15), and (2.17) in Ref. [12] we obtain approximately (see Fig. 1)

$$\underline{F}(r_3,r_0)=\begin{bmatrix} 1 & o(1)\exp(+2\pi i\bar{\gamma}_{32}) \\ i\exp(-2\pi i\bar{\gamma}_{32})f(\bar{\gamma}_{32}) & 1 \end{bmatrix}, \tag{3.5a}$$

and

$$\begin{aligned}\underline{F}(r_2',r_3)&=\begin{bmatrix} 1 & i\exp(+2\pi i\bar{\gamma}_{32})f(\bar{\gamma}_{32}e^{-i\pi}) \\ o(1)\exp(-2\pi i\bar{\gamma}_{32}) & 1 \end{bmatrix}, \\ &=\begin{bmatrix} 1 & i\exp(+2\pi i\bar{\gamma}_{32})[1+\exp(+2\pi i\bar{\gamma}_{32})]/f(\bar{\gamma}_{32}) \\ o(1)\exp(-2\pi i\bar{\gamma}_{32}) & 1 \end{bmatrix},\end{aligned} \tag{3.5b}$$

where

$$\bar{\gamma}_{32}=\frac{1}{\pi}\int_{(t_3)}^{(t_2)} q(r)dr\ , \tag{3.6}$$

and $f(\bar{\gamma})$ is a simplified notation for the function $f(\bar{\gamma}_0,\ldots,\bar{\gamma}_{2N})$ given up to the fifth-order approximation by Eqs. (2.14), (2.13a), (2.13b), and (2.11a)–(2.11c) in Ref. [12] and up to the 13th-order approximation by Eqs. (2.7), (2.9), and (2.10a)–(2.10g) in Ref. [14]. With matrix multiplication we obtain from (3.5a) and (3.5b) approximately

$$\begin{aligned}\underline{F}(r_2',r_0)&=\underline{F}(r_2',r_3)\underline{F}(r_3,r_0) \\ &=\begin{bmatrix} -\exp(+2\pi i\bar{\gamma}_{32}) & i\exp(+2\pi i\bar{\gamma}_{32})\{[1+\exp(+2\pi i\bar{\gamma}_{32})]/f(\bar{\gamma}_{32})+o(1)\} \\ i\exp(-2\pi i\bar{\gamma}_{32})[f(\bar{\gamma}_{32})+o(1)] & 1+o(1) \end{bmatrix}.\end{aligned} \tag{3.7}$$

Using the approximate formula (3.7) and the exact formula (see Fig. 1)

$$\underline{F}(r_2,r_2')=\begin{bmatrix} 0 & -i\exp(+2\pi i\bar{\gamma}_{32}) \\ -i\exp(-2\pi i\bar{\gamma}_{32}) & 0 \end{bmatrix}, \tag{3.8}$$

we obtain with the aid of matrix multiplication the approximate formula

$$\underline{F}(r_2,r_0)=\underline{F}(r_2,r_2')\underline{F}(r_2',r_0)=\begin{bmatrix} f(\bar{\gamma}_{32})+o(1) & -i\exp(+2\pi i\bar{\gamma}_{32})[1+o(1)] \\ i & [1+\exp(+2\pi i\bar{\gamma}_{32})]/f(\bar{\gamma}_{32})+o(1) \end{bmatrix}. \tag{3.9}$$

To be able to proceed further, we now introduce the assumption that neither t_2 nor t_3 lies too far above the first-order anti-Stokes line emerging from t_1 toward the right. Noting that this assumption implies that the absolute value of neither $\exp(-2\pi i\bar{\gamma}_{12})$ nor $\exp[2\pi i(\bar{\gamma}_{32}-\bar{\gamma}_{12})]$ is too large, and using matrix multiplication, we finally obtain from (3.2a), (3.2b), (3.4), and (3.9) in the limit as $r_2\to+\infty$ the approximate formulas

$$F_{11}(+\infty,+0)=f(\bar{\gamma}_{32})+\exp[+2\pi i(\bar{\gamma}_{32}-\bar{\gamma}_{12})]\ , \tag{3.10a}$$

$$\begin{aligned}&F_{21}(+\infty,+0) \\ &\quad=i\{1+\exp(-2\pi i\bar{\gamma}_{12}) \\ &\qquad\times[1+\exp(+2\pi i\bar{\gamma}_{32})]/f(\bar{\gamma}_{32})\}\ .\end{aligned} \tag{3.10b}$$

