Correlated wave functions and hyperfine splittings of the 2s state of muonic ^{3,4}He atoms

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(Received 27 December 1990; revised manuscript received 20 August 1991)

The correlated wave functions of the 2s state, $(1s)_e(2s)_\mu$ of muonic ^{4}He and ^{3}He atoms are calculated by the variational approach, employing an improved set of the Hylleraas-type basis functions to study the 2s state of muonic He atoms. The energy and radial expectation values are calculated. Our proposed wave functions for the muonic ^{4}He and ^{3}He atoms are good enough to describe the electron and muon at small, intermediate, and large values of radial coordinates. The hyperfine splittings of the $(1s)_e(2s)_\mu$ muonic ^{4}He and ^{3}He atoms are also presented. Including relativistic, QED, and other corrections up to $O(\alpha^2)$, we arrive at the values $\Delta\nu=(4287.01\pm0.10$ and 4052.64 ± 0.10 MHz for the total hyperfine splittings of the 2s state of muonic ^{4}He and ^{3}He atoms, respectively.

PACS number(s): 31.10.+z, 31.30.Gs, 31.30.Jv, 36.10.Dr

I. INTRODUCTION

The muonic helium atom was produced in a Larmor-precession experiment at the Space Radiation Effects Laboratory [1,2]. This simple system can provide a sensitive test for the three-body Schrödinger wave function, the electromagnetic interaction of the electron and muon, and also the precise direct determination of the magnetic moment mass of the μ^- as a test *CPT* invariance. The precise measurement of the ground-state hyperfine structure of the muonic ^{4}He has been made [3,4], and that of muonic ^{3}He atoms has also been made [5]. Various theoretical studies of these topics have been made, and they are consistent with the experimental results. However, there are few works [6–8] corresponding to the excited-state wave functions and hyperfine structure of these systems. Huang [6] has only calculated the wave functions for the excited states of the muonic ^{4}He atom, $(1s)_e(2p)_\mu$, $(1s)_e(3d)_\mu$, $(1s)_e(4f)_\mu$, and $(1s)_e(5g)_\mu$, which are automatical orthogonal to the ground state. Amusia, Kuchiev, and Yakhontov [7] and Drachman [8] have calculated the hyperfine splitting of the $(1s)_e(2s)_\mu$ state of muonic He atoms through a first-order perturbation. They used only simple hydrogenic wave functions to describe the excited-state muonic He atoms. However, even in the ground state of muonic He atoms, the hyperfine splitting is very sensitive to the e-μ^- correlation because of the singular character of the operators $\delta(\mathbf{r}_{e\mu})$ and $\delta(\mathbf{r}_e)$. Therefore, it is necessary to calculate the correlated wave functions for the 2s state of muonic He atoms.

In the present work, we will calculate the $(1s)_e(2s)_\mu$ correlated wave functions of muonic He atoms in Sec. II, in order to examine whether the wave functions, which are similar to what we proposed [9,10] for the ground state of muonic He atoms, are also good for studying the 2s state of muonic He atoms. In Sec. III we calculate the lowest-order hyperfine splittings of the 2s state of muonic He atoms using the correlated wave functions obtained in Sec. II. We then consider the relativistic, radiative, recoil, and nuclear finite-size corrections up to $O(\alpha^2)$ in Sec. IV. Last, we summarize the results in Sec. V.

II. WAVE FUNCTION

The nonrelativistic Hamiltonian of muonic He atoms is given in the atomic units by

$$H=-\frac{1}{2m_e}\nabla_e^2-\frac{1}{2m_\mu}\nabla_\mu^2-\frac{1}{2m_N}(\nabla_e+\nabla_\mu)^2+\frac{1}{r_{e_\mu}}-\frac{2}{r_e}-\frac{2}{r_\mu}\ , \tag{2.1}$$

where m_e, m_μ, and m_N are the masses of the electron, muon, and nucleus, respectively. r_e (or r_μ) is the distance between the electron (or muon) and nucleus.

It is well known that the muon is tightly bound to the ^{4}He (^{3}He) nucleus with mean radius about 400 times smaller than that of the electron cloud for the 1s state of muonic ^{4}He (^{3}He) atoms, because of the large ratio of the muon and electron masses, m_μ/m_e, which is about 200. The muon is also tightly bound to the ^{4}He (^{3}He) nucleus, but with a mean radius about 200 times smaller than that of the electron cloud for the 2s state of muonic (^{4}He (^{3}He) atoms. Therefore, the wave functions for the 2s state of muonic He atoms can also be separated into two parts as for the ground state of muonic He atoms [9–11]. The wave functions

$$\Psi(\mathbf{r}_e,\mathbf{r}_\mu)=\sum_{l_1,m_1,n_1=0}^{l_1+m_1+n_1\le\omega_1} C_{l_1m_1n_1}\Phi_{l_1m_1n_1}(\mathbf{r}_e,\mathbf{r}_\mu)+\sum_{l_2,m_2,n_2=0}^{l_2+m_2+n_2\le\omega_2} C_{l_2m_2n_2}\tilde{\Phi}_{l_2m_2n_2}(\mathbf{r}_e,\mathbf{r}_\mu)\ , \tag{2.2}$$

where the summations are taken over non-negative integers of l_1,m_1,n_1 and l_2,m_2,n_2 with given selected bounds of ω_1 and ω_2. The basis functions

$$\Phi_{l_1 m_1 n_1}(\mathbf{r}_e,\mathbf{r}_\mu)=\frac{1}{4\pi}\exp\left[-\frac{A}{2}r_e-\frac{B}{2}r_\mu\right]r_{e\mu}^{l_1}r_e^{m_1}r_\mu^{n_1}\ , \tag{2.3}$$

$$\tilde{\Phi}_{l_2 m_2 n_2}(\mathbf{r}_e,\mathbf{r}_\mu)=\frac{1}{4\pi}\exp\left[-\frac{\alpha}{2}r_e-\frac{\beta}{2}r_\mu\right]r_{e\mu}^{l_2}r_e^{m_2}r_\mu^{n_2} \tag{2.4}$$

are Hylleraas-type basis functions. The constants A and B are given by

$$A=2m_e\frac{m_N}{m_e+m_N} \tag{2.5}$$

and

$$B=2m_\mu\frac{m_N}{m_\mu+m_N}\ . \tag{2.6}$$

The basis functions $\Phi_{l_1 m_1 n_1}(\mathbf{r}_e,\mathbf{r}_\mu)$ essentially account for the hydrogenlike electron cloud around the pseudonucleus $(\mu\text{-He})^+$. We except that only a few terms of the basis functions in Eq. (2.3) are needed to give the main contribution to the energy. The two groups of the basis functions used in constructing the wave functions in Eq. (2.2) are similar to that successfully employed by Kono and Hattori [12] and Drake [13] for the Rydberg states of helium.

The constants α, β, C_{l_1,m_1,n_1}, and $\tilde{C}_{l_2 m_2 n_2}$ are determined by the variational method. $\Phi_{l_2 m_2 n_2}$ will modify the lowest-order hyperfine splitting $(\Delta\nu)_F$, which is calculated from the basis functions Φ_{l_1,m_1,n_1}, because $\tilde{\Phi}_{l_2,m_2,n_2}$ describes most of the electron-muon correlation. Therefore, the proposal of the wave functions in Eq. (2.2) is reasonably expected to be good, not only for calculating $\Delta\nu_F$, but also for studying the $2s$ state of muonic He atoms.

We add a few terms to $\Psi(\mathbf{r}_e,\mathbf{r}_\mu)$, which are given as follows:

$$\sum_{l_0,m_0,n_0=0}^{l_0+m_0+n_0\le\omega_0}\bar{\Phi}_{l_0,m_0,n_0}(\mathbf{r}_e,\mathbf{r}_\mu)\ , \tag{2.7}$$

with

$$\bar{\Phi}_{l_0 m_0 n_0}(\mathbf{r}_e,\mathbf{r}_\mu)=\frac{1}{4\pi}\exp\left[-\frac{a_0}{2}r_e-\frac{b_0}{2}r_\mu\right]r_{e\mu}^{l_0}r_e^{m_0}r_\mu^{n_0}\ , \tag{2.8}$$

where l_0,m_0,n_0 are non-negative integers with a selected bound ω_0. The constants a_0 and b_0 are given by

$$a_0=2m_e\frac{m_N}{m_e+m_N} \tag{2.9}$$

and

$$b_0=4m_\mu\frac{m_N}{m_\mu+m_N}\ . \tag{2.10}$$

The coefficients $C^0_{l_0 m_0 n_0}$ are determined by the variational method. These terms in Eq. (2.7) will help to get the converged hyperfine splittings quickly. Because the ground state of muonic He atoms can be mostly described by a few terms of $\bar{\Phi}_{l_0 m_0 n_0}$, we thus have better ground-state wave functions if we include $\bar{\Phi}_{l_0 m_0 n_0}$ in the trial functions in Eq. (2.2). We therefore expect the excited wave functions, which are orthogonal to the improved ground-state wave functions, are better than those which are calculated only from Eq. (2.2). It is obvious that we need more terms to construct good wave functions purely using the trial functions in Eq. (2.2).

