Geometric phases in two-level atoms excited by pulses propagating without loss

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We consider the geometrical phases arising in the state vector of two-level atoms due to their interaction with a self-consistently generated classical electrical field propagating without loss through the atomic medium. Three conservation laws are shown to exist generally and are used to solve for the individual quantum amplitudes, phases, and the electric field. We calculate the geometrical phases in two situations: (a) where the atoms are initally in the ground state and (b) where the initial state is a coherent superposition of the ground and excited states. In both cases the geometrical phase is the Aharonov-Anandan phase resulting from the atomic state vector tracing out a closed curve on the projective Hilbert space—here the Bloch sphere. We show that geometric quantities associated with the curve on the Bloch sphere are directly related to physical observables. The solid angle subtended by the closed curve (shown to equal twice the geometric phase) is a measure of the maximum atomic inversion, while the speed with which the curve is traced is related to the energy uncertainty in the state. An experimental method to observe the total phase change in a two-level subsystem is outlined, using photon echoes in a three-level medium.

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I. INTRODUCTION

The paradigm for Berry's phase [1] has been the system of a spin-$\frac{1}{2}$ particle in a magnetic field. We prepare the system in one of the energy eigenstates. The time evolution of the magnetic field is chosen to be both adiabatic and periodic, the period being much greater than the period of precessional motion of the particle. The state vector then remains in the eigenstate at all times and picks up a phase in addition to the usual dynamical phase when the magnetic field completes a cycle. The additional phase depends only on the solid angle subtended by the closed loop, traced by the magnetic field, at the origin and hence is called geometric. Consider a more general situation where the time evolution of the Hamiltonian is not necessarily adiabatic but the state vector varies periodically in time. In this case Aharonov and Anandan [2] showed the state vector again picks up a geometric phase after each period. This Aharonov and Anandan (AA) phase depends on the closed loop traced by the ray representative of the state vector in the projective Hilbert space. In particular, the state vector for a spin-$\frac{1}{2}$ particle in a *constant* magnetic field undergoes a cyclic evolution and if initially the state vector is not an energy eigenstate, the geometric phase equals one-half the solid angle subtended by the closed loop traced out by the spin vector. Both the Berry phase for an energy eigenstate with an adiabatically varying magnetic field and the Aharonov-Anandan phase for the state vector have been measured in nuclear magnetic resonance (NMR) experiments [3, 4].

There are close analogies between magnetic resonance and optical resonance. By the result of Feynman, Vernon, and Hellwarth [5], the interaction of two-level atoms with resonant electromagnetic radiation can be mapped onto the problem of a spin-$\frac{1}{2}$ particle in a magnetic field. This has been used recently to show that the energy eigenstate of a two-level atom develops Berry's phase when placed in an external, nearly resonant electric field and that the phase affects the Rabi frequency of oscillation [6]. It has also been generalized to study Berry's phase arising in a two-level atomic system described by a density matrix (either because it is prepared in a mixed state or to include the effects of dissipation) and interacting with an external classical laser field [7]. However there is one significant aspect in which optical resonance differs from magnetic resonance and that is the emission of radiation by the atomic dipoles at wavelengths close to that of the applied field. Under the influence of the resonant field the atoms may emit radiation which then propagates through the medium composed of two-level atoms.

In this work we will consider the phenomenon of self-induced transparency (SIT) [8–10] where optical radiation propagates without loss in an absorbing medium and show that the state vector for the atom may develop a geometric phase as a consequence of the interaction. In contrast to most other work on the geometric phase, the phase we study develops as a consequence of the self-consistent interaction between the medium of two-level atoms and the electric field and not due to the variation of some external parameters. As a consequence of this, the state and the electric field vary on the same time scale and the assumptions for the appearance of Berry's phase are not met. However, for special choices of initial conditions the state vector goes through a cyclic evolution tracing out a closed curve on the projective Hilbert

space leading to the appearance of the AA phase. Therefore in this paper we will concentrate on the study of this AA phase for SIT. We will show explicitly the intimate connection between the geometry of the curves traced on the projective Hilbert space (here the Bloch sphere) and physical observables of the system such as the maximum atomic inversion and the instantaneous energy uncertainty.

In Sec. II we review the derivation of the Maxwell-Bloch equations describing the interaction of the two-level atoms with the nearly resonant classical electric field. Losses leading to homogeneous broadening of spectral lines are neglected. In other words we restrict ourselves to electrical pulses which last for a time much shorter than the time given by the homogeneous decay width of the Lorentzian spectral line. With the slowly-varying-envelope approximation, the coupled equations reduce to a system of first-order differential equations. We show that there exist three conservation laws, generalizing the forms of the laws as usually stated. The solution of these equations can only be obtained by making further assumptions on the nature of the inhomogeneous broadening present. For the rest of the paper we consider only the simplest choice, that the spectral lines are infinitely sharp. In Sec. III we assume that initially the atoms are in their ground state and obtain the well-known hyperbolic secant pulse solution (first found by McCall and Hahn [8]) as the propagating electric field. After the passage of this pulse the atoms are returned to their ground state and we calculate the geometric phase that develops in the atomic state at the end of this cyclical evolution. We show that this phase is directly related to the maximum atomic inversion during the course of the evolution. A study of the curve traced on the projective Hilbert space in Sec. III A shows (a) the geometrical phase equals minus half the solid angle subtended by the closed curve and (b) the velocity along this curve is directly related to the instantaneous energy uncertainty in the state. Section IV deals with another initial condition where we start with a coherent superposition of the ground state and the excited state. Here, instead of a single pulse, a continuous-pulse train propagates through the medium taking the state vector through a cyclical evolution only for certain choices of initial parameters. The geometrical phase is calculated and again shown to equal minus half the solid angle subtended by the closed loop on the projective Hilbert space. To emphasize the point made earlier, this loop is determined by the dynamics of our self-consistent system and the initial conditions. This is in contrast to previous studies of the geometrical phase, where the closed loop was determined by the choice of the external field. In Sec. V we propose a method using photon echoes in a three-level system by which the phase of the state vector may be measured following the setup used to observe the AA phase in magnetic resonance [4]. We end with a summary in Sec. VI.

II. COUPLED MAXWELL-SCHRÖDINGER EQUATIONS

The phenomenon of self-induced transparency was first observed nearly two decades ago by McCall and Hahn [8] and later followed up by Slusher and Gibbs [11], Matulic and Eberly [12], and others. When light at low intensities is incident upon a resonant medium it gets absorbed within a characteristic length (known as Beer's length) of the medium. However as the intensity is increased beyond a threshold value, the medium behaves as though it were transparent to the light pulse which passes through practically unabsorbed. In addition, the velocity of the pulse is much slower than the phase velocity in the medium. This behavior is nonlinear in nature and cannot be understood as perturbations around the behavior at low intensities. Instead this can be understood by modeling the system as a collection of two-level atoms whose dynamics is governed by Schrödinger's equation in the presence of a classical electric field which itself evolves according to Maxwell's equation. Our point of departure from previous work is that we focus on both the amplitude and the *phase* developed by the state vector and not just the bilinear combinations of the amplitudes and the relative phase occurring in the components of the Bloch vector. Consequently, the system of equations we study are somewhat different from that considered in the earlier literature.

We consider the interaction of a medium of two-level atoms with an electric field whose frequency is nearly resonant with the frequency of transition between the two levels. Let $|g\rangle$ and $|e\rangle$ be the lower and upper energy eigenstates differing in energy by $\hbar\omega_0$, in the absence of the electric field. Dipole-dipole interactions between the atoms are neglected so that the atoms interact with one another only via the electric field. We assume that all other atomic levels are very distant in energy and hence do not couple resonantly to the electric field. The electric field is taken to be a circularly polarized plane wave, propagating in the positive z direction, the field vector being

$$\mathbf{E} = \mathcal{E}(z,t)[\hat{\mathbf{x}}\cos\Phi(z,t) + \hat{\mathbf{y}}\sin\Phi(z,t)] , \tag{2.1}$$

where $\mathcal{E}(z,t)$ is the real envelope of the field and

$$\Phi(z,t) = \omega t - Kz + \phi(z,t) \tag{2.2}$$

is the total field phase with ω being the frequency of the light pulse in the medium. Circularly polarized light couples to a $|\Delta m| = 1$ transition between the two levels and the dipole moment vector for an individual atom can be written as $\mathbf{p} = p(\hat{\mathbf{x}}\sigma_1 + \hat{\mathbf{y}}\sigma_2)$ where $p = p_x = p_y$. The Hamiltonian for an atom in the presence of the electric field is

$$H = \frac{\hbar\omega_0}{2}\sigma_3 - \mathbf{p}\cdot\mathbf{E} = \begin{pmatrix} \frac{\hbar\omega_0}{2} & -p\mathcal{E}e^{-i\Phi} \\ -p\mathcal{E}e^{i\Phi} & -\frac{\hbar\omega_0}{2} \end{pmatrix} . \tag{2.3}$$

The electric-field amplitude $\mathcal{E}$ is assumed to be sufficiently strong that the field may be regarded as classical, yet small enough so that $|p\mathcal{E}| \ll \hbar\omega_0$. This last condition is necessary so that the atom does not get ionized in the presence of the electric field and is well satisfied in practice. The wave number of the field is

$$K = (\omega\eta)/c + \delta k ,$$

where η is the refractive index of the medium and δk reflects the possible existence of dispersion due to the interaction between light and the atoms. The refractive index is assumed to be unaffected by the light pulse intensity so that no Kerr-like effects are present. Note that since we wish to identify ω as the carrier frequency, any constant term arising in $\partial_t\phi$ must be set to zero.