IV. PHASE-INTEGRAL FORMULAS FOR THE S MATRIX AND THE REGGE POLES

Inserting (3.10a) and (3.10b) into (2.1a), we obtain the approximate formula

$$S_l=\frac{f(\bar{\gamma}_{32})+\exp[+2\pi i(\bar{\gamma}_{32}-\bar{\gamma}_{12})]}{1+\exp(-2\pi i\bar{\gamma}_{12})[1+\exp(2\pi i\bar{\gamma}_{32})]/f(\bar{\gamma}_{32})}\times\exp(2i\delta_l)\ , \tag{4.1}$$

from which, in the limit when $f(\bar{\gamma}_{23})$ tends to unity, we obtain the S-matrix formula derived by Thylwe [11] for the case when the three relevant transition zeros are all well separated from each other.

Inserting (3.10b) into (2.2), we obtain approximately

$$\exp[+2\pi i(\bar{\gamma}_{12}-\tfrac{1}{2})]=\frac{1+\exp(+2\pi i\bar{\gamma}_{32})}{f(\bar{\gamma}_{32})}\quad\text{for } l=l_m\ , \tag{4.2}$$

i.e.,

$$\bar{\gamma}_{12}=m+\tfrac{1}{2}+\frac{1}{2\pi i}\ln\frac{1+\exp(+2\pi i\bar{\gamma}_{32})}{f(\bar{\gamma}_{32})}\quad\text{for } l=l_m\ , \tag{4.2'}$$

where m is an integer. When t_3 lies far away from t_1 and t_2, the quantum number m must, according to Eq. (3.5) in Ref. [12], be a *non-negative* integer, and if t_3 approaches t_2, the value of m must for continuity reasons remain unchanged and thus be a *non-negative* integer also in (4.2'). Using (4.2), we obtain from (3.10a) and (3.10b) the approximate formulas

$$F_{11}(+\infty,+0)=\exp[-2\pi i(\bar{\gamma}_{12}-\tfrac{1}{2})]=\frac{f(\bar{\gamma}_{32})}{1+\exp(2\pi i\bar{\gamma}_{32})}\quad\text{for } l=l_m \tag{4.3a}$$

and

$$\frac{\partial F_{21}(+\infty,+0)}{\partial l}=-2\pi\frac{\partial\bar{\gamma}_{12}}{\partial l}+\frac{1}{1+\exp(-2\pi i\bar{\gamma}_{32})}\left[2\pi\frac{\partial\bar{\gamma}_{32}}{\partial l}-i\exp[+2\pi i(\bar{\gamma}_{12}-\bar{\gamma}_{32})]\frac{\partial f(\bar{\gamma}_{32})}{\partial l}\right]$$
$$=-2\pi\frac{\partial\bar{\gamma}_{12}}{\partial l}+\frac{2\pi\exp(2\pi i\bar{\gamma}_{32})}{1+\exp(2\pi i\bar{\gamma}_{32})}\frac{\partial\bar{\gamma}_{32}}{\partial l}+i\frac{\partial\ln f(\bar{\gamma}_{32})}{\partial l}\quad\text{for } l=l_m\ . \tag{4.3b}$$