We show the energy of the $2s$ state of muonic ^{4}He and ^{3}He atoms, which converged to within an accuracy of $\approx 10^{-9}$ a.u., in Table I and II, respectively. It is found that β/B, which is defined in Eqs. (2.3) and (2.4), is approximately equal to 1, the same as that found for the ground state of muonic He atoms [9,10]. In other words, the muon can be essentially described in terms of $2s$-state hydrogenic wave functions. The terms $\bar{\Phi}_{l_0 m_0 n_0}$ were add-

TABLE I. Energy of the $2s$ state of muonic ^{4}He for the correlated wave functions with $l_0+m_0+n_0\le\omega_0$, $l_1+m_1+n_1\le\omega_1$, $l_2+m_2+n_2\le\omega_2$, and $m_0,n_0,m_1,n_1,m_2,n_2\le 3$. The format $N[M]$ means $\omega=N$; number of terms equals M.

ω_0	ω_1	ω_2	Energy (a.u.)	α/A	β/B
1[4]	3[20]	3[20]	−101.035 490 242	6.340	1.092
1	3	4[33]	−101.035 490 339	5.658	1.063
1	3	5[44]	−101.035 490 356	4.777	1.142
1	3	6[64]	−101.035 490 362	4.130	1.135
1	3	7[80]	−101.035 490 367	3.627	1.137
1	3	8[96]	−101.035 490 368	3.242	1.113
1	3	9[112]	−101.035 490 370	2.937	1.132
3	3	3	−101.035 490 262	6.395	1.099
3	3	5	−101.035 490 358	4.762	1.157
3	3	6	−101.035 490 364	4.111	1.157
3	3	7	−101.035 490 368	3.625	1.157
3	3	8	−101.035 490 370	3.233	1.154
3	3	9	−101.035 490 371	2.925	1.143

TABLE II. Energy of the 2s state of muonic ^{3}He for the correlated wave functions with $l_0+m_0+n_0 \le \omega_0$, $l_1+m_1+n_1 \le \omega_1$, $l_2+m_2+n_2 \le \omega_2$, and $m_0,n_0,m_1,m_2,n_2 \le 3$.

ω_0	ω_1	ω_2	Energy (a.u.)	α/A	β/B
1	3	3	−100.137 298 734	6.318	1.092
1	3	4	−100.137 298 835	5.637	1.063
1	3	5	−100.137 298 852	4.760	1.142
1	3	6	−100.137 298 858	4.116	1.135
1	3	7	−100.137 298 863	3.615	1.137
1	3	8	−100.137 298 864	3.231	1.113
1	3	9	−100.137 289 866	2.927	1.132
3	3	3	−100.137 298 755	6.372	1.099
3	3	5	−100.137 298 854	4.746	1.157
3	3	6	−100.137 298 860	4.097	1.157
3	3	7	−100.137 298 864	3.613	1.157
3	3	8	−100.137 298 866	3.223	1.154
3	3	9	−100.137 298 868	2.915	1.143

TABLE III. Average value of $1/r_e$, $1/r_\mu$, and $1/r_{e\mu}$ of the 2s state of muonic ^{4}He atoms in atomic units with the bound $l_0+m_0+n_0 \le \omega_0$, $l_1+m_1+n_1 \le \omega_1$, $l_2+m_2+n_2 \le \omega_2$, and $m_0,n_0,m_1,n_1,m_2,n_2 \le 3$. The format $R[n]$ means $R \times 10^n$.

ω_0	ω_1	ω_2	$1/r_e$	$1/r_\mu$	$1/r_{e\mu}$
1	3	3	1.000 618 21	1.005 349 750 3[2]	1.000 205 792
1	3	5	1.000 618 51	1.005 349 747 9[2]	1.000 205 868
1	3	6	1.000 618 53	1.005 349 747 7[2]	1.000 205 875
1	3	7	1.000 618 54	1.005 349 747 7[2]	1.000 205 878
1	3	8	1.000 618 55	1.005 349 747 6[2]	1.000 205 880
1	3	9	1.000 618 56	1.005 349 747 5[2]	1.000 205 882
3	3	3	1.000 618 26	1.005 349 749 4[2]	1.000 205 799
3	3	5	1.000 618 52	1.005 349 747 9[2]	1.000 205 870
3	3	6	1.000 618 54	1.005 349 747 7[2]	1.000 205 876
3	3	7	1.000 618 55	1.005 349 747 7[2]	1.000 205 879
3	3	8	1.000 618 57	1.005 349 747 6[2]	1.000 205 882
3	3	9	1.000 618 56	1.005 349 747 5[2]	1.000 205 883

TABLE IV. Average value or r_e, r_μ, and $r_{e\mu}$ of the 2s state of muonic ^{4}He atoms in atomic units with the bound $l_0+m_0+n_0 \le \omega_0$, $l_1+m_1+n_1 \le \omega_1$, $l_2+m_2+n_2 \le \omega_2$, and $m_0,n_0,m_1,n_1,m_2,n_2 \le 3$. The format $R[n]$ means $R \times 10^n$.

ω_0	ω_1	ω_2	r_e	r_μ	$r_{e\mu}$
1	3	3	1.499 387 786	1.492 003 504 4[−2]	1.499 585 33
1	3	5	1.499 387 234	1.492 003 512 2[−2]	1.499 584 78
1	3	6	1.499 387 186	1.492 003 513 0[−2]	1.499 584 73
1	3	7	1.499 387 182	1.492 003 513 0[−2]	1.499 584 73
1	3	8	1.499 387 175	1.492 003 513 2[−2]	1.499 584 72
1	3	9	1.499 387 165	1.492 003 513 6[−2]	1.499 584 71
3	3	3	1.499 387 632	1.492 003 505 6[−2]	1.499 584 18
3	3	5	1.499 387 222	1.492 003 511 9[−2]	1.499 584 77
3	3	6	1.499 387 185	1.492 003 512 8[−2]	1.499 584 73
3	3	7	1.499 387 180	1.492 003 512 9[−2]	1.499 584 73
3	3	8	1.499 387 169	1.492 003 513 2[−2]	1.499 584 71
3	3	9	1.499 387 164	1.492 003 513 4[−2]	1.499 584 71

TABLE V. Average value of r_e^2, r_μ^2, and $r_{e\mu}^2$ of the 2s state of muonic ^{4}He atoms in atomic units with the bound $l_0+m_0+n_0 \le \omega_0$, $l_1+m_1+n_1 \le \omega_1$, $l_2+m_2+n_2 \le \omega_2$, and $m_0,n_0,m_1,n_1,m_2,n_2 \le 3$. The format $R[n]$ means $R \times 10^n$.

ω_0	ω_1	ω_2	r_e^2	r_μ^2	$r_{e\mu}^2$
1	3	3	2.997 777	2.597 094 87[−4]	2.998 367
1	3	5	2.997 778	2.597 094 90[−4]	2.998 368
1	3	6	2.997 784	2.597 094 92[−4]	2.998 374
1	3	7	2.997 784	2.597 094 90[−4]	2.998 374
1	3	8	2.997 781	2.597 094 96[−4]	2.998 371
1	3	9	2.997 784	2.597 094 91[−4]	2.998 374
3	3	3	2.997 777	2.597 094 88[−4]	2.998 367
3	3	5	2.997 776	2.597 094 91[−4]	2.998 366
3	3	6	2.997 786	2.597 094 91[−4]	2.998 376
3	3	7	2.997 781	2.597 094 92[−4]	2.998 371
3	3	8	2.997 782	2.597 094 92[−4]	2.998 372
3	3	9	2.997 783	2.597 094 92[−4]	2.998 373

TABLE VI. Average value of $1/r_e$, $1/r_\mu$, and $1/r_{e\mu}$ of the 2s state of muonic ^{3}He atoms in atomic units with the bound $l_0+m_0+n_0 \le \omega_0$, $l_1+m_1+n_1 \le \omega_1$, $l_2+m_2+m_2+n_2 \le \omega_2$, and $m_0,n_0,m_1,n_1,m_2,n_2 \le 3$. The format $R[n]$ means $R \times 10^n$.