It is assumed that the optical resonance line in the medium is inhomogeneously broadened, caused by Doppler broadening in a gas and by a distribution of static crystalline electric and magnetic fields in solids. We neglect the effect of other linewidth contributions due to homogeneous broadening, e.g., caused by radiative decay, collisions with other atoms, etc. This is a valid approximation as long as we consider pulse widths of a few nanoseconds, that being somewhat less than the time scale over which homogeneous broadening occurs. The distribution of transition frequencies ω_0 of the atoms in the medium is described by the spectral density function $g(\gamma)$ where $\gamma = \omega_0 - \bar{\omega}$. To a good approximation, $g(\gamma)$ may be taken to be symmetrical about a central frequency $\bar{\omega}$. It is normalized as

$$\int_{-\infty}^{\infty} g(\gamma)d\gamma = 1 \ .$$

For the spin-$\frac{1}{2}$ particle in a time-dependent rotating magnetic field, the solution of Schrödinger's equation is obtained very simply by going to a frame rotating with the magnetic field. A similar procedure can be followed here. We go from the laboratory coordinate system $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ to a rotating coordinate system whose unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ rotate about $\hat{\mathbf{e}}_3 \equiv \hat{\mathbf{z}}$ with angular speed $\partial_t\Phi$. This transformation is effected by the unitary matrix

$$U(z,t) = e^{(i\Phi\sigma_3)/2} \ .$$

The Hamiltonian in the rotating frame is

$$H_r = UHU^{-1} + i\hbar\frac{\partial U}{\partial t}U^{-1}$$

$$= \frac{\hbar}{2}(\omega_0 - \partial_t\Phi)\sigma_3 - p\mathcal{E}\sigma_1 \qquad (2.4)$$

while the transformed state vector is

$$|\Psi_r\rangle \equiv U|\Psi\rangle = A|e\rangle + B|g\rangle \ . \qquad (2.5)$$

We have expressed the state vector as a superposition of the unperturbed energy eigenstates with (A, B) as the complex amplitudes in the rotating frame. Schrödinger's equation in the rotating coordinate system reduces to the coupled equations:

$$\frac{\partial A}{\partial t} = -\frac{i}{2}(\omega_0 - \partial_t\Phi)A + \frac{ip}{\hbar}\mathcal{E}B \ , \qquad (2.6)$$

$$\frac{\partial B}{\partial t} = \frac{ip}{\hbar}\mathcal{E}A + \frac{i}{2}(\omega_0 - \partial_t\Phi)B \ . \qquad (2.7)$$

In order to write the complete set of coupled equations in real form we introduce the real amplitudes Q_a, Q_b and phases ϕ_a, ϕ_b to describe the atomic states. They are defined as

$$A = Q_a(z,t;\gamma)e^{i\phi_a(z,t;\gamma)} \ , \qquad (2.8)$$

$$B = Q_b(z,t;\gamma)e^{i\phi_b(z,t;\gamma)} \ . \qquad (2.9)$$

The macroscopic polarization density vector of the medium is

$$\mathbf{P} = Np\int_{-\infty}^{\infty} g(\gamma)[\hat{\mathbf{e}}_1\langle\sigma_1\rangle_r + \hat{\mathbf{e}}_2\langle\sigma_2\rangle_r]d\gamma \ , \qquad (2.10)$$

where $\langle\sigma_i\rangle_r$ are the expectation values calculated in the rotating frame and N is the number of atoms per unit volume. We define a pseudospin vector or Bloch vector $\mathbf{s}$ whose components are

$$u \equiv \langle\sigma_1\rangle_r = 2Q_aQ_b\cos(\phi_b - \phi_a) \ , \qquad (2.11)$$

$$v \equiv \langle\sigma_2\rangle_r = 2Q_aQ_b\sin(\phi_b - \phi_a) \ , \qquad (2.12)$$

$$w \equiv \langle\sigma_3\rangle_r = Q_a^2 - Q_b^2 \ . \qquad (2.13)$$

From the conservation of probability it follows that the pseudospin vector has a constant magnitude equal to unity, i.e., $u^2 + v^2 + w^2 = 1$. u and v are the components of the polarization which are in phase and out of phase (by $\pi/2$), respectively, with the electric field. w is known as the *inversion* and measures the difference in population levels.

In terms of the unit vectors in the rotating frame, the electric field simplifies to $\mathbf{E} = \mathcal{E}\hat{\mathbf{e}}_1$. In this frame the electric-field vector has a constant direction and changes slowly in time compared to the variation in the laboratory frame. The electric field induces a polarization in the medium of two-level atoms which in turn acts as a source for the propagation of the field. The evolution equations for the electric field and the above Schrödinger equation together form a self-consistent set. Assuming that the field is propagating in the z direction, Maxwell's equation for the electric field reduces to

$$\left(\frac{\partial^2}{\partial z^2} - \frac{4\pi\sigma}{c^2}\frac{\partial}{\partial t} - \frac{\eta^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P} \ , \qquad (2.14)$$

where σ is the conductivity of the medium and η is the refractive index. Here Ohmic losses of energy are assumed to be negligible so that we can set $\sigma = 0$. We will now make the slowly varying approximation on the amplitude and phase of the complex envelope, viz.

$$\frac{\partial\mathcal{E}}{\partial t} << \omega\mathcal{E} \ , \quad \frac{\partial\mathcal{E}}{\partial z} << K\mathcal{E} \ , \quad \frac{\partial\phi}{\partial t} << \omega\phi \ , \quad \frac{\partial\phi}{\partial z} << K\phi \ ,$$

i.e., $\mathcal{E}, \phi$ vary slowly on the time and length scale set by the carrier oscillations. We will drop all second derivatives and products of first derivatives of $\mathcal{E}$ and ϕ. With these approximations and also using the condition $\omega_0 >> \frac{2p\mathcal{E}}{\hbar}$ we obtain the following equations on equating the components of $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$:

$$\frac{\partial\mathcal{E}}{\partial z} + \frac{\eta}{c}\frac{\partial\mathcal{E}}{\partial t} = \frac{2\pi Np}{\omega\eta c}\int_{-\infty}^{\infty}\omega_0^2 v(z,t;\gamma)g(\gamma)d\gamma \ , \qquad (2.15)$$

$$\left(\frac{\partial\phi}{\partial z} + \frac{\eta}{c}\frac{\partial\phi}{\partial t} - \delta k\right)\mathcal{E} = -\frac{2\pi Np}{\omega\eta c}\int_{-\infty}^{\infty}\omega_0^2 u(z,t;\gamma)g(\gamma)d\gamma \ . \qquad (2.16)$$

Now we look for traveling-wave solutions for the electric field. We are interested in pulses which maintain a constant shape as they propagate through the medium. Thus we assume that all the quantities to be determined depend on the space-time coordinates through the single variable ξ, defined as

$$\xi = t - \frac{z}{v_0} , \tag{2.17}$$

where v_0 is the steady velocity of the pulse. With this ansatz, our system of coupled Schrödinger-Maxwell equations reduces to a system of six differential equations,

$$\frac{d}{d\xi}Q_a = -\frac{p}{\hbar}\mathcal{E}Q_b \sin(\phi_b - \phi_a) , \tag{2.18}$$

$$\frac{d}{d\xi}Q_b = \frac{p}{\hbar}\mathcal{E}Q_a \sin(\phi_b - \phi_a) , \tag{2.19}$$

$$\frac{d}{d\xi}\mathcal{E} = -\alpha\int_{-\infty}^{\infty}\omega_0^2 Q_a Q_b \sin(\phi_b - \phi_a)g(\gamma)d\gamma , \tag{2.20}$$

$$\frac{d}{d\xi}\phi_a + \frac{1}{2}\left(\Delta - \frac{d}{d\xi}\phi\right) = \frac{p}{\hbar}\frac{\mathcal{E}Q_b}{Q_a}\cos(\phi_b - \phi_a) , \tag{2.21}$$

$$\frac{d}{d\xi}\phi_b - \frac{1}{2}\left(\Delta - \frac{d}{d\xi}\phi\right) = \frac{p}{\hbar}\frac{\mathcal{E}Q_a}{Q_b}\cos(\phi_b - \phi_a) , \tag{2.22}$$

$$\frac{d}{d\xi}\phi + \Gamma = \alpha\int_{-\infty}^{\infty}\omega_0^2\frac{Q_a Q_b}{\mathcal{E}}\cos(\phi_b - \phi_a)g(\gamma)d\gamma , \tag{2.23}$$

where we have defined the parameters

$$\Delta = \omega_0 - \omega , \quad \alpha = \frac{4\pi N p}{\omega\eta c}\left(\frac{1}{v_0} - \frac{\eta}{c}\right)^{-1} ,$$

$$\Gamma = \delta k\left(\frac{1}{v_0} - \frac{\eta}{c}\right)^{-1} .$$

We recall that these equations are valid only over time intervals much shorter than the characteristic relaxation time set by the inverse of the homogeneous decay width. Equations (2.21) to (2.23) lead to the following equation for the relative phase $\phi_{ba} = \phi_b - \phi_a$

$$\frac{d}{d\xi}\phi_{ba} = \frac{p}{\hbar}\mathcal{E}\left(\frac{Q_a}{Q_b} - \frac{Q_b}{Q_a}\right)\cos\phi_{ba} - \frac{\alpha}{\mathcal{E}}\int_{-\infty}^{\infty}\omega_0^2 Q_a Q_b \cos\phi_{ba}g(\gamma)d\gamma + \Delta + \Gamma . \tag{2.24}$$

This equation for the relative phase together with the three equations (2.18)–(2.20) for the amplitudes form a closed system by themselves. From these equations we can derive with complete generality the following three conservation laws:

$$1 = Q_a^2 + Q_b^2 , \tag{2.25}$$

$$I_2 = \alpha\int_{-\infty}^{\infty}\omega_0^2 Q_a^2 g(\gamma)d\gamma - \frac{p}{\hbar}\mathcal{E}^2 , \tag{2.26}$$

$$I_3 = \int_{-\infty}^{\infty}\omega_0^2\left(p\mathcal{E}Q_a Q_b\cos\phi_{ba} - \frac{\hbar(\Delta+\Gamma)}{2}Q_a^2\right)g(\gamma)d\gamma , \tag{2.27}$$

where I_2 and I_3 are constants. The first law is a statement of the normalization of the state vector. In terms of the components of the Bloch vector, the other two conservation laws may be written as

$$I_2' = \int_{-\infty}^{\infty}\omega_0^2 w(\xi,\gamma)g(\gamma)d\gamma - \frac{2p}{\hbar\alpha}\mathcal{E}^2(\xi) , \tag{2.28}$$

$$I_3' = \int_{-\infty}^{\infty}\omega_0^2\left(p\mathcal{E}u(\xi,\gamma) - \frac{\hbar(\Delta+\Gamma)}{2}w(\xi,\gamma)\right)g(\gamma)d\gamma , \tag{2.29}$$

where we have renamed the constants as I_2' and I_3'. The second conservation law is the Manley-Rowe relation, indicating the transfer of energy from the field to the excited state [13]. The third conservation law relates the energy of interaction between the electric field and the atoms to the energy stored in the atoms which depends on the inversion. Matulic and Eberly [12] do obtain particular forms of the second and third conservation laws but only after making the "factorization assumption," viz. that the solution for the second component of the Bloch vector $v(\xi,\gamma)$ factorizes in the form

$$v(\xi,\gamma) = F(\gamma)v(\xi,0) , \tag{2.30}$$

an assumption which is not valid in general. Here, we have shown that (2.30) is not a requirement to obtain the conservation laws and (2.25)–(2.27) are valid in a more general context.