Inserting (4.3a) and (4.3b) into (2.3), we obtain approximately

$$r_m=\left[\frac{\dfrac{f(\bar{\gamma}_{32})}{1+\exp(2\pi i\bar{\gamma}_{32})}\exp(2i\delta_l)}{2\pi i\dfrac{\partial\bar{\gamma}_{12}}{\partial l}+\dfrac{\partial\ln f(\bar{\gamma}_{32})}{\partial l}-\dfrac{2\pi i\exp(2\pi i\bar{\gamma}_{32})}{1+\exp(2\pi i\bar{\gamma}_{32})}\dfrac{\partial\bar{\gamma}_{32}}{\partial l}}\right]_{l=l_m} \tag{4.4}$$

$$=\left[\frac{f(\bar{\gamma}_{32})\exp(2i\delta_l)}{[1+\exp(2\pi i\bar{\gamma}_{32})]\left[\dfrac{\partial\ln f(\bar{\gamma}_{32})}{\partial l}+2\pi i\dfrac{\partial(\bar{\gamma}_{12}-\bar{\gamma}_{32})}{\partial l}\right]+2\pi i\dfrac{\partial\bar{\gamma}_{32}}{\partial l}}\right]_{l=l_m}, \tag{4.4'}$$

where it should be recalled that $f(\bar{\gamma}_{32})$ is a simplified notation, as was explained directly following (3.6). In fact, $f(\bar{\gamma}_{32})$ depends not only on $\bar{\gamma}_{32}$ itself, which is a sum of contributions from the successive orders of approximation, viz. $\bar{\gamma}_{32}=(\bar{\gamma}_{32})_0+\cdots+(\bar{\gamma}_{32})_{2N}$, but depends explicitly also on the contributions $(\bar{\gamma}_{32})_{2\nu}$ to $\bar{\gamma}_{32}$ from the various orders of approximation; see Eqs. (2.14), (2.13a), (2.13b), and (2.11a)–(2.11c) in Ref. [12]. As a consequence of this, it is only for the first-order approximation that the derivative $\partial\ln f/\partial l$ in (4.4) can be obtained from the simple formula

$$\frac{\partial\ln f(\bar{\gamma}_{32})}{\partial l}=\frac{\partial\ln f(\bar{\gamma}_{32})}{\partial\bar{\gamma}_{32}}\frac{\partial\bar{\gamma}_{32}}{\partial l}\ , \tag{4.5}$$

while, when a higher-order approximation is used, the calculation of the derivative in question is more complicated.

When t_3 moves away from t_2 in such a way that $\exp(2\pi i\bar{\gamma}_{32})\to 0$ and $f(\bar{\gamma}_{32})\to 1$, it is seen that, in the limit, (4.2') goes over into Eq. (3.5) in Ref. [12] with $\bar{\gamma}$ replaced by $\bar{\gamma}_{12}$, and (4.4) goes over into Eq. (3.6) in Ref. [12] with $f(\bar{\gamma}_0,\ldots,\bar{\gamma}_{2N})$ replaced by unity and $\bar{\gamma}$ replaced by $\bar{\gamma}_{12}$. Thus, as is to be expected, (4.2') and (4.4) transform into the two-turning-point formulas pertaining to the case that t_1 and t_2 are well separated and that all other transition points lie far away from the cluster consisting of t_1 and t_2.

V. COMPARISON WITH RESULTS OBTAINED BY PREVIOUS AUTHORS

Using semiclassical techniques, Delos and Carlson [7] obtained approximate results for the positions and residues of the Regge poles. The pole positions were accurate, but there was a discrepancy of about $\pi/2$ in the

phase of the residue, as was pointed out by Connor, Jakubetz, and Sukumar [8]. These authors also resolved the discrepancy and brought the semiclassical results into good agreement with accurate numerical results. For the S matrix Connor, Jakubetz, and Sukumar [8] obtained a formula [their Eq. (19a)] which in their notation reads

$$S(l)=\frac{\exp(i\alpha)A^{-}(\epsilon)+\exp(-i\alpha-\pi\epsilon)}{\exp(-i\alpha)A^{+}(\epsilon)+\exp(i\alpha-\pi\epsilon)}\exp[2i\delta(l)]$$
$$=\frac{\exp(\pi\epsilon)A^{-}(\epsilon)+\exp(-2i\alpha)}{1+\exp(\pi\epsilon-2i\alpha)A^{+}(\epsilon)}\exp[2i\delta(l)]\ . \quad (5.1)$$