ω_0	ω_1	ω_2	$1/r_e$	$1/r_\mu$	$1/r_{e\mu}$
1	3	3	1.000 587 25	9.963 679 570[1]	1.000 168 23
1	3	5	1.000 587 56	9.963 679 546[1]	1.000 168 31
1	3	6	1.000 587 58	9.963 679 544[1]	1.000 168 31
1	3	7	1.000 587 59	9.963 679 542[1]	1.000 168 32
1	3	8	1.000 587 60	9.963 679 543[1]	1.000 168 32
1	3	9	1.000 587 61	9.963 679 542[1]	1.000 168 32
3	3	3	1.000 587 30	9.963 679 561[1]	1.000 168 23
3	3	5	1.000 587 57	9.963 679 545[1]	1.000 168 31
3	3	6	1.000 587 59	9.963 679 543[1]	1.000 168 32
3	3	7	1.000 587 60	9.963 679 543[1]	1.000 168 32
3	3	8	1.000 587 61	9.963 679 542[1]	1.000 168 32
3	3	9	1.000 587 61	9.963 679 542[1]	1.000 168 32

TABLE VII. Average value of r_e, r_μ and $r_{e\mu}$ of the 2s state of muonic ^{3}He atoms in atomic units with the bound $l_0+m_0+n_0 \le \omega_0$, $l_1+m_1+n_1 \le \omega_1$, $l_2+m_2+n_2 \le \omega_2$, and $m_0,n_0m_1,n_1,m_2,n_2 \le 3$. The format $R[n]$ means $R \times 10^n$.

ω_0	ω_1	ω_2	r_e	r_μ	$r_{e\mu}$
1	3	3	1.499 438 8	1.505 452 705 5[−2]	1.499 639 42
1	3	5	1.499 438 2	1.505 452 713 7[−2]	1.499 638 79
1	3	6	1.499 438 1	1.505 452 714 5[−2]	1.499 638 74
1	3	7	1.499 438 1	1.505 452 714 4[−2]	1.499 638 73
1	3	8	1.499 438 1	1.505 452 714 7[−2]	1.499 638 73
1	3	9	1.499 438 1	1.505 452 715 0[−2]	1.499 638 71
3	3	3	1.499 438 6	1.505 452 706 7[−2]	1.499 639 24
3	3	5	1.499 438 2	1.505 452 713 3[−2]	1.499 638 77
3	3	6	1.499 438 1	1.505 452 714 3[−2]	1.499 638 73
3	3	7	1.499 438 1	1.505 452 714 3[−2]	1.499 638 72
3	3	8	1.499 438 1	1.505 452 714 7[−2]	1.499 638 71
3	3	9	1.499 438 1	1.505 452 714 9[−2]	1.499 638 71

ed with two selected bounds of ω_0, 1 or 3. The wave functions with $\omega_0=3$ give a lower expectation value of energy than the wave functions with $\omega_0=1$.

The radial expectation values of $\langle 1/r_e \rangle$, $\langle 1/r_\mu \rangle$, $\langle 1/r_{e\mu} \rangle, \langle r_e \rangle, \langle r_\mu \rangle, \langle r_{e\mu} \rangle$ and $\langle r_e^2 \rangle, \langle r_\mu^2 \rangle, \langle r_{e\mu}^2 \rangle$ are given in Tables III, IV, and V and Tables VI, VII, and VIII, respectively, for the muonic ^{4}He and muonic ^{3}He atoms. As shown in Tables V and VIII, the wave functions with $\omega_0=3$ make the series $\langle r_e^2 \rangle$ and $\langle r_\mu^2 \rangle$ converge more quickly that the wave functions with $\omega_0=1$.

These radial expectation values give some indication of the accuracy of the wave functions at small, intermediate, and large values of radial coordinates. Comparing with our previous results [9,10] of the 1s state of muonic He atoms, the present series $\langle r_e \rangle, \langle r_\mu \rangle, \langle r_{e\mu} \rangle$ converge at the same rate as the previous ones. We also find the convergences of the series $1/r_\mu$, r_μ, and r_μ^2 are almost indistinguishable. However, the convergences of the series $1/r_e$, r_e, and $1/r_e^2$ are different; the series $1/r_e$ converges the most quickly, but the series r_e^2 converges the most slowly. In other words, the wave functions in Eqs. (2.2) and (2.7) describe the muon well at small, intermediate, and large values of the radial coordinates. However, it becomes worse for our proposed wave functions to describe the electron as the electron becomes farther away from the nucleus. This is due to the fact that the muon is almost confined to a small region closer to the nucleus, but the electron is spread over a relatively large region.

We also find in Tables I and II that α/A approaches almost 3. α represents the size of the electron correlated wave functions which describe the penetration of the electron into the pseudonucleus $(\mu\text{-He})^+$ and the extension of the electron. It is also found in Tables IV and VII that the expectation values $\langle r_e \rangle$ of the 2s state of muonic He atoms are smaller than those of the ground state of muonic He atoms [9,10]. It represents that the electron penetrates into the 2s μ^- cloud more than the 1s μ^- cloud, and the e^--μ^- correlation, which is described by $\Phi_{l_2 m_2 n_2}$ in Eq. (2.2), extends to a region farther from nucleus than those of the ground state of He atoms [9,10], because the present α/A is smaller than the previous ones. However, we will show in Sec. III that the main contributions of the hyperfine splittings come from the region, which is half an order larger than the μ^- cloud, but 40 times smaller than the e^- cloud. Therefore, our proposed wave functions can still give a good converged value of the hyperfine splitting.

III. LOWEST-ORDER HYPERFINE SPLITTING

By limiting $m_1, n_1, m_2, n_2 \leq 3$ and $\omega_1=3$, we have calculated $\Delta\nu_F$ for muonic ^{4}He and ^{3}He atoms, which are presented in Table IX. We expected that the series $\Delta\nu_F$ for the 2s state of muonic He atoms converges less quickly than those for the 1s state of muonic He atoms, because the muon and electron are more closely correlated in the 2s state of muonic He atoms. However, it is surprising that the present series converges as quickly as the previous series [9,10] for the 1s state of muonic He atoms, if the number of terms $\tilde{\Phi}_{l_0 m_0 n_0}$ is left out of account.

The present converged values (all in MHz) are

$$(\Delta\nu)_F = 4276.69 \pm 0.10 \tag{3.1}$$

and

$$(\Delta\nu)_F^{e\text{-}\mu} = 3205.63 \pm 0.08 , \tag{3.2}$$

$$(\Delta\nu)_F^{e} = 838.89 \pm 0.02 , \tag{3.3}$$

$$(\Delta\nu)_F = 4044.02 \pm 0.10 , \tag{3.4}$$

for the 2s state of muonic ^{4}He and ^{3}He atoms, respectively. We find in Table IX that the wave functions with $\omega_0=3$ and 4 help the series converge more quickly than those with $\omega_0=1$.

We also give the probability of finding the electron and muon at the same position in Tables X and XI. The major contributions to $(\Delta\nu)_F, (\Delta\nu)_F^{e\text{-}\mu}$ come from the region of the muon cloud, which extends to almost half an order of magnitude larger than the muon cloud. The size of the region, which gives major contributions for the 2s state of muonic He atoms, is twice as large as that of the 1s state

TABLE VIII. Average value of r_e^2, r_μ^2, and $r_{e\mu}^2$ of the 2s state of muonic ^{3}He atoms in atomic units with the bound $l_0+m_0+n_0 \leq \omega_0$, $l_1+m_1+n_1 \leq \omega_1$, $l_2+m_2+n_2 \leq \omega_2$, and $m_0, n_0, m_1, n_1, m_2, n_2 \leq 3$. The format $R[n]$ means $R \times 10^n$.