The complete solution of the system of equations (2.18)–(2.23) will give us the amplitudes and the phases occurring in the state vector at any arbitrary time as well as the polarization and the inversion. In the literature on self-induced transparency, the only solutions obtained for an arbitrary choice of the spectral density function $g(\gamma)$ have been by use of the "factorization assumption" (2.30). This is the case because the system (2.18)–(2.23) corresponds to an infinite set of coupled equations that decouple only for specific choices of $g(\gamma)$. For these choices, analytical solutions have been found. By assuming $g(\gamma)$ to be a sum of δ functions located at different values of γ, Menyuk, Chen, and Lee [14] proved that for certain values of the parameters the system exhibits the Painlevé property and consequently is likely to be integrable. Here, however, we avoid the complications of constructing a more general solution and consider only the simplest choice for $g(\gamma)$, viz. a single Dirac δ function, in what follows.

We assume therefore that the effects of inhomogeneous broadening on the width of the optical resonance line can also be ignored. All the atoms have the same transition frequency and consequently each atom experiences the same detuning from the applied field frequency ω. The spectral density function $g(\gamma)$ reduces to a Dirac delta function

$$g(\gamma) = \delta(\gamma - \Delta) ,$$

where $\Delta = \omega_0 - \omega$. In a gas inhomogeneous broadening would be absent if we are dealing with atoms either stationary or moving with the same velocity. Sharp-line SIT has been observed in a beam of atomic rubidium with a few nanoseconds pulse from a ^{202}Hg II laser [15].

The Doppler width was about one-fourth the homogeneous width of the pulse. It has also been found that the difference between broad-line and sharp-line SIT is not very marked physically although the analysis of the latter case is considerably simpler.

This assumption leaves the equations for the individual quantum amplitudes and phases unchanged. However the equations for the relative quantum phase $\phi_{ba} = \phi_b - \phi_a$ and the electric-field amplitude and phase simplify to

$$\frac{d}{d\xi}\phi_{ba} = \left[\frac{p}{\hbar}\mathcal{E}\left(\frac{Q_a}{Q_b} - \frac{Q_b}{Q_a}\right) - \frac{\alpha_s}{\mathcal{E}}Q_aQ_b\right]\cos\phi_{ba} + \Delta + \Gamma \ , \tag{2.31}$$

$$\frac{d}{d\xi}\mathcal{E} = -\alpha_s Q_a Q_b \sin\phi_{ba} \ , \tag{2.32}$$

$$\frac{d}{d\xi}\phi = \alpha_s \frac{Q_aQ_b}{\mathcal{E}}\cos\phi_{ba} - \Gamma \ , \tag{2.33}$$

where

$$\alpha_s \equiv \omega_0^2\alpha = \frac{4\pi\omega_0^2 Np}{\omega\eta c}\left(\frac{1}{v_0} - \frac{\eta}{c}\right)^{-1} > 0 \ .$$

The solution of this system of atoms interacting with a classical electric field depends crucially on the choice of initial conditions. We solve it first for the usual case of self-induced transparency, viz. all atoms are initially in the ground state, and second in the more general situation where the initial state vector is a coherent superposition of the ground and excited states. We remark in passing that precisely the same form of the equations as above occurs in the description of three-wave interactions in plasmas [13].

III. SINGLE-PULSE SOLUTIONS

With all the atoms initially in the ground state, the initial amplitudes for the states and electric field are

$$Q_a^2(-\infty) = 0 \ , \quad Q_b^2(-\infty) = 1 \ , \quad \mathcal{E}(-\infty) = 0 \ ,$$

respectively. Consequently the conservation laws (2.25)–(2.27) simplify to

$$Q_a^2 + Q_b^2 = 1 \ , \tag{3.1}$$

$$\mathcal{E}^2 = \frac{\hbar\alpha_s}{p}Q_a^2 \ , \tag{3.2}$$

$$\mathcal{E}Q_aQ_b\cos\phi_{ba} = \frac{\hbar(\Delta+\Gamma)}{2p}Q_a^2 \ . \tag{3.3}$$

Defining the variable

$$r(\xi;\Delta) = Q_a^2(\xi;\Delta)$$

and the parameters

$$r_0 = 1 - \frac{(\Delta+\Gamma)^2}{4\kappa} \ , \quad \kappa = \frac{p\alpha_s}{\hbar} = \frac{4\pi\omega_0^2Np^2}{\hbar\omega\eta c}\left(\frac{1}{v_0} - \frac{\eta}{c}\right)^{-1}$$

we find that Eqs. (2.18), (3.2), and (3.3) lead to

$$\frac{dr}{d\xi} = -2r\sqrt{\kappa(r_0 - r)} \ . \tag{3.4}$$

This equation has the solution

$$r = r_0\,\mathrm{sech}^2(\sqrt{\kappa r_0}\xi) \ . \tag{3.5}$$

It follows that the excited- and ground-state amplitudes are, respectively,

$$Q_a = \sqrt{r_0}\,\mathrm{sech}(\sqrt{\kappa r_0}\xi) \ , \tag{3.6}$$

$$Q_b = \sqrt{1 - r_0\,\mathrm{sech}^2(\sqrt{\kappa r_0}\xi)} \tag{3.7}$$

while the envelope of the electric field is of the form

$$\mathcal{E} = \frac{\hbar}{p}\sqrt{\kappa r_0}\,\mathrm{sech}(\sqrt{\kappa r_0}\xi) \ . \tag{3.8}$$

The secant hyperbolic form of the electric field in (3.8) is the well-known "soliton" pulse of self-induced transparency, first found by McCall and Hahn. It is also known as a 2π pulse since the area under the envelope of the electric field is 2π. This is the only possible single pulse which can propagate without appreciable loss through the medium. From the argument of the sech function in (3.8) we see that $\sqrt{\kappa r_0}$ can be identified with the inverse of the pulse width (τ_p) so that (3.8) reads

$$\mathcal{E} = \frac{\hbar}{p\tau_p}\mathrm{sech}\frac{\xi}{\tau_p} \ . \tag{3.9}$$

Substituting for κ and r_0 into $\kappa r_0 = 1/\tau_p^2$ leads to the following expression for the steady velocity v_0 of the secant hyperbolic pulse in terms of the other parameters:

$$\frac{1}{v_0} = \frac{\eta}{c} + \frac{4\pi\omega_0^2Np^2\tau_p^2}{\hbar\omega\eta c}(1 + \tau_p^2\Delta^2)^{-1} \ . \tag{3.10}$$

With typical values as $\omega_0 = \omega = 10^{15}$ Hz, $N = 10^{13}$ atoms/cm^3, $p = 10^{-18}$ esu, $\Delta = 0$, and $\tau_p = 10^{-9}$ sec, the above expression leads to $v_0/c \approx 0.01$. Thus the interaction of the light pulse with the atoms slows it down considerably. In terms of the two independent parameters Δ and τ_p, we can express r_0 and κ as

$$r_0 = [1 + (\Delta\tau_p)^2]^{-1} \ , \quad \kappa = \Delta^2 + 1/(\tau_p)^2 \ .$$

Having solved for the amplitudes we can now integrate Eqs. (2.21), (2.22), and (2.33) to find the phases. The phases associated with the excited and ground states, are, respectively

$$\phi_a = \tfrac{1}{4}(\Delta+\Gamma)\xi + c_a \ , \tag{3.11}$$

$$\phi_b = \tfrac{1}{4}(\Delta+\Gamma)\xi + \arctan\left[\frac{1}{\Delta\tau_p}\tanh\left(\frac{\xi}{\tau_p}\right)\right] + c_b \ . \tag{3.12}$$

The constant c_a occurring in the phase ϕ_a of the excited state at the initial instant has no physical significance and for convenience we can choose $c_a = c_b = c_{-\infty}$ to be such that $\phi_b(-\infty) = 0$. The additional phase developed in the electric field is

$$\phi = \tfrac{1}{2}(\Delta - \Gamma)\xi + \phi_0 \ . \tag{3.13}$$

Earlier we had stated that in order to consistently identify ω as the carrier frequency of the pulse we must set any constant term in $\partial_t\phi$ to zero. Doing that for the solution in (3.13) yields $\Gamma = \Delta$ or equivalently the dispersion relation

$$\delta k = \Delta\left(\frac{1}{v_0} - \frac{\eta}{c}\right) . \tag{3.14}$$

Now the expressions for the excited-state amplitude Q_a and the electric field $\mathcal{E}$ show that as $\xi \to \infty$, $Q_a \to 0$, and $\mathcal{E} \to 0$. In other words, after the pulse has left the atoms return to their initial state—the ground state. Thus the pulse takes the atoms through a cyclical evolution. Since the complete solution of the Schrödinger equation has been obtained, we can calculate the dynamical and geometric phases developed by the state vector at the end of this cyclic evolution. The dynamical phase is defined as [2]

$$\theta = -\frac{1}{\hbar}\int_{-\infty}^{\infty}\langle\Psi_r|H_r|\Psi_r\rangle dt , \tag{3.15}$$

this being the generalization of the usual dynamical phase $-\int dt\, E/\hbar$ to the case where the state vector is not an energy eigenstate. Integrating the expectation value of the Hamiltonian over time, we find that at the end of the cyclical evolution, the dynamical phase for any group of atoms is

$$\theta = \frac{2\Delta\tau_p}{1+(\Delta\tau_p)^2} + \lim_{\xi\to\infty}\Delta\xi . \tag{3.16}$$

For atoms on resonance ($\Delta = 0$) the dynamical phase vanishes at the end of the cyclical evolution and also at all intermediate times.