Putting in the definition of $A^{\pm}(\epsilon)$ on p. 1788 in the paper by Connor, Jakubetz, and Sukumar [8]

$$\ln(-\epsilon)=\ln(i\epsilon)+i\pi/2\ , \quad (5.2)$$

and using the reflexion formula for the Γ function, we obtain

$$A^{+}(\epsilon)=\exp(+\pi\epsilon)\frac{1+\exp[2\pi i(i\epsilon)]}{f(i\epsilon)} \quad (5.3a)$$

and

$$A^{-}(\epsilon)=\exp(-\pi\epsilon)\ \ f(i\epsilon)\ , \quad (5.3b)$$

where

$$f(\bar{\gamma})=\frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2}+\bar{\gamma})}\exp[\bar{\gamma}(\ln\bar{\gamma}-1)] \quad (5.4)$$

is our function $f(\bar{\gamma})$ for the first-order approximation, as is easily seen from Eqs. (2.14), (2.13a), (2.13b), and (2.11a) in Ref. [12]. Inserting (5.3a) and (5.3b) into (5.1), we obtain

$$S(l)=\frac{f(i\epsilon)+\exp(-2i\alpha)}{1+\exp[-2i(\alpha+i\epsilon\pi)]\{1+\exp[2\pi i(i\epsilon)]\}/f(i\epsilon)}\ . \quad (5.5)$$

Putting in (5.5)

$$\alpha=\pi(\bar{\gamma}_{12}-\bar{\gamma}_{32})\ , \quad (5.6a)$$

$$i\epsilon=\bar{\gamma}_{32}\ , \quad (5.6b)$$

we obtain our formula (4.1) with S_l replaced by $S(l)$ and δ_l replaced by $\delta(l)$. It is thus seen that the formula for the S matrix given by Connor, Jakubetz, and Sukumar [8] agrees with the first-order version of our formula (4.1).

Brink and Takigawa [9] have also obtained a formula for the S matrix [their Eq. (2.10)] which in their notation reads

$$\eta=\frac{1+\bar{N}(i\epsilon)\exp(2iS_{32})}{N(i\epsilon)+\exp(2iS_{32})}\exp(2i\delta_l)$$
$$=\frac{\bar{N}(i\epsilon)+\exp(-2iS_{32})}{1+\exp(-2iS_{32})N(i\epsilon)}\exp(2i\delta_l)\ . \quad (5.7)$$

Using the reflexion formula for the Γ function, we can rewrite Eqs. (2.11a) and (2.11b) in Ref. [9] as follows:

$$N(z)=f(z)=\frac{1+\exp(-2\pi ize^{-i\pi})}{f(ze^{-i\pi})}=\exp(2\pi iz)\frac{1+\exp(+2\pi ize^{-i\pi})}{f(ze^{-i\pi})}\ , \quad (5.8a)$$

$$\bar{N}(z)=f(ze^{-i\pi})\ , \quad (5.8b)$$

where $f(z)$ is defined according to (5.4) with $\bar{\gamma}$ replaced by z. The function f in (5.8a) and (5.8b) is thus our function f for the first-order approximation. Using (5.8a) and (5.8b), we can rewrite (5.7) into the form

$$\eta=\frac{[f(i\epsilon e^{-i\pi})+\exp(-2iS_{32})]\exp(2i\delta_l)}{1+\exp[-2i(S_{32}-i\pi\epsilon)]\{1+\exp[+2\pi i(i\epsilon e^{-i\pi})]\}/f(i\epsilon e^{-i\pi})}\ . \quad (5.9)$$

Inserting into (5.9)

$$i\epsilon=\bar{\gamma}_{32}e^{+i\pi} \quad (5.10a)$$

and (since the transition zeros in Ref. [9] are related to those in the present paper such that r_1, r_2, and r_3 correspond to t_2, t_3, and t_1, respectively)

$$S_{32}=\pi\bar{\gamma}_{13}=\pi(\bar{\gamma}_{12}-\bar{\gamma}_{32})\ , \quad (5.10b)$$

we obtain our formula (4.1) with S_l replaced by η. It is thus seen that also the formula for the S matrix given by Brink and Takigawa [9] agrees with the first-order version of our formula (4.1).

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