ω_0	ω_1	ω_2	r_e^2	r_μ^2	$r_{e\mu}^2$
1	3	3	2.997 985	2.644 127 57[−4]	2.998 584
1	3	5	2.997 985	2.644 127 61[−4]	2.998 584
1	3	6	2.997 991	2.644 127 63[−4]	2.998 590
1	3	7	2.997 991	2.644 127 60[−4]	2.998 590
1	3	8	2.997 988	2.644 127 67[−4]	2.998 587
1	3	9	2.997 991	2.644 127 61[−4]	2.998 590
3	3	3	2.997 984	2.644 127 58[−4]	2.998 583
3	3	5	2.997 983	2.644 127 62[−4]	2.998 582
3	3	6	2.997 992	2.644 127 63[−4]	2.998 592
3	3	7	2.997 988	2.644 127 62[−4]	2.998 587
3	3	8	2.997 989	2.644 127 62[−4]	2.998 588
3	3	9	2.997 990	2.644 127 62[−4]	2.998 589

TABLE IX. Lowest-order hyperfine splittings of the 2s state of muonic ^{4}He and ^{3}He atoms for the correlated wave functions with $l_0+m_0+n_0 \le \omega_0$, $l_1+m_1+n_1 \le \omega_1$, $l_2+m_2+n_2 \le \omega_2$, and $m_0, n_0, m_1, n_1, m_2, n_2 \le 3$.

			^{4}Heμ^-e^-	^{3}Heμ^-e^-		
ω_0	ω_1	ω_2	$\Delta\nu_F$ (MHz)	$\Delta\nu_F^{e\text{-}\mu}$ (MHz)	$\Delta\nu_F^{e}$ (MHz)	$\Delta\nu_F$ (MHz)
1	3	3	4280.271	3208.340	837.581	4045.921
1	3	5	4277.515	3206.214	838.156	4044.384
1	3	7	4277.018	3205.867	838.294	4044.161
1	3	8	4276.944	3205.812	838.329	4044.141
1	3	9	4276.834	3205.728	838.342	4044.070
3	3	3	4279.931	3208.084	837.657	4045.741
3	3	5	4277.509	3206.241	838.218	4044.459
3	3	6	4277.187	3205.995	838.297	4044.292
3	3	7	4277.077	3205.913	838.324	4044.237
3	3	8	4276.875	3205.758	838.362	4044.120
3	3	9	4276.791	3205.695	838.369	4044.064
4	3	8	4276.773	3205.682	838.390	4044.072
4	3	9	4276.723	3205.642	838.388	4044.030
4	3	10	4276.701	3205.627	838.387	4044.014
4	3	11	4276.684			

of muonic He atoms [9,10]. However, from Tables X and XI, it is shown that the present series converges almost as quickly as those for the 1s-state case in the region, which give the major contributions. Therefore, it is not accidental for the hyperfine splittings of the 2s state of muonic He atoms to converge similarly as those of the 1s state of muonic He atoms.

Amusia, Kuchiev, and Yakhontov [7] have calculated the hyperfine splitting for the 2s state of ^{4}He atoms including the second-order perturbed terms to arrive at the value $\Delta V = 4260.4$ MHz ($\Delta\nu_F \approx 4250$ MHz). Drachman [8] employed the global-operator technique to show that the hyperfine splittings in the muonic 2s states are found to be 4257.2 MHz for ^{4}He and 4027.6 MHz for ^{3}He ($\Delta\nu_F \approx 4247$ MHz for ^{4}He and 4018 MHz for ^{3}He). We also calculated $\Delta\nu_F$ by the global-operator method including only $\overline{\Phi}_{l_0 m_0 n_0}$ and $\Phi_{l_1 m_1 n_1}$, which are shown in Eqs. (2.7) and (2.3), respectively. In Table XII we find that the lowest-order hyperfine splittings come closer to the converged values in Eqs. (3.1)–(3.4) as the number of terms increases, although the trial functions we employed in the global approach are poor for calculating the expectation values of the local operators, $\langle \delta(\mathbf{r}_{e\mu}) \rangle$ and $\langle \delta(\mathbf{r}_e) \rangle$. In Table XII we also find that our results deviated from the results of Drachman [8]. It is worth mentioning that we give the results in the first column of Table XII by using trial functions approximately equal to what Drachman [8] used. The deviations may come from the mass-polarization operator, which we employed both

TABLE X. Probability density of simultaneous finding the electron and muon of the 2s state of muonic ^{4}He atoms at the same position as a function of distance (in atomic units) from the nucleus. The format $R[n]$ means $R \times 10^n$.

			2[−3]	4[−3]	6[−3]	8[−3]
ω_0	ω_1	ω_2	1[−3]	2[−2]	4[−2]	6[−2]
3	3	7	8.362 13[1]	1.613 99[1]	1.721 02[1]	1.196 16[2]
			2.294 27[2]	1.416 16[2]	9.165 31[−1]	1.512 14[−3]
3	3	8	8.359 36[1]	1.612 62[1]	1.722 11[1]	1.196 31[2]
			2.294 30[2]	1.416 07[2]	9.165 98[−1]	1.514 74[−3]
3	3	9	8.358 70[1]	1.612 30[1]	1.722 34[1]	1.196 33[2]
			2.294 28[2]	1.416 03[2]	9.166 63[−1]	1.509 61[−3]
4	3	8	8.358 48[1]	1.611 97[1]	1.722 73[1]	1.196 41[2]
			2.294 32[2]	1.416 03[2]	9.164 93[−1]	1.515 16[−3]
4	3	9	8.356 76[1]	1.611 35[1]	1.723 09[1]	1.196 44[2]
			2.294 33[2]	1.416 02[2]	9.166 42[−1]	1.512 31[−3]
4	3	10	8.359 53[1]	1.612 61[1]	1.722 23[1]	1.196 32[2]
			2.294 26[2]	1.415 97[2]	9.169 26[−1]	1.515 32[−3]
4	3	11	8.360 70[1]	1.612 96[1]	1.722 03[1]	1.196 28[2]
			2.294 17[2]	1.415 99[2]	9.171 59[−1]	1.512 33[−3]

TABLE XI. Probability density of simultaneous finding the electron and muon of the 2s state of muonic ^{3}He atoms at the same position as a function of distance (in atomic units) from the nucleus. The format $R[n]$ means $R \times 10^n$.

ω_0	ω_1	ω_2	2[−3] 1[−2]	4[−3] 2[−2]	6[−3] 4[−2]	8[−3] 6[−2]
3	3	7	8.297 46[1]	1.712 18[1]	1.535 43[1]	1.142 05[2]
			2.233 07[2]	1.446 14[2]	1.009 18	1.791 20[−3]
3	3	8	8.294 76[1]	1.710 78[1]	1.536 47[1]	1.142 20[2]
			2.233 10[2]	1.446 05[2]	1.009 25	1.794 20[−3]
3	3	9	8.293 96[1]	1.710 39[1]	1.536 73[1]	1.142 23[2]
			2.233 09[2]	1.446 01[2]	1.009 33	1.788 30[−3]
4	3	8	8.293 79[1]	1.710 07[1]	1.537 09[1]	1.142 31[2]
			2.233 13[2]	1.446 00[2]	1.009 15	1.794 58[−3]
4	3	9	8.292 13[1]	1.709 45[1]	1.537 43[1]	1.142 34[2]
			2.233 14[2]	1.445 99[2]	1.009 31	1.791 41[−3]
4	3	10	8.295 04[1]	1.710 82[1]	1.536 55[1]	1.142 21[2]
			2.233 06[2]	1.445 95[2]	1.009 63	1.794 84[−3]

in calculating the wave functions and replacing the local operators with global operators.

We recommend the calculation of the 2s-state hyperfine splittings through the global-operator approach as we have done [11] for the ground state of muonic He atoms. Comparison of the results of the two methods, the local- and global-operator methods, will check the convergence of the lowest-order hyperfine splittings. Of course, experimental results will help to clarify this point. We hope that our work will stimulate further experimental work.

IV. RELATIVISTIC AND RADIATIVE CORRECTIONS

Amusia, Kuchiev, and Yakhontov [7] and Drachman [8] have calculated lowest-order hyperfine splittings of the 2s state of muonic He atoms, but they gave a rough estimate of the relativistic and radiative corrections. We will calculate these corrections up to α^2 by the first-order perturbation method. The unperturbed wave functions can be approximated to be the product of 1s hydrogenic wave functions with effective nucleus charge $Z_e=1$ and 2s hydrogenic wave functions with $Z_\mu=2$.