The geometric phase can be found by subtracting the dynamical phase from the total phase change undergone by the state vector. The state vector is

$$|\Psi_r\rangle = Q_a e^{i\phi_a}|e\rangle + Q_b e^{i\phi_b}|g\rangle . \tag{3.17}$$

We have as an initial condition $|\Psi_r(-\infty)\rangle = |g\rangle$ and our choice of the initial phase is $\phi_b(-\infty) = 0$ which corresponds to setting the integration constant $c_{-\infty}$ in ϕ_b to

$$c_{-\infty} = \tfrac{1}{2}\lim_{\xi\to\infty}\Delta\xi + \arctan\left(\frac{1}{\Delta\tau_p}\right) . \tag{3.18}$$

After the pulse has vanished at the location of an atom, the state vector is related to the ground state by a phase,

$$|\Psi_r(\infty)\rangle = e^{i\phi_b(\infty)}|g\rangle , \tag{3.19}$$

where the total phase change in the state vector is [using Eq. (3.12)]

$$\phi_b(\infty) = \lim_{\xi\to\infty}\Delta\xi + 2\arctan\left(\frac{1}{\Delta\tau_p}\right) . \tag{3.20}$$

Hence the geometric phase at the end of the cyclical evolution is

$$\beta \equiv \phi_b(\infty) - \theta \tag{3.21}$$

$$= 2\arctan\left(\frac{1}{\Delta\tau_p}\right) - \frac{2\Delta\tau_p}{1+(\Delta\tau_p)^2} . \tag{3.22}$$

This is one of the main results in the paper. It relates the geometrical phase developed in the ground state of the two-level atom to the detuning Δ and the pulse width τ_p. Figure 1 shows the variation of β with the dimensionless

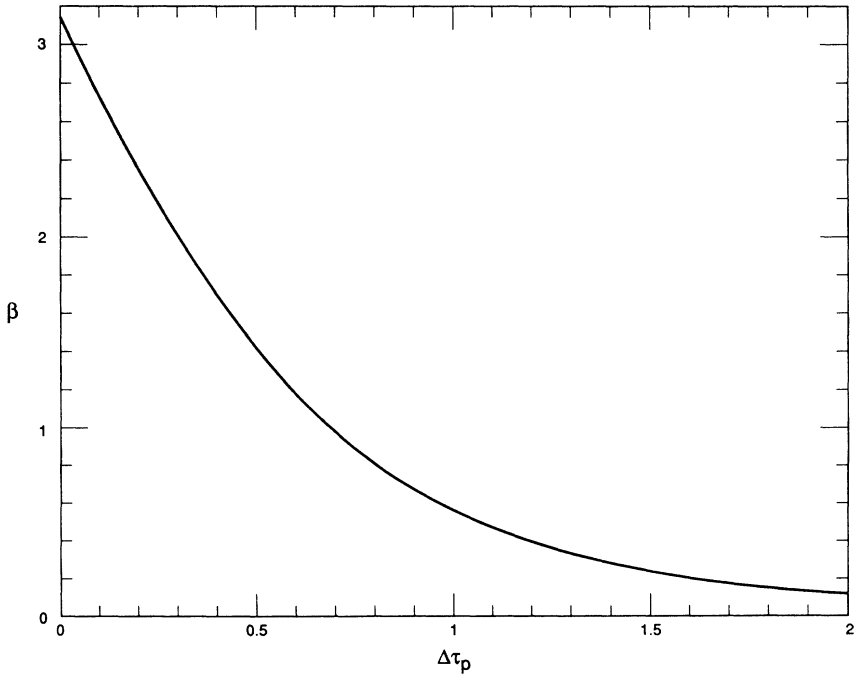

FIG. 1. Variation of the geometric phase β with $\Delta\tau_p$ for the case where the propagating electric field is a sech pulse. Δ is the detuning and τ_p is the pulse width. At resonance ($\Delta = 0$), $\beta(0) = \pi$ and this equals the entire phase change in the ground state of the atom.

quantity $\Delta\tau_p$. If the electric field is exactly at resonance with the atoms, i.e., $\Delta = 0$, then

$$\beta_{\rm res} = \pi \ , \tag{3.23}$$

and this is the entire phase change undergone by the ground state since the dynamical phase is zero for these atoms. This phase change encodes the history of the atom which makes it differ from an atom that has not been excited and deexcited by a pulse.

At resonance ($\Delta = 0$), the tip of the pseudospin vector traces a great circle on the Bloch sphere. As the detuning Δ increases, the size of the loop traced by this vector (and the solid angle subtended by this loop) shrinks correspondingly. The extent of the excursion on the Bloch sphere is a measure of the maximum value of the inversion $w_{\rm max}$. Since

$$w_{\rm max} = \frac{1-(\Delta\tau_p)^2}{1+(\Delta\tau_p)^2}$$

we can relate the geometric phase β to $w_{\rm max}$ as

$$\beta = 2\arctan\left(\sqrt{\frac{1+w_{\rm max}}{1-w_{\rm max}}}\right) - \sqrt{1-w_{\rm max}^2} \ . \tag{3.24}$$

This expression relates the geometric phase developed by the ground state at the end of the cyclical evolution to the maximum excitation, due to energy absorbed from the electric field by the atom during the course of the evolution. Figure 2 shows that β is a monotonically increasing function of $w_{\rm max}$. The above relation between β and the $w_{\rm max}$ is true only in this particular frame rotating with the field, since the value of the geometric phase depends on the particular frame while the inversion is invariant under unitary transformations.

A. Geometry associated with motion on the Bloch sphere

Now we will show that the phase β calculated above is indeed a geometrical quantity. It is straightforward to show that the density matrix can be parametrized by the three components (u, v, w) of the Bloch vector. Consequently, a closed loop in the projective space corresponds to the loop traced by the Bloch vector on the unit sphere (recall that $u^2+v^2+w^2 = 1$ due to conservation of probability). The components of the Bloch vector are

$$u = 2Q_aQ_b\cos\phi_{ba} = \frac{2\Delta\tau_p}{1+(\Delta\tau_p)^2}\,{\rm sech}\frac{\xi}{\tau_p} \ , \tag{3.25}$$

$$v = 2Q_aQ_b\sin\phi_{ba} = \frac{2}{1+(\Delta\tau_p)^2}\tanh\frac{\xi}{\tau_p}\,{\rm sech}\frac{\xi}{\tau_p} \ , \tag{3.26}$$

$$w = Q_a^2 - Q_b^2 = \frac{2}{1+(\Delta\tau_p)^2}\,{\rm sech}^2\frac{\xi}{\tau_p} - 1 \ . \tag{3.27}$$

If we label the polar and azimuthal angles on the sphere by μ and ν, respectively, then we find

$$\mu \equiv \arccos w = \arccos\left(\frac{2}{1+(\Delta\tau_p)^2}\,{\rm sech}^2\frac{\xi}{\tau_p} - 1\right) \ , \tag{3.28}$$

$$\nu \equiv \phi_{ba} = \arctan\left(\frac{1}{\Delta\tau_p}\tanh\frac{\xi}{\tau_p}\right) \ . \tag{3.29}$$

The equation of the loop defines the polar angle μ to be a function of the azimuthal angle ν and ν_0 is the initial azimuthal angle determined by our choice of the initial phase $\phi_b(-\infty)$ to be $\nu_0 = -\arctan[1/(\Delta\tau_p)]$. This is a mathematical value picked by continuity of the function ν defined in (3.29), the actual physical value of the initial

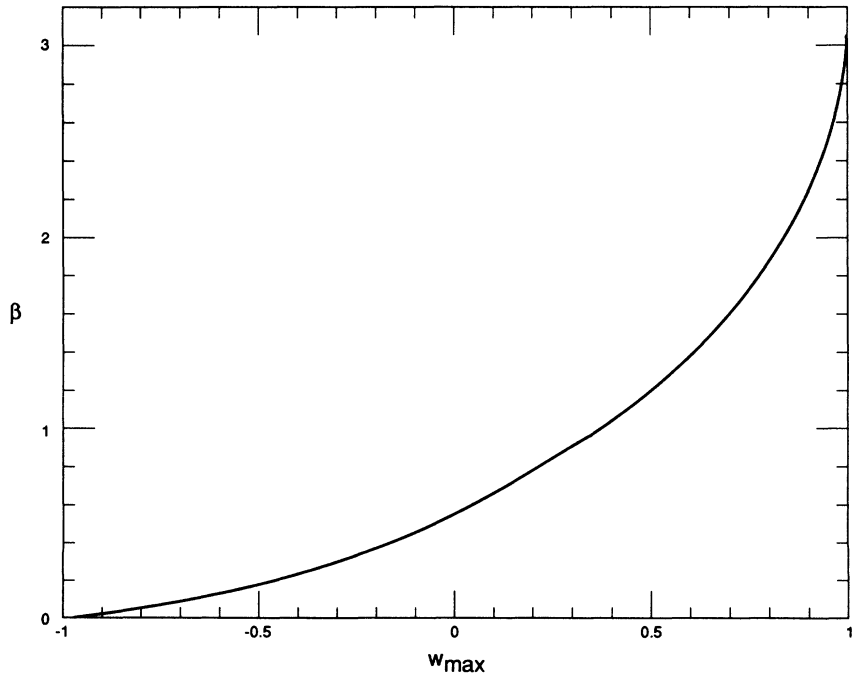

FIG. 2. The geometric phase β as a function of $w_{\rm max}$, the maximum value of the atomic inversion, for the sech pulse. Since this is an invertible function, β can be obtained from a measurement of $w_{\rm max}$ or vice versa.

azimuthal angle being undefined since the initial Bloch vector is along the negative $\hat{e}_3$ axis.

The solid angle subtended by the closed loop is

$$\Omega = \int_{\nu_0}^{-\nu_0} d\nu \int_{\pi}^{\mu(\nu)} \sin\mu \, d\mu \tag{3.30}$$

$$= -4\arctan\left(\frac{1}{\Delta\tau_p}\right) + \frac{4\Delta\tau_p}{1+(\Delta\tau_p)^2} \, . \tag{3.31}$$

Hence on comparing (3.22) and (3.31) we find that the geometric phase is related to the solid angle as

$$\beta = -\tfrac{1}{2}\Omega \, . \tag{3.32}$$

It is interesting to observe that precisely the same solution for the Bloch vector as given in (3.25)–(3.27) is obtained by making the factorization assumption (2.30) when inhomogeneous broadening is included [8]. This implies that the closed loop traced on the Bloch sphere would be identical leading to the same solid angle and hence the same geometric phase. This suggests then that the geometric phase is relatively insensitive to the presence of inhomogeneous broadening. At present this is only a conjecture and would have to be either confirmed or disproved by detailed calculations or experiment.

The closed curve traced on the Bloch sphere can be characterized by the area it encloses and its circumference. We have seen that the enclosed area (here equal in magnitude to the solid angle subtended) is related to the geometric phase developed in the state vector. Does the length along the path of the loop or the rate at which it is traversed give any information about physical observables? Indeed Anandan and Aharonov have recently shown [17] that the speed on the projective space is related to the uncertainty in energy as

$$\frac{ds}{dt} = \frac{2}{\hbar}\Delta E \, . \tag{3.33}$$

Here ds^2 is the Fubini-Study metric on the projective Hilbert space given by

$$ds^2 = d\langle\Psi|\, d|\Psi\rangle$$

or equivalently may be defined as the difference from unity of the squared modulus of the overlap of two neighboring states. The energy uncertainty squared is defined as usual to be the variance of the Hamiltonian,

$$\Delta E^2 = \langle H^2\rangle - \langle H\rangle^2 \, .$$

From (3.33) it follows that the time integral of the uncertainty in energy for any quantum evolution is the same for all systems which trace out the same curve on the projective Hilbert space.

We can provide an explicit test of this claim by calculating both sides of Eq. (3.33) independently. The projective space for our two-level system is the unit Bloch sphere for which the Fubini-Study metric reduces to the natural metric (see, e.g., Ref. [16])

$$ds^2 = d\mu^2 + (\sin^2\mu)\, d\nu^2 \, . \tag{3.34}$$

Using our results above for the polar and azimuthal angles, we obtain the speed along the curve traced by the Bloch vector on the Bloch sphere to be

$$\frac{ds}{d\xi} = \frac{2}{\tau_p[1+(\Delta\tau_p)^2]}\operatorname{sech}\frac{\xi}{\tau_p}\sqrt{1+(\Delta\tau_p)^2\tanh^2\frac{\xi}{\tau_p}} \, . \tag{3.35}$$

In Fig. 3 we show the variation of the velocity as a function of the dimensionless time ξ/τ_p for three different values of $\Delta\tau_p$. If ξ^* denotes the instant at which the velocity reaches a maximum, a plot of ξ^* versus $|\Delta\tau_p|$ will show a bifurcation at $|\Delta\tau_p| = 1.0$. For $|\Delta\tau_p| \le 1.0$, $\xi^* = 0.0$ while for $|\Delta\tau_p| > 1.0$ there are two values of ξ^*, these being the solutions of $\operatorname{sech}^2\xi^*/\tau_p = \frac{1}{2} + \frac{1}{2}(\Delta\tau_p)^2$.

We can also calculate the energy uncertainty. In calculating the dynamical phase we found the expectation value of the Hamiltonian to be

$$\langle H_r\rangle = \frac{\hbar\Delta}{2}w - p\mathcal{E}u \tag{3.36}$$

while the expectation value of H_r^2 is

$$\langle H_r^2\rangle = \left(\frac{\hbar\Delta}{2}\right)^2 + (p\mathcal{E})^2 \, . \tag{3.37}$$

Substituting the expressions for u, w, and $\mathcal{E}$, it is seen that $(2/\hbar)\Delta E_r$ agrees exactly with the speed $ds/d\xi$ calculated above. This system thus provides a nontrivial verification of the claim (3.33) made in [17]. With this identification, the variation of the energy uncertainty with time is also shown by Fig. 3. At first glance it might appear strange that the energy uncertainty for $\Delta\tau_p \le 1$ is a maximum at $\xi = 0.0$ which is where the inversion is also a maximum. This is easily resolved once we recall that for a time-dependent system the instantaneous energy eigenstates are also time dependent and the inversion as calculated here is between the unperturbed energy eigenstates. Let the higher and lower instantaneous eigenstates of the Hamiltonian be denoted $|+\rangle$ and $|-\rangle$, respectively, with corresponding eigenenergies E_+ and E_-. With the Hamiltonian given by (2.4) we find

$$E_\pm(t) = \pm\sqrt{p^2\mathcal{E}^2 + \left(\frac{\hbar\Delta}{2}\right)^2} = \pm\sqrt{\langle H_r^2\rangle} \, .$$

If $P_+(t)$ is the probability of being in the state $|+\rangle$, then the energy uncertainty as a function of time is given by

$$\Delta E^2(t) = 4E_+^2(t)P_+(t)[1 - P_+(t)] \, . \tag{3.38}$$

Consequently the time at which ΔE reaches its maximum depends on the variation of both E_+ and P_+ with time. By contrast, for a time-independent Hamiltonian where E_+ is independent of time, ΔE is a maximum when $P_+ = \frac{1}{2}$, i.e., when the inversion is zero. Comparing Eq. (3.33) with (3.38), one might say (in a rough sense) that the velocity on the projective Hilbert space keeps track of both the time variation in the eigenenergies and the transition rate between the two instantaneous eigenstates.

We can also use the above expression for the speed along the curve to calculate the circumference C of the closed loop traced by the Bloch vector at the end of a

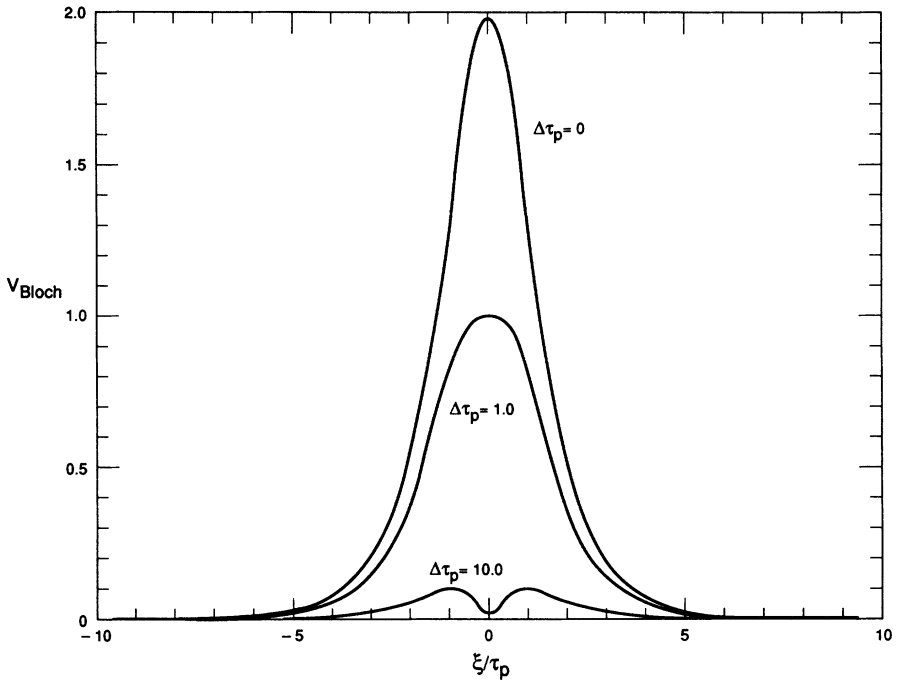

FIG. 3. Velocity $[v_{\text{Bloch}} = ds/d(\xi/\tau_p)]$ on the Bloch sphere of the ray representative of the state vector as a function of the dimensionless "time" ξ/τ_p for different values of $\Delta\tau_p$. By relation (3.33) this figure also represents the variation of the energy uncertainty with time.

cyclical evolution. We obtain

$$C = \oint ds = \frac{4}{\sqrt{1+(\Delta\tau_p)^2}}\mathfrak{E}\left(\frac{\Delta\tau_p}{\sqrt{1+(\Delta\tau_p)^2}}\right) , \quad (3.39)$$

where $\mathfrak{E}$ is the complete elliptic integral of the second kind. At resonance with $\Delta = 0$, the circumference reduces to 2π, i.e., the Bloch vector traces out a great circle on the sphere. Figure 4 shows that the circumference C decreases as the parameter $\Delta\tau_p$ increases. Figures of the closed loops traced by the Bloch vector for various values of $\Delta\tau_p$ can be found in the original paper of McCall and Hahn [8].

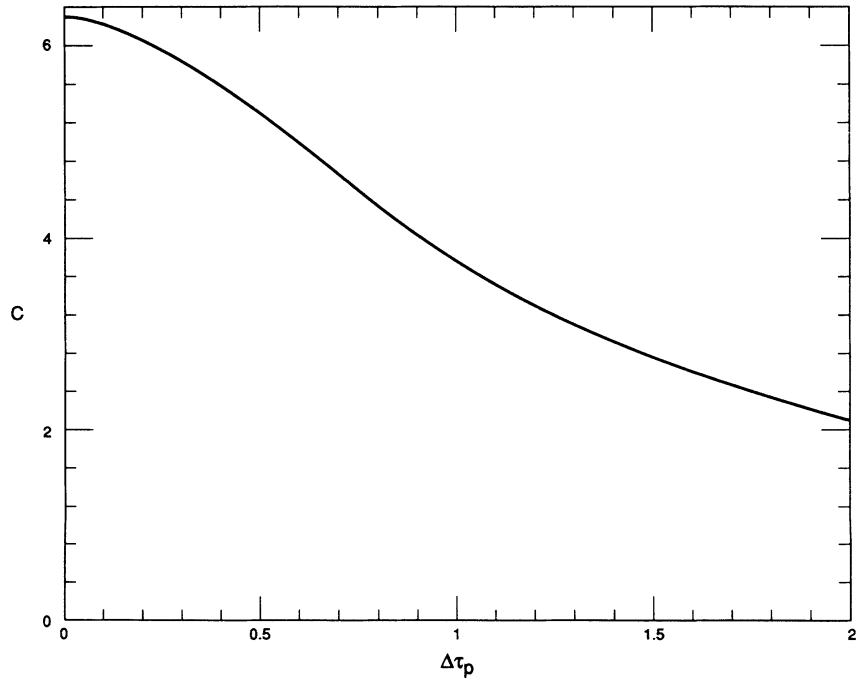

FIG. 4. The circumference C of the closed loop traversed on the Bloch sphere when the atom returns to the ground state, as a function of $\Delta\tau_p$. At resonance, $C(0) = 2\pi$ showing that the closed loop is a great circle.