TABLE XII. Lowest-order hyperfine splittings of the 2s state of muonic ^{4}He and ^{3}He atoms by the global-operator method for the correlated wave functions with $l_0+m_0+n_0 \le 0$, $l_1+m_1+n_1 \le \omega_1$, and $m_1, n_1 \le 3$.[a]

ω_1	^{4}Heμ^-e^- $\Delta\nu_F$ (MHz)	$\Delta\nu_F^{e\text{-}\mu}$ (MHz)	^{3}Heμ^-e^- $\Delta\nu_F^e$ (MHz)	$\Delta\nu_F$ (MHz)
1[a]	4264.31	3196.33	835.55	4031.88
1	4264.66	3196.59	837.03	4033.62
2	4269.72	3200.32	838.09	4038.40
3	4270.10	3201.27	838.25	4039.51

[a] $m_1=0$, $1_l=0$.

A. Relativistic correction

We treat the Breit interactions as a small perturbation, which is given by

$$\delta H=\frac{\alpha}{2r_{e\mu}}\left[\boldsymbol{\alpha}_e\cdot\boldsymbol{\alpha}_\mu+(\boldsymbol{\alpha}_e\cdot\mathbf{r}_{e\mu})\frac{(\boldsymbol{\alpha}_\mu\cdot\mathbf{r}_{e\mu})}{r_{e\mu}^2}\right] . \tag{4.1}$$

Thus the relativistic hyperfine splitting of the 2s state of muonic He atoms can be written as

$$\Delta\nu=\left\langle\begin{matrix}J & M\\ 0 & 0\end{matrix}\middle|\delta H\middle|\begin{matrix}J & M\\ 0 & 0\end{matrix}\right\rangle-\left\langle\begin{matrix}J & M\\ 1 & 1\end{matrix}\middle|\delta H\middle|\begin{matrix}J & M\\ 1 & 1\end{matrix}\right\rangle . \tag{4.2}$$

In the $(1s)_e(2s)_\mu$ state of muonic He atoms, the total angular momenta j_e and j_μ of the electron and muon, respectively, are coupled to singlet and triplet states of the total angular momentum J.

The 1s hydrogenic wave function with $Z_e=1$ for the electron and the 2s hydrogenic wave function with $Z_\mu=2$ for the muon can be written, respectively, as

$$\Psi_e=\begin{bmatrix} g_e(r) & \chi_\kappa^m \\ if_e(r) & \chi_{-\kappa}^m\end{bmatrix} \tag{4.3}$$

and

$$\Psi_\mu=\begin{bmatrix} g_\mu(r) & \chi_\kappa^m \\ if_\mu(r) & \chi_{-\kappa}^m\end{bmatrix} . \tag{4.4}$$

The angular functions χ_κ^m in Eqs. (4.3) and (4.4) are normalized spherical spinors defined as

$$\chi_\kappa^m=\sum_{M,\mu}\langle lM\tfrac{1}{2}\nu|jm\rangle Y_{lM}\chi_\mu , \tag{4.5}$$

where Y_{lM} are spherical harmonics, $l=0$ and $M=0$ in the present case, and χ_μ are spin eigenfunctions with $s=\frac{1}{2}$ and $s_Z=\mu$. We will evaluate the expectation values of δH in Eq. (4.2) in terms of radial integrals [14–16] and thus give $\Delta\nu$ as

$$\Delta\nu=\tfrac{32}{9}\int_0^\infty\int_0^\infty dr_e dr_\mu r_e^2 r_\mu^2 g_e f_e g_\mu f_\mu \frac{r_<}{r_>^2} . \tag{4.6}$$

After calculating the above radial integrals and keeping the terms up to $O(\alpha^2)$, we have

$$\Delta\nu=\delta^{\rm rel}\Delta\nu_F+\Delta\nu_F$$
$$=\Delta\nu_F\left[1+\frac{3}{2}Z_e^2\alpha^2-\frac{41}{96}Z_\mu^2\alpha^2+O\left[\alpha^2\frac{m_e}{m_\mu}\right]\right] . \quad (4.7)$$

B. Vacuum polarization

As Borie [17] showed in the ground state of muonic ^{4}He atoms, the vacuum polarization correction for the $2s$ state of muonic He atoms has two contributions.

The first correction is induced because the magnetic potential is corrected [18–20]. Its contribution to the hyperfine splitting is

$$\delta^1_{VP}\Delta\nu_F=-\frac{8\pi}{3}\frac{\alpha}{m_e m_\mu}\int d^3r_e\int d^3r_\mu|\Phi_{e,1s}|^2|\Phi_{\mu,2s}|^2 \times\frac{1}{(2\pi)^3}\int e^{i\mathbf{q}\cdot\mathbf{r}_{e\mu}}\Pi(q^2) , \quad (4.8)$$

with

$$\Pi(q^2)=\frac{\alpha}{\pi}\frac{q^2}{2m_e^2}\int_0^1\frac{2v^2(1-v^2/3)}{4+q^2/m_e^2(1-v^2)} , \quad (4.9)$$

and $\Phi_{e,ls}\Phi_{\mu,2s}$ are the $(1s)_e(2s)_\mu$-state uncorrelated wave functions for muonic He atoms, which are given as

$$\Phi_{e,ls}=\left[\frac{a^3}{8\pi}\right]^{1/2}e^{(-a/2)r_e} , \quad (4.10)$$

$$\Phi_{\mu,2s}=\left[\frac{b^3}{8\pi}\right]^{1/2}\left[1-\frac{b}{2}r_\mu\right]e^{(-b/2)r_\mu} , \quad (4.11)$$

with

$$a=2m_e\alpha , \quad (4.12)$$

$$b=2m_\mu\alpha , \quad (4.13)$$

if the spin-wave functions are neglected.

In calculating $\delta^1_{VP}\Delta\nu_F$, we integrated directly instead of numerically [17]. Using the property of $\delta'(x)$ and some simple integrations we can express Eq. (4.7) as

$$\delta^1_{VP}\Delta\nu_F=-\frac{1}{8\pi^2}\alpha\Delta\nu_F G\left[1+\frac{3a}{a+b}\right] \quad (4.14)$$

and

$$G=-2\pi\int_0^1 dv\frac{v^2(1-v^2/3)}{(1-v^2)^{1/2}}\left[\frac{1}{1+\alpha(1-v^2)^{1/2}}\left[-8\alpha-27\alpha\frac{a}{a+b}\right]+\frac{22\alpha^2(1-v^2)^{1/2}}{[1+\alpha(1-v^2)^{1/2}]^2}\right.$$
$$+\frac{1}{1+\tilde{B}(1-v^2)^{1/2}}\left[8\alpha+27\alpha\frac{a}{a+b}\right]+\frac{(1-v^2)^{1/2}}{[1+\tilde{B}(1-v^2)^{1/2}]^2}\left[6\alpha\tilde{B}+9\alpha^2+27\alpha\tilde{B}\frac{a}{a+b}\right]$$
$$\left.+\frac{2(1-v^2)}{[1+\tilde{B}(1-v^2)^{1/2}]^3}\left[2\alpha\tilde{B}^2+9\alpha\tilde{B}^2\frac{a}{a+b}-5\alpha^2\tilde{B}\right]+\frac{6(1-v^2)^{3/2}}{[1+\tilde{B}(1-v^2)^{1/2}]^4}\alpha\tilde{B}^3\right]+\cdots , \quad (4.15)$$

where $\tilde{B}=m_\mu\alpha/m_e$, α is the fine-structure constant, and the ellipsis represents higher-order terms.

Keeping the terms up to α^2, Eq. (4.14) can be rewritten as

$$\delta^1_{VP}\Delta\nu_F=\frac{\Delta\nu_F}{16}\alpha^2\left[-6+\frac{1}{\pi}\left[32\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{m_e}{b+s}+48\tilde{B}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{m_e^2}{(b+s)^2}\right.\right.$$
$$\left.\left.+64\tilde{B}^2\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{m_e^3}{(b+s)^3}+192\tilde{B}^3\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{m_e^4}{(b+s)^4}\right]\right] , \quad (4.16)$$

with

$$s=\frac{2}{(1-v^2)^{1/2}} , \quad (4.17)$$

and can be integrated to be

$$\delta^1_{VP}\Delta\nu_F=0.055\alpha^2\Delta\nu_F . \quad (4.18)$$

The other corrections of the vacuum polarization arise from the magnetic correction to the electron and muon wave functions. Treating the hyperfine interaction between the electron and muon as a small perturbation, the magnetic corrections $\delta_M^{e\text{-}\mu}\Phi_e$ and $\delta_M^{e\text{-}\mu}\Phi_\mu$ to the uncorrelated wave functions Φ_e and Φ_μ can be found by the method of Dalgarno and Lewis [21] (as used, for example, by Drachman [22]), from the equations

$$\left[-\frac{1}{2m_e}\nabla_e^2-\frac{\alpha}{r_e}+\frac{1}{m_e}\alpha^2\right]\delta_M^{e\text{-}\mu}\Phi_e=(E_e^M-H_e^M)\Phi_e \quad (4.19)$$

and

$$\left[-\frac{1}{2m_\mu}\nabla_\mu^2-\frac{\alpha}{r_\mu}+\frac{1}{2m_\mu}\alpha^2\right]\delta_M^{e\text{-}\mu}\Phi_\mu=(E_\mu^M-H_\mu^M)\Phi_\mu , \quad (4.20)$$

where

$$H_e^M = -\frac{2}{3}\pi\alpha^2\frac{m_e}{m_\mu}\boldsymbol{\sigma}_e\cdot\boldsymbol{\sigma}_\mu\int d^3r_\mu\Phi_\mu^+\delta(\mathbf{r}_{e\mu})\Phi_\mu\,, \tag{4.21}$$

$$E_e^M = -\frac{2}{3}\pi\alpha^2\frac{m_e}{m_\mu}\boldsymbol{\sigma}_e\cdot\boldsymbol{\sigma}_\mu\int d^3r_e d^3r_\mu\Phi_e^+\Phi_\mu^+\delta(\mathbf{r}_{e\mu})\Phi_e\Phi_\mu\,, \tag{4.22}$$

and H_μ^M and E_μ^M are defined similarly as in Eqs. (4.21) and (4.22).

Requiring $\int d^3r_e\Phi_e\delta_M^{e\text{-}\mu}\Phi_e=0$ and $\int d^3r_\mu\Phi_\mu\delta_M^{e\text{-}\mu}\Phi_\mu=0$, the magnetic corrections $\delta_M^{e\text{-}\mu}\Phi_e$ and $\delta_M^{e\text{-}\mu}\Phi_\mu$ are given, respectively, by

$$\begin{aligned}\delta_M^{e\text{-}\mu}\Phi_e = &\left[\frac{2}{\pi a^3}\right]^{1/2} m_e E_E^M e^{-(a/2)r_e}\\ &\times\left\{\frac{a^2r_e}{2}+a\left[\ln\left[\frac{ab}{a+b}r_e\right]+\gamma-\mathrm{Ei}(-br_e)\right]-\frac{1}{r_e}+\frac{e^{-br_e}}{r_e}-\frac{(a-b)(a+b)^3}{4(a^2-ab+b^2)}r_e e^{-br_e}\right.\\ &\left.+\frac{b(a+b)^4r_e^2e^{-br_e}}{8(a^2-ab+b^2)}+\frac{(3b^2-2ab+a^2)(a+b)^2}{4b(a^2-ab+b^2)}e^{-br_e}-\frac{5}{2}a+\frac{a^2[4b^3-a^3+ab(9a-16a)]}{4b(a+b)(a^2-ab+b^2)}\right\},\end{aligned} \tag{4.23}$$

$$\begin{aligned}\delta_M^{e\text{-}\mu}\Phi_\mu = &\left[\frac{2}{\pi b^3}\right]^{1/2} m_\mu E_\mu^M e^{-(b/2)r_\mu}\left[1-\frac{b}{2}r_\mu\right]\\ &\times\left\{\frac{b^2r_\mu}{2}-\frac{7b}{2(br_\mu/2-1)}+2b\left[\ln\left[\frac{ab}{a+b}r_\mu\right]+\gamma-\mathrm{Ei}(-ar_\mu)\right]-\frac{1}{r_\mu}+\frac{e^{-ar_\mu}}{r_\mu}\right.\\ &+\frac{(a+b)^4}{2a(a^2-ab+b^2)}e^{-ar_\mu}+\frac{b(a^2+5ab+7b^2)}{2(br_\mu/2-1)(a^2+ab+b^2)}e^{-ar_\mu}-\frac{11}{2}b-\frac{b^4}{2a(a+b)^2}+\frac{2b^2}{a+b}\\ &\left.-\frac{9}{2}\frac{b^6}{(a+b)^3(a^2-ab+b^2)}+\frac{3}{2}\frac{b^4(a^2+ab+3b^2)}{(a+b)^3(a^2-ab+b^2)}\right\},\end{aligned} \tag{4.24}$$

where γ is the Euler's constant and $\mathrm{Ei}(x)$ is the exponential-integral function.

Thus the vacuum-polarization corrections to the hyperfine splitting can be written as

$$\delta_{e,vp}^{e\text{-}\mu}\Delta\nu_F=2\left[\int d^3r_e\overline{\Phi}_e\gamma_0 eA_{e,0}^{vp}\delta_M^{e\text{-}\mu}\Phi_e\right]_{\text{singlet}}-2\left[\int d^3r_e\overline{\Phi}_e\gamma_0 eA_{e,0}^{vp}\delta_M^{e\text{-}\mu}\Phi_e\right]_{\text{triplet}} \tag{4.25}$$

and

$$\delta_{\mu,vp}^{e\text{-}\mu}\Delta\nu_F=2\left[\int d^3r_\mu\overline{\Phi}_\mu\gamma_0 eA_{\mu,0}^{vp}\delta_M^{e\text{-}\mu}\Phi_\mu\right]_{\text{singlet}}-2\left[\int d^3r_\mu\overline{\Phi}_\mu\gamma_0 eA_{\mu,0}^{vp}\delta_M^{e\text{-}\mu}\Phi_\mu\right]_{\text{triplet}}, \tag{4.26}$$

with

$$\begin{aligned}e,A_{e,0}^{vp}(\mathbf{r}) = &\frac{\alpha^2}{2\pi}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\\ &\times\left[-\frac{2}{r}e^{-sr}+\frac{b^4e^{-sr}}{r(b^2-s^2)^2}+\frac{b^2e^{-sr}}{2r(b^2-s^2)}-\frac{b^4e^{-br}}{r(b^2-s^2)^2}-\frac{b^2e^{-br}}{2r(b^2-s^2)}-\frac{13b^6e^{-sr}}{2r(b^2-s^2)^3}\right.\\ &+\frac{13b^6e^{-br}}{2r(b^2-s^2)^3}+\frac{6b^8e^{-sr}}{r(b^2-s^2)^4}+\frac{6b^8e^{-br}}{r(b^2-s^2)^4}-\frac{b^3e^{-br}}{2(b^2-s^2)}+\frac{11b^5e^{-br}}{4(b^2-s^2)^2}\\ &\left.-\frac{3b^7e^{-br}}{(b^2-s^2)^3}+\frac{b^4re^{-br}}{2(b^2-s^2)}-\frac{3b^6re^{-br}}{4(b^2-s^2)^2}-\frac{b^5r^2e^{-br}}{8(b^2-s^2)}\right],\end{aligned} \tag{4.27}$$

and $eA_{\mu,0}^{vp}(\mathbf{r})$, which has been calculated by Huang and Hughes [23].