Thus from the calculations above, we see that the geometry of the curve on the Bloch sphere, namely, the rate at which it is traversed and the solid angle subtended by it (for a closed curve), contains information about physical observables of the system. The study of the underlying geometry is particularly useful in cases where we are able to extract these geometric quantities from the symmetries of the system alone and without having to solve the Schrödinger equation ([18, 19]).

IV. CONTINUOUS PULSE TRAINS

Apart from the single pulse solution found in Sec. III, infinite pulse trains may also propagate through the medium if we choose the initial conditions differently. Here we take the initial-state vector to be a superposition of the ground state and the excited state. Our initial conditions on the amplitudes and the electric field are

$$Q_a^2(\xi_0) = \sin^2\frac{\lambda}{2}\ , \quad Q_b^2(\xi_0) = \cos^2\frac{\lambda}{2}\ , \quad \mathcal{E}(\xi_0) = 0\ ,$$

where ξ_0 denotes the initial value of the independent variable while the initial-state vector is

$$|\Psi_r(\xi_0)\rangle = \sin\frac{\lambda}{2}e^{i\phi_a(\xi_0)}|e\rangle + \cos\frac{\lambda}{2}e^{i\phi_b(\xi_0)}|g\rangle\ , \quad (4.1)$$

where $\phi_a(\xi_0)$ and $\phi_b(\xi_0)$ are the initial phases of the excited and ground states, respectively. A calculation of the Bloch vector components shows that λ is the angle between the initial Bloch vector and the negative $\hat{e}_3$ axis. Much of the development in this section is similar to that in Sec. III. What is qualitatively different is that the state vector undergoes a cyclical evolution only for a particular set of initial parameters.

Defining the variable

$$r(\xi) = Q_a^2(\xi) - Q_a^2(\xi_0) = Q_a^2 - \sin^2\frac{\lambda}{2} \quad (4.2)$$

and using the conservation laws we find that $r(\xi)$ obeys

$$\frac{dr}{d\xi} = -2\sqrt{\chi(r)}\ , \quad (4.3)$$

where $\chi(r)$ is a cubic polynomial

$$\chi(r) = \kappa r(r_+ - r)(r - r_-) \quad (4.4)$$

whose roots are

$$r_\pm = \frac{1}{2}[\zeta \pm \sqrt{\zeta^2 + \sin^2\lambda}]\ , \quad \zeta = \cos\lambda - \frac{(\Delta+\Gamma)^2}{4\kappa}\ .$$

Since $\chi(r)$ has negative slope at both $r = r_+$ and $r = r_-$ and positive slope at the origin, it follows that $\chi(r)$ has the shape shown in Fig. 5. Furthermore, from (4.3) it follows that the region of physical interest is where $\chi(r) \geq 0$. In addition, from the equality $r = p\mathcal{E}^2/\hbar\alpha_s$, we can exclude the possibility of r being negative since α_s is a positive parameter. The region of interest is thus restricted to $0 \leq r \leq r_+$. Integrating the equation for $r(\xi)$ we obtain the solution

$$r(\xi) = r_+ \mathrm{cn}^2[\sqrt{\kappa(r_+ - r_-)}(\xi - \xi_0) + K(l), l]\ , \quad (4.5)$$

where cn is one of the Jacobi elliptic functions and $K(l)$ is the complete elliptic integral of the first kind with modulus $l = \sqrt{r_+/(r_+ - r_-)}$. We obtain the amplitudes for the ground and excited states and for the electric field from the expression for r. The phases too are found by using the solution for r. It is straightforward to find the

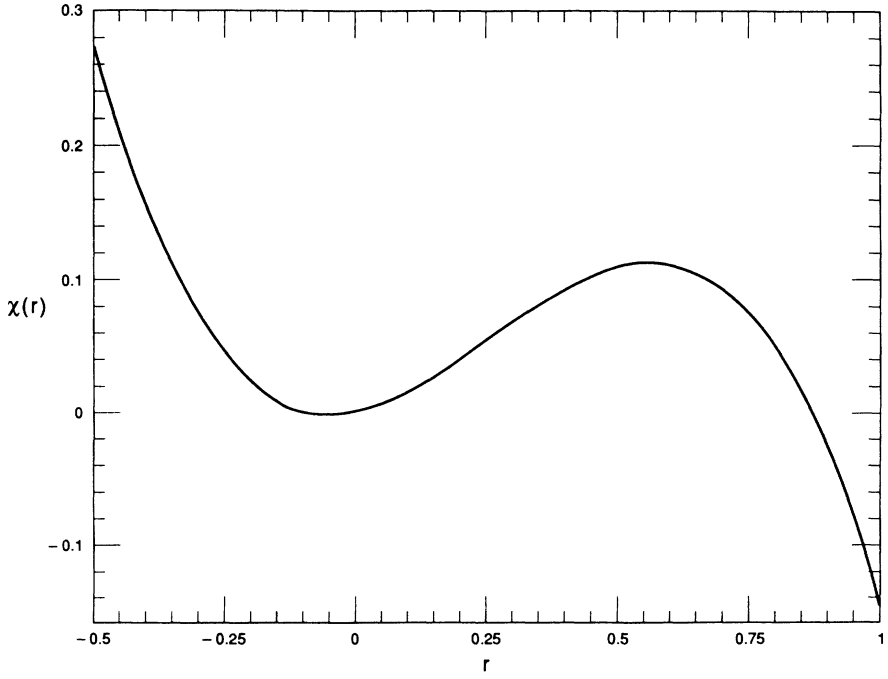

FIG. 5. Plot of the cubic polynomial $\chi(r)$, given by Eq. (4.4), as a function of the variable r defined in Eq. (4.2). $\chi(r) = 0$ at $r = r_-$, 0, r_+.

phase in the electric field to be

$$\phi = \tfrac{1}{2}(\Delta - \Gamma)\xi + \phi_0 \ . \tag{4.6}$$

As in Sec. III, setting the constant in $\partial_t\phi$ to zero yields $\Gamma = \Delta$ or equivalently the dispersion relation

$$\delta k = \Delta\left(\frac{1}{v_0} - \frac{\eta}{c}\right) \ . \tag{4.7}$$

This expression for δk is the same as that obtained in Sec. III as it should be, since the dispersion relations do not depend on the initial conditions. For the phases of the excited and ground states we obtain the following results:

$$\phi_a(\xi) = \frac{\Delta}{\sqrt{\kappa(r_+ - r_-)}}(q_a^2 - 1)[\Pi(\text{am}\ U, q_a^2, l)]_{K(l)}^{U} + \frac{\Delta}{2}(\xi - \xi_0) + \phi_a(\xi_0) \ , \tag{4.8}$$

$$\phi_b(\xi) = \frac{\Delta}{\sqrt{\kappa(r_+ - r_-)}}(1 - q_b^2)[\Pi(\text{am}\ U, q_b^2, l)]_{K(l)}^{U} - \frac{\Delta}{2}(\xi - \xi_0) + \phi_b(\xi_0) \ . \tag{4.9}$$

$\Pi(\text{am}\ U, q^2, l)$ is the incomplete elliptic integral of the third kind and is a function of the variable

$$U = \sqrt{\kappa(r_+ - r_-)}(\xi - \xi_0) + K(l)$$

and the constants

$$q_a^2 = \frac{r_+}{r_+ + \sin^2\frac{\lambda}{2}} > 0 \ , \quad q_b^2 = -\frac{r_+}{\cos^2\frac{\lambda}{2} - r_+} < 0 \ .$$

am U is the amplitude function defined as am U=arcsin(sn U). From Eq. (4.5) it follows that the amplitudes are

$$Q_a^2 = r_+\text{cn}^2(U,l) + \sin^2\frac{\lambda}{2} \ , \tag{4.10}$$

$$Q_b^2 = \cos^2\frac{\lambda}{2} - r_+\text{cn}^2(U,l) \ , \tag{4.11}$$

$$\mathcal{E}^2 = \frac{\hbar^2}{p^2}\kappa r_+\text{cn}^2(U,l) \ . \tag{4.12}$$

We introduce the parameter τ as a measure of the pulse width of the electric field. Expressed in terms of the other parameters

$$\tau = \frac{1}{\sqrt{\kappa(r_+ - r_-)}} \ . \tag{4.13}$$

The electric field can then be written as

$$\mathcal{E} = \mathcal{E}_m\ \text{cn}\left(\frac{\xi - \xi_0}{\tau} + K(l), l\right) \ . \tag{4.14}$$

$\mathcal{E}_m$ is the maximum value of the electric field which depends on the initial parameters as

$$\mathcal{E}_m = \frac{\hbar}{\sqrt{2}p\tau}\sqrt{1 - (\Delta\tau)^2\sin^2\lambda + \cos\lambda\sqrt{1 - (\Delta\tau)^4\sin^2\lambda}} \ . \tag{4.15}$$

An arbitrarily shaped pulse injected into the atomic medium satisfying the chosen initial conditions will eventually settle down into the pulse shape given by Eq. (4.14) after traveling through several absorption lengths. The velocity v_0 of the pulse can also be expressed in terms of the initial angle λ , the detuning Δ, and the pulse width τ as

$$\frac{1}{v_0} = \frac{\eta}{c} + \frac{4\pi N\omega_0^2 p^2\tau^2}{c\hbar\omega\eta}\left[(\Delta\tau)^2\cos\lambda + \sqrt{1 - (\Delta\tau)^4\sin^2\lambda}\right]^{-1} \ . \tag{4.16}$$

As functions of λ, both the amplitude $\mathcal{E}_m$ and the velocity v_0 are symmetric about $\lambda = \pi$ and decrease as the initial excitation increases or as λ increases from 0 to π. At the particular value $\lambda = \pi$, $\mathcal{E}_m$ vanishes, i.e., with all atoms initially in the excited state there is no interaction between the injected field and the atoms in this semiclassical theory, and the quantum state amplitudes Q_a and Q_b stay at their initial values 1 and 0, respectively.