Here the lowest-order hyperfine splitting $\Delta\nu_F$ should be replaced by $\Delta\nu_F^{e\text{-}\mu}$, which is induced by the hyperfine interaction between the electron and muon. The magnetic correction $\delta_M^e\Phi_e$ to the electron wave function (which arises from the hyperfine interaction between the electron and ^{3}He nucleus) can be calculated similarly as the magnetic corrections in Eqs. (4.23) and (4.24) and the result is

$$\delta_M^e\Phi_e=-m_e\left[\frac{2}{a^3\pi}\right]^{1/2}e^{-ar/2}\Delta\nu_F^e\left[\frac{1}{r}-a\ln r+a\left[\frac{5}{2}-\gamma-\ln a\right]-\frac{a^2}{2}r\right]. \tag{4.28}$$

The magnetic correction in Eq. (4.28) is the same as what Brodsky and Erickson gave [18]. Thus the induced vacuum-polarization correction to the hyperfine splitting can be written as

$$\delta^{e}_{e,vp}\Delta\nu^{e}_{F}=2\left[\int d^3r_e\overline{\Phi}_e\gamma_0 e A^{vp}_{e,0}\delta^{e}_{M}\Phi_e\right]_{\text{singlet}}-2\left[\int d^3r_e\overline{\Phi}_e\gamma_0 e A^{vp}_{e,0}\delta^{e}_{M}\Phi_e\right]_{\text{triplet}} \tag{4.29}$$

After calculating Eqs. (4.25), (4.26), and (4.29) similarly as (4.15), we have

$$\begin{aligned}\delta^{e\text{-}\mu}_{\mu,vp}\Delta\nu_F=-\frac{2\alpha^2}{\pi}\Delta\nu_F\Bigg[&-6\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{bm_e}{(b+s)^2}+18\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^2m_e}{(b+s)^3}\\&-12\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^3m_e}{(b+s)^4}-18\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^4m_e}{(b+s)^5}\\&+15\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^5m_e}{(b+s)^6}\Bigg],\end{aligned} \tag{4.30}$$

$$\begin{aligned}\delta^{e}_{e,vp}\Delta\nu^{e}_F=-\frac{2\alpha^2}{\pi}\Delta\nu^{e}_F\Bigg[&-2\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{1}{s}-\frac{3m_e}{4b}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^2}{b^2-s^2}\\&+\frac{m_e}{4b}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^4}{(b^2-s^2)^2}+\frac{7m_e}{2b}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^6}{(b^2-s^2)^3}\\&-6\frac{m_e}{b}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^8}{(b^2-s^2)^4}+\frac{1}{2}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^2m_e}{(b^2-s^2)s}\\&+\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^4m_e}{(b^2-s^2)^2s}-\frac{13}{2}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^6m_e}{(b^2-s^2)^3s}\\&+6\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^8m_e}{(b^2-s^2)^4s}\Bigg],\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}\delta^{e\text{-}\mu}_{e,vp}\Delta\nu_F=-\frac{2\alpha^2}{\pi}\Delta\nu_F\Bigg[&-2\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{1}{s}+\frac{1}{2}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^2m_e}{(b^2-s^2)s}\\&+\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^4m_e}{(b^2-s^2)^2s}-\frac{13}{2}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^6m_e}{(b^2-s^2)^3s}\\&+6\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^8m_e}{(b^2-s^2)^4s}-\frac{141}{256}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^2m_e}{(b^2-s^2)b}\\&-\frac{53}{256}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^4m_e}{(b^2-s^2)^2b}+\frac{1}{128}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^6m_e}{(b^2-s^2)^3b}\\&-\frac{91}{64}\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^8m_e}{(b^2-s^2)^4b}\\&+\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\left[\frac{m_e}{b+s}+\frac{3bm_e}{4(b+s)^2}+\frac{b^2m_e}{2(b+s)^3}+\frac{3b^3m_e}{4(b+s)^4}\right]\\&\times\left[1-\frac{b^2}{2(b^2-s^2)}-\frac{b^4}{(b^2-s^2)^2}+\frac{13b^6}{2(b^2-s^2)^3}-\frac{6b^8}{(b^2-s^2)^4}\right]\Bigg].\end{aligned} \tag{4.32}$$

The integrals in Eqs. (4.30)–(4.32) can be calculated analytically. We present some of them in the following:

$$\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^2}{b^2-s^2}=-2\left[\frac{5}{9}\frac{\tilde{b}^2}{3}+\frac{(2+\tilde{b}^2)(1-\tilde{b}^2)^{1/2}}{6}\ln\left[\frac{1-(1-\tilde{b}^2)^{1/2}}{(1-\tilde{b}^2)^{1/2}}\right]\right], \tag{4.33}$$

$$\int_0^1 dv\frac{2v^2(1-v^2/3)}{1-v^2}\frac{b^2m_e}{(b^2-s^2)s}=\frac{\pi}{2}\left[\frac{3}{8}-\frac{\tilde{b}^2}{2}-\frac{\tilde{b}^4}{3}\right], \tag{4.34}$$

and

$$\int_0^1 dv \frac{2v^2(1-v^2/3)}{1-v^2}\frac{m_e}{b+s} = -\frac{1}{3}\left[\tilde{b}\left[\frac{5}{3}+\tilde{b}^2\right]-\tilde{b}^2\left[\frac{3}{2}+\tilde{b}^2\right]\frac{\pi}{2}+\left[\tilde{b}^2\left[\tilde{b}^2+1\right]-2\right]\frac{\tilde{b}}{(1-\tilde{b}^2)^{1/2}}\ln\left[\frac{\sqrt{1-\tilde{b}}+\sqrt{1+\tilde{b}}}{-\sqrt{1-\tilde{b}}+\sqrt{1+\tilde{b}}}\right]\right], \tag{4.35}$$

where $\tilde{b}=2m_e/b$, and others can be calculated through them.

Thus we have

$$\delta^{e\text{-}\mu}_{\mu,vp}\Delta\nu_F=0.066\alpha^2\Delta\nu_F , \tag{4.36}$$

$$\delta^{e}_{e,vp}\Delta\nu^e_F=0.755\alpha^2\Delta\nu^e_F , \tag{4.37}$$

and

$$\delta^{e\text{-}\mu}_{e,vp}\Delta\nu_F=0.886\alpha^2\Delta\nu_F . \tag{4.38}$$

C. Recoil and self-energy corrections

Nonrelativistic recoil effects are included in $\langle\delta(\mathbf{r}_{e\mu})\rangle$ and $\langle\delta(\mathbf{r}_e)\rangle$, since we used the correlated wave functions and included an exact nonrelativistic reduced-mass correction in calculating the lowest-order hyperfine splitting. Other corrections are produced by processes of second-order perturbation in quantum electrodynamics, in which the electron and muon (or the ^{3}He nucleus) interact twice either through the exchange of two transverse photons or through the exchange of one transverse photon and one instantaneous Coulomb interaction. We used the estimation of the recoil effect [9,10,24–27].

$$\delta^{(e\text{-}\mu)}_{\text{rec}}=\Delta\nu_F\frac{3\alpha}{\pi}\frac{m_e}{m_\mu}\ln\frac{m_\mu}{m_e} \tag{4.39}$$

and

$$\delta^{e}_{\text{rec}}=-\Delta\nu^e_F\frac{3\alpha}{\pi}\frac{m_e}{m_{^3\text{He}}}\ln\frac{m_{^3\text{He}}}{m_e} . \tag{4.40}$$

The self-energy corrections are induced by the emission and reabsorption of virtual photons by the electrons or muons themselves. The anomalous magnetic moment of the muon and electron are well known as [28]

$$\frac{g_\mu}{2}=1+\frac{\alpha}{2\pi}+0.765\,782\frac{\alpha^2}{\pi^2}+O(\alpha^3) \tag{4.41}$$

and

$$\frac{g_e}{2}=1+\frac{\alpha}{2\pi}-0.3285\frac{\alpha^2}{\pi^2}+O(\alpha^3) . \tag{4.42}$$

The radiative correction at the electron vertex, which is referred to as the "binding correction," is given by [18,19,29]

$$\delta_{\text{binding}}=\tfrac{3}{2}\alpha(\ln 2-\tfrac{13}{4}) . \tag{4.43}$$

D. Nuclear finite-size correction

We follow the calculations of the hyperfine shift in hydrogen due to the finite size of the proton, which were done by Zemach [30] and Grotch and Yennie [31], because the wave functions for $(2s)_\mu(1s)_e$ muonic ^{3}He are approximately equal to the $1s$-state hydrogenic wave functions with $Z_e=1$. The nuclear finite-size correction to $\Delta\nu^e_F$ is therefore given as

$$-2u\alpha R\,\Delta\nu^e_F , \tag{4.44}$$

where u is the reduced mass of the electron and ^{3}He nucleus and

$$R=\int r\rho_M(\mathbf{u})\rho_E(\mathbf{r}-\mathbf{u})d^3u\,d^3r . \tag{4.45}$$

The electric and magnetic form factors ρ_E and ρ_M are normalized to unity. The experimental determination of ^{3}He form factors, obtained from the electron-^{3}He scattering, indicated that the Gaussian fit provided a good description of both the electric and magnetic form factors. We therefore use

$$\rho_E(\mathbf{r})=\rho_0(\mathbf{r})+\Delta\rho(\mathbf{r}) , \tag{4.46}$$

$$\rho_0=\frac{1}{8\pi^{3/2}}\left[\frac{1}{a^3}e^{-r^2/4a^2}-\frac{b^2(6c^2-r^2)}{4c^7}e^{-r^2/4c^2}\right] , \tag{4.47}$$

$$\Delta\rho=\frac{pdq_0^2}{2\pi^{3/2}}\left[\frac{\sin(q_0r)}{q_0r}+\frac{p^2}{2q_0^2}\cos(q_0r)\right]e^{-p^2r^2/4} , \tag{4.48}$$

and

$$\rho_M=\frac{1}{8\pi^{3/2}}\left[\frac{1}{a'^3}e^{-r^2/4a'^2}-\frac{b'^2(6c'^2-r^2)}{4c'^7}\right] . \tag{4.49}$$

The constants used in Eqs. (4.47)–(4.49) were determined by McCarthy, Sick, and Whitney [32].