Let us examine the evolution of the atomic amplitudes Q_a and Q_b. Since $\text{cn}^2(U,l)$ is a periodic function with a real period $2K(l)$ the amplitudes will return to their initial values at

$$\xi_{P_n} = \xi_0 + 2nK(l)\tau \ , \tag{4.17}$$

where n is an integer. For cyclical evolution the Bloch vector s must be periodic. The atomic inversion w is certainly periodic at intervals of $2K(l)\tau$ but the transverse components u, v are periodic only if the relative phase $\phi_{ba} = \phi_b - \phi_a$ is periodic. In order for that to happen, the initial parameters $(\lambda, \Delta\tau)$ may only take on certain values. Let us denote

$$C_a^{(n)} = \phi_a(\xi_{P_n}) - \phi_a(\xi_0) \ , \quad C_b^{(n)} = \phi_b(\xi_{P_n}) - \phi_b(\xi_0) \ .$$

The state vector at $\xi = \xi_{P_n}$ may be written as

$$|\Psi_r(\xi_{P_n})\rangle = e^{iC_a^{(n)}}\left(e^{i\phi_a(\xi_0)}\sin\frac{\lambda}{2}|e\rangle + e^{i(C_b^{(n)} - C_a^{(n)})}e^{i\phi_b(\xi_0)}\cos\frac{\lambda}{2}|g\rangle\right) \ . \tag{4.18}$$

If at $\xi = \xi_{P_n}$ the components of the transverse polarization return to their initial values then $|\Psi_r(\xi_{P_n})\rangle$ is related to $|\Psi_r(\xi_0)\rangle$ by a phase. That occurs if the following condition is satisfied:

$$C_b^{(n)} - C_a^{(n)} = \phi_{ba}(\xi_{P_n}) - \phi_{ba}(\xi_0) = 0 \mod 2\pi \ . \tag{4.19}$$

Using the relation

$$\Pi\left(n\pi + \frac{\pi}{2}, q^2, l\right) = (2n+1)\Pi\left(\frac{\pi}{2}, q^2, l\right) = (2n+1)\Pi(q^2, l)$$

[obtained from Eq. (110.04) in [20]] where $\Pi(q^2, l)$ is the complete elliptic integral of the third kind, the above

condition (4.19) for cyclical evolution may be written as

$$2n\Delta\tau[(1-q_b^2)\Pi(q_b^2,l)+(1-q_a^2)\Pi(q_a^2,l)+K(l)] = 0 \mod 2\pi . \tag{4.20}$$

This equation expresses the fact that a chosen value of the initial angle λ allows only certain values of the quantity $\Delta\tau$ for which the state vector undergoes a cyclical evolution. We solve this equation numerically to find the dimensionless parameter $\Delta\tau$ for a given λ. Figure 6 is a plot of these solutions of the cyclical evolution condition as a function of λ. As we expect, these solutions are symmetric about $\lambda = \pi$; the initial populations and the evolution of the system both possess this same symmetry. Cyclical evolution in this system comes about not by taking the parameters through a closed loop (as is the case for Berry's phase), but by choosing the initial values to lie on the one-dimensional curve shown in Fig. 6.

We also examine how the relative phase $\phi_{ba}(\xi)$ evolves with time for a particular choice of the parameters $(\lambda, \Delta\tau)$ lying on this curve. From (4.8) and (4.9) it follows that

$$\phi_{ba}(\xi) - \phi_{ba}(\xi_0) = \Delta\tau \left([(1-q_b^2)\Pi(\mathrm{am}\, U, q_b^2, l) + (1-q_a^2)\Pi(\mathrm{am}\, U, q_a^2, l)]_{K(l)}^{U} - \frac{\xi-\xi_0}{\tau} \right) . \tag{4.21}$$

Figure 7 shows the variation of $\phi_{ba}(\xi) - \phi_{ba}(\xi_0)$ with the dimensionless scaled variable $(\xi-\xi_0)/\tau$ for the arbitrarily chosen value of $\lambda = \pi/10$. Plots for other values of λ are similar. Superimposed on the linear variation (with slope $-\Delta\tau$) is the behavior due to the elliptic integrals. This relative phase difference goes through integral multiples of 2π at various times and at each of these times the Bloch vector completes a closed loop on the Bloch sphere.

If $\lambda = 0$, i.e., we return to the case dealt with in Sec. III, the condition for cyclical evolution becomes irrelevant since the amplitude for the excited state vanishes at the end of the evolution. If this condition is fulfilled for $\lambda \neq 0$ then the total phase change after n periods is

$$C_a^{(n)} = n\Delta\tau[-2(1-q_a^2)\Pi(q_a^2,l) + K(l)] . \tag{4.22}$$

The dynamical part of this total phase is evaluated as before:

$$\theta(\xi_{P_n}) = -\frac{1}{\hbar}\int_{\xi_0}^{\xi_{P_n}} \langle\Psi_r|H_r|\Psi_r\rangle d\xi . \tag{4.23}$$

The expectation value of the Hamiltonian in terms of the variable r defined in (4.2) is

$$\langle\Psi_r|H_r|\Psi_r\rangle = -\frac{\hbar}{2}\Delta(2r+\cos\lambda) . \tag{4.24}$$

Integrating the above expression and using the relation

$$\mathfrak{E}\left(n\pi+\frac{\pi}{2},l\right) = (2n+1)\mathfrak{E}\left(\frac{\pi}{2},l\right) = (2n+1)\mathfrak{E}(l)$$

[Eq. (113.02) in [20]] where $\mathfrak{E}(l)$ is the complete elliptic integral of the second kind, we find the dynamical phase to be

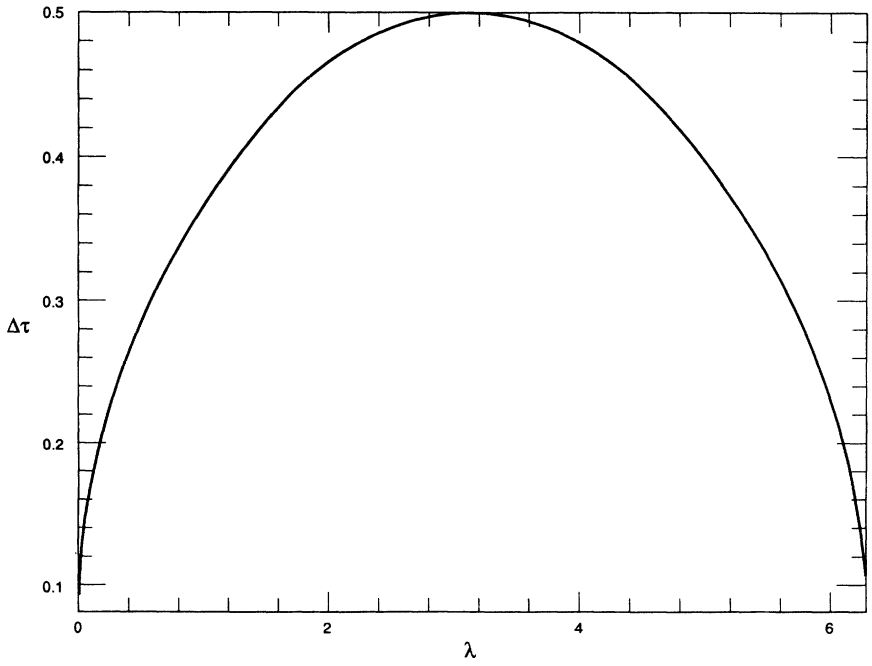

FIG. 6. This curve determines the value of $\Delta\tau$ for a given λ, the initial angle made by the Bloch vector with the negative $\hat{e}_3$ axis, in order that the evolution be cyclic when the initial state is not the ground state. It is obtained as the numerical solution of the condition (4.20).

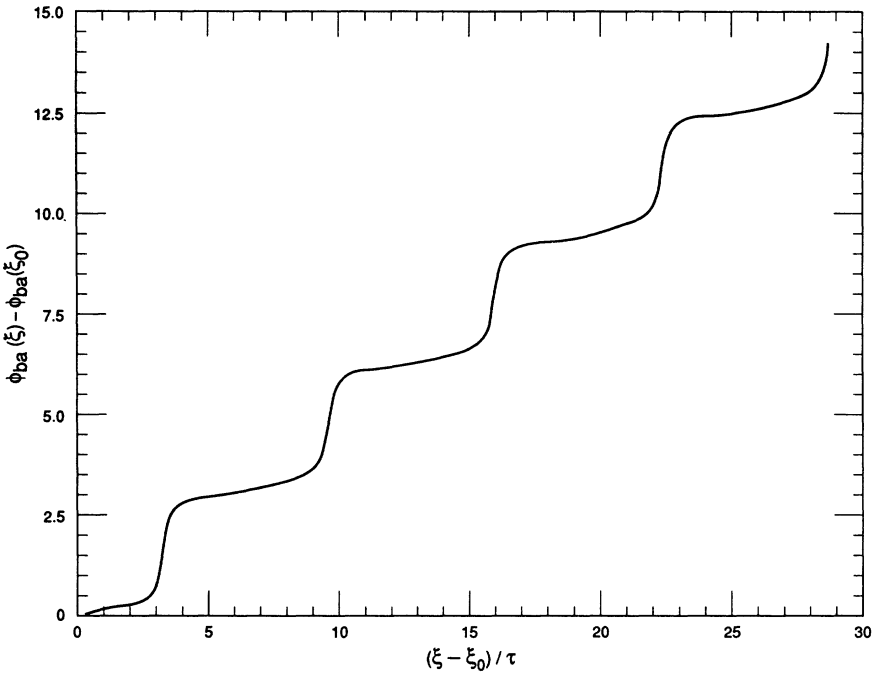

FIG. 7. Evolution of the relative phase change $\phi_{ba}(\xi) - \phi_{ba}(\xi_0)$, where ϕ_{ba} is the phase of the ground state relative to the excited state, with the dimensionless "time" variable $\frac{\xi-\xi_0}{\tau}$ for $\lambda = \pi/10$. When $\phi_{ba}(\xi) - \phi_{ba}(\xi_0) = 2n\pi$, the atomic system completes a cyclic evolution.

$$\theta(\xi_{P_n}) = 2n\Delta\tau[(r_+ - r_-)\mathfrak{E}(l) + (r_- + \tfrac{1}{2}\cos\lambda)K(l)] . \tag{4.25}$$

Hence, the geometrical phase is obtained by subtracting off the dynamical phase from the total phase:

$$\begin{aligned}\beta_n &= C_a^{(n)} - \theta(\xi_{P_n}) \\ &= -2n\Delta\tau[(1-q_a^2)\Pi(q_a^2,l) + (r_+ - r_-)\mathfrak{E}(l) \\ &\quad + \left(r_- - \sin^2\frac{\lambda}{2}\right)K(l)] .\end{aligned} \tag{4.26}$$

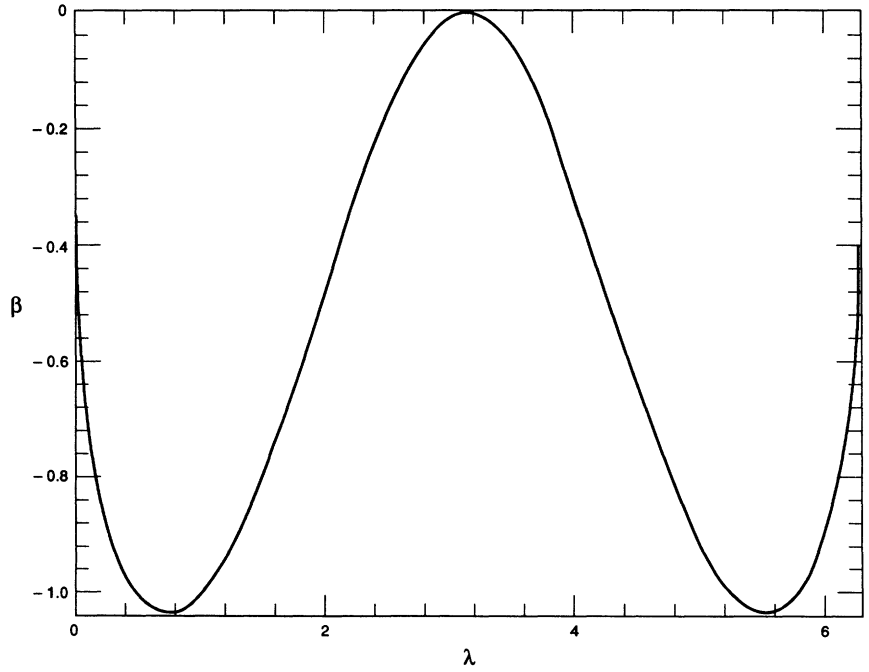

FIG. 8. Variation of the geometric phase with λ. The corresponding value of $\Delta\tau_p$ must be found from Fig. 6. β vanishes when $\lambda = \pi$, i.e., when the atoms are initially in the excited state. When $\lambda = 0$ this figure is not applicable but instead the results of Sec. III hold.

We list the expressions for the various symbols occurring in the above equation in terms of the parameters $\Delta\tau$ and λ in Appendix A. Figure 8 shows the variation of β with λ. It is also symmetric about $\lambda = \pi$ at which value the geometric phase vanishes. This corresponds to the state vector remaining constant at all times, which is the expected behavior if all the atoms are initially in the excited state . At $\lambda = 0 \pmod{2\pi}$ the considerations of Sec. III hold with β varying as a smooth function of $\Delta\tau$. Between the values of $\lambda = 0$ and $\lambda = \pi$ the absolute value of β reaches a maximum at $\lambda \cong 0.76$ which corresponds to the maximum inversion that can be achieved during the course of a cyclical evolution with $\lambda \neq 0$.

We can prove by an explicit calculation that this phase β_n is minus half of the solid angle subtended by the loop. The method is similar to that in Sec. III and the details of this calculation may be found in Appendix B.

In both this and Sec. III we have chosen the amplitude of the electric field to be zero initially. As a consequence the slowly varying phase ϕ of the electric field did not develop in time. We point out here that the electric field will be chirped if we choose nonzero values for the initial amplitude of the electric field. In addition, Lamb [21] has obtained multipulse solutions which do not preserve their shape but still propagate without loss. Thus there exist other possible solutions for the amplitudes and phases in this two-level system but we do not consider them in this work.

V. OBSERVING THE PHASES

Here we consider the question of observing the phases associated with the state experimentally. The state vector for the two-level system at any time may be written as

$$|\Psi_r(\xi)\rangle = e^{i\phi_a}[Q_a(\xi)|e\rangle + e^{i\phi_{ba}}Q_b(\xi)|g\rangle]. \tag{5.1}$$

The relative phase difference ϕ_{ba} between the eigenstates can be known by measuring the components (u, v) of the polarization vector, since

$$\phi_{ba} = \arctan\left(\frac{v}{u}\right) .$$

However the overall phase would appear to be unobservable since it disappears in calculating the expectation value of any physical quantity. As was pointed out by Bouchiat and Gibbons [22], the overall phase after a cyclical evolution is observable either if the state vector above represents only some of the degrees of freedom and is made to interfere with an identical system whose corresponding degrees of freedom have evolved unchanged, or by subsuming the system as part of a larger system. Indeed the first experiment (by Suter, Mueller, and Pines [4]) to report the observation of the Aharonov-Anandan phase opted for the second scheme by considering a two-level system as part of a three-level system. They studied the NMR transitions of a spin-1 system with three levels 1,2,3 and experimentally observed the geometric phase associated with the cyclic evolution of the subsystem 2-3 by its effect on the magnetization of 1-2. We will consider the optical analog of that experiment, the crucial difference here being that the cyclical evolution of 2-3 will be governed by its own dynamics and not externally controlled.

When considering three-level atomic systems, there are three possible configurations known as the cascade, V, and Λ configurations, all of which have level 2 in common. These are shown in Fig. 9. For definiteness we will consider only the phases arising in the cascade configuration by the propagation of sech pulses in what follows. The other two configurations and the phases due to continuous pulse trains may also be studied but we do not consider them here.

The basic idea as enunciated, for example, by Stoll, Wolff, and Mehring [23] is as follows. Assuming that our three-level system starts in the ground state $|1\rangle$, we prepare a coherent superposition of states $|1\rangle$ and $|2\rangle$ by applying a $\pi/2$ pulse resonant with the 1-2 transition. Now if a pulse resonant with the 2-3 transition (we have in mind a SIT pulse) propagates through the system, it will create a superposition of states $|2\rangle$ and $|3\rangle$ causing the population and phase of $|2\rangle$ to change relative to that of $|1\rangle$. A measurement of the 1-2 transverse polarization (or equivalently of the off-diagonal elements in the 1-2 block of the 3×3 density matrix) yields information about the phase change induced in the 2-3 subsystem by the propagating soliton pulse. Since we are considering the case of sharp-line self-induced transparency, all spectral lines are assumed to be only homogeneously broadened. Consequently the decoherence of the 1-2 pulse due to inhomogeneous broadening can be neglected. With linewidths around 10 MHz, the decay time for the 1-2 pulse is close to 30 ns. Typical pulse lengths for SIT pulses are $\sim$5 ns. Following the passage of the SIT pulse, we inject another $\pi/2$ pulse to the 1-2 transition. This pulse creates an echo pulse at a later time, this echo pulse is then measured with a phase-sensitive detector to extract the total phase change undergone by state $|2\rangle$. The sequence of pulses is shown in Fig. 10.

To see how the phase change in the 2-3 subsystem is extracted from the echo pulse we provide a brief sketch of the analysis. The results given below are derived in Appendix C. We start initially with all atoms in the ground state $|1\rangle$. The first $\pi/2$ pulse distributes the atoms equally among levels $|1\rangle$ and $|2\rangle$. Following this pulse at some time t_1, the state vector can be written as

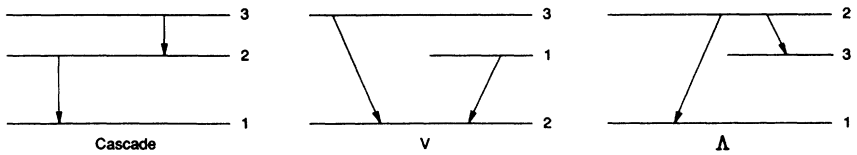

FIG. 9. All possible configurations of optical transitions in a three-level atom.

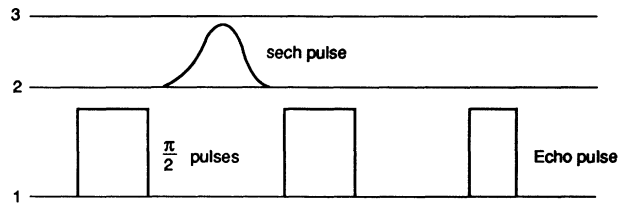

FIG. 10. Sequence of pulses applied to the 1-2 and 2-3 transitions in a cascade configuration. Time increases along the horizontal axis to the right.

$$|\Psi\rangle(t_1) = \frac{1}{\sqrt{2}}(e^{i\phi_1(t_1)}|1\rangle + |2\rangle) , \tag{5.2}$$

where $\phi_1(t_1)$ is an arbitrary phase of state $|1\rangle$. Next the soliton pulse is applied to the 2-3 levels. We observe the polarization of the 1-2 levels in a frame rotating with the frequency of this pulse. If T is the period of this pulse, then following its passage, the transverse polarization components of the two-level subsystem 1-2 are

$$\langle P_x\rangle^{1-2}(t_1+T) = c\cos[\phi_2(t_1+T) - \phi_1(t_1)] ,$$
$$\langle P_y\rangle^{1-2}(t_1+T) = c\sin[\phi_2(t_1+T) - \phi_1(t_1)] , \tag{5.3}$$

where $\phi_2(t_1+T)$ is the total phase change undergone by state $|2\rangle$ and c is a constant. In principle, by measuring the free induction decay of the 1-2 levels, the total phase change $\phi_2(t_1+T) - \phi_1(t_1)$ of state $|2\rangle$ with respect to the arbitrary initial phase of $|1\rangle$ can be obtained. In practice however, homogeneous broadening effects will cause a loss of the 1-2 polarization during the propagation of the 2-3 pulse. To counteract this, another $\pi/2$ pulse may be applied to the 1-2 transition to equalize their populations and thus enhance the signal. This pulse is represented by the field

$$\mathbf{E}(t_2+t) = 2\mathcal{E}_0\cos(\omega_1 t + \alpha)\hat{\mathbf{x}} , \tag{5.4}$$

where $\mathcal{E}_0$ is a constant amplitude field which lasts for an interval τ and we have added a phase shift of α to the field. $t_2 \geq t_1 + T$ is the instant at which this pulse is applied. We find then that the echo signal following the pulse has the following polarization components:

$$\langle P_x\rangle^{1-2}(t_2+\tau) = c\cos[\phi_2(t_1+T) - \phi_1(t_1) + \alpha]\cos\alpha ,$