Integrating Eq. (4.45), we have $R=2.456$ fm. Thus we obtain the correction of the hyperfine splitting as

$$\delta_v\Delta\nu^e_F\approx-91.98\times10^{-6}\Delta\nu^e_F . \tag{4.50}$$

The nuclear finite-size correction of $\Delta\nu_F(\Delta\nu^{e\text{-}\mu}_F)$ can be estimated for singlet and triplet states, which depend on $\langle\Phi_e|\delta V|\delta^{e\text{-}\mu}_M\Phi_e\rangle$ and $\langle\Phi_\mu|\delta V|\delta^{e\text{-}\mu}_M\Phi_\mu\rangle$. Here we treat δV as a small perturbation, which is given as

$$\delta V=\begin{cases}\dfrac{Z\alpha}{r}-\dfrac{Z\alpha}{R}\left[\dfrac{3}{2}-\dfrac{r^2}{2R^2}\right] & \text{for } r\le R ,\\ 0 \quad \text{for } r>R .\end{cases} \tag{4.51}$$

Here we consider the ^{3}He and ^{4}He nuclei as a sphere of radius R with the charge uniformly spreading over its volume.

In calculating $\langle\Phi_e|\delta V|\delta^{e\text{-}\mu}_M\Phi_e\rangle$ and $\langle\Phi_\mu|\delta V|\delta^{e\text{-}\mu}_M\Phi_\mu\rangle$, we need only find $\delta^{e\text{-}\mu}_M\Phi_e$ and $\delta_M^{e\text{-}\mu\Phi_\mu}$, which are shown in Eqs. (4.23) and (4.24), inside the nucleus. They are simply given as

TABLE XIII. Contributions of the hyperfine splitting (HFS) of the 2s state of muonic ^{4}He atoms.[a]

	$\Delta\nu_F$ (MHz)	(ppm)
Lowest-order HFS: $\Delta\nu_F$	4276.69±0.1	
Relativistic correction	−0.047	−11.09
Radiative correction:		
Anomalous magnetic moment [28]	9.9497	2326.5
Vacuum polarization:		
e^--induced muon magnetic polarization potential ($\delta^1_{vp}\Delta\nu_F$)	0.0125	2.93
Magnetic correction to the wave function:		
Electron ($\delta^{e\text{-}\mu}_{e,vp}\Delta\nu_F$)	0.2018	47.18
Muon ($\delta^{e\text{-}\mu}_{\mu,vp}\Delta\nu_F$)	0.0150	3.515
Binding [18,19,29]	−0.5816	−136.0
Recoil correction:		
Electron-muon (4.39)	0.7685	179.7
Total calculated HFS: $\Delta\nu=4287.01\pm0.1$		

[a]The fundamental physical constants used here are the same as our previous work [9,10] except $m_\mu=105.659\,48$ MeV.

$$\delta^{e\text{-}\mu}_M\Phi_e \approx \left[\frac{2}{\pi a^3}\right]^{1/2} E_e^m m_e \left[1-\frac{a}{2}r\right]\left[\frac{a^2}{2}r-b-\frac{5}{2}a+a\left[\ln\frac{a}{a+b}-br\right]+\frac{b^2r}{2}-\frac{(a-b)(a+b)^3}{4(a^2-ab+b^2)}r\right.$$
$$\left.+\frac{(3b^2-2ab+a^2)(a+b)^2}{4b(a^2-ab+b^2)}(1-br)+\frac{a^2[4b^3-a^2+ab(9a-16b)]}{4b(a^2-ab+b^2)(a+b)}\right] \quad (4.52)$$

TABLE XIV. Contributions of the hyperfine splitting of the 2s state of muonic ^{3}He atoms.

	$\Delta\nu^{e\text{-}\mu}$ (MHz)	$\Delta\nu^e$ (MHz)	(ppm)
Lowest-order HFS:	3205.63±0.08	838.39±0.02	
Relativistic correction (4.7):			
Electron	0.2561	0.0670	79.9
Muon	−0.2916		−90.97
Nuclear finite-size correction (4.50)		−0.0771	−91.98
Radiative correction:			
Anomalous magnetic moment [28]	7.4579		2326.5
		0.972	1159.6
Vacuum polarization:			
e^--induced ^{3}He^{2+} magnetic polarization potential [18,19,20]		0.0166	19.8
Magnetic correction to the wave function:			
Electron ($\delta^e_{e,vp}\Delta\nu_F^{(e)}$)		0.0337	40.204
e^--induced muon magnetic polarization potential ($\delta^1_{vp}\Delta\nu_F^{e\text{-}\mu}$)	0.0094		2.93
Magnetic correction to the wave function:			
Electron ($\delta^{e\text{-}\mu}_{e,vp}\Delta\nu_F^{e\text{-}\mu}$)	0.1512		47.18
Muon ($\delta^{e\text{-}\mu}_{\mu,vp}\Delta\nu_F^{e\text{-}\mu}$)	0.0113		3.515
Binding [18,19,20]	−0.4360	−0.144	−136.0
Recoil correction:			
Electron-muon (4.39)	0.5761		179.7
Electron-^{3}He^{2+}(4.40)		−0.0091	−10.9
Total calculated HFS: $\Delta\nu=4052.64\pm0.10$			

and

$$\delta_M^{e\text{-}\mu}\Phi_\mu \approx \left[\frac{2}{\pi b^3}\right]^{1/2} E_\mu^M m_\mu \left[1-\frac{b}{2}r\right]^2$$
$$\times\left[-\frac{5}{4}b^2 r+\frac{a^2}{2}r+2b\left[\ln\frac{b}{a+b}-r\right]+\frac{(a+b)^4}{2a(a^2-ab+b^2)}(1-ar)\right.$$
$$-\frac{b(a^2+5ab+7b^2)}{2(a^2-ab+b^2)}(1-b^2r)(1-ar)-2b+2\frac{b^2}{a+b}-a-\frac{b^4}{2a(a+b)^2}$$
$$\left.-\frac{9b^6}{2(a+b)^3(a^2-ab+b^2)}+\frac{3b^4(a^2+ab+3b^2)}{2(a^2-ab+b^2)(a+b)^3}\right] . \tag{4.53}$$

Therefore, the hyperfine splitting will be reduced by an amount $\Delta\nu_F O(aR)(bR) \approx 10^{-6}\Delta\nu_F$ which can be neglected for the present.

V. SUMMARY

Lowest-order hyperfine splittings have been calculated in Sec. III by using correlated wave functions, which are shown in Sec. II. The corrections to the lowest-order hyperfine splittings are calculated in Sec. IV up to order α^2. They are given as

$$\begin{aligned} \delta^{\text{rel}} &= -0.208\alpha^2 ,\\ \delta^1_{vp} &= 0.055\alpha^2 ,\\ \delta^{e\text{-}\mu}_{\mu,vp} &= 0.066\alpha^2 ,\\ \delta^e_{e,vp} &= 0.755\alpha^2 ,\\ \delta^{e\text{-}\mu}_{e,vp} &= 0.886\alpha^2 . \end{aligned} \tag{5.1}$$

Including the recoil, self-energy, and nuclear finite-size corrections, the details of each contribution being given in Tables XIII and XIV, we obtained, for the total hyperfine splitting,

$$\Delta\nu = 4287.01 \pm 0.10 \text{ MHz} , \tag{5.2}$$

for the 2*s* state of muonic ^{4}He atoms, and

$$\Delta\nu = 4052.64 \pm 0.10 \text{ MHz} , \tag{5.3}$$

for the 2*s* state of muonic ^{3}He atoms.

ACKNOWLEDGMENT

This work is supported by the National Science Council of the Republic of China under Grant No. NSC 79-0208-M005-19.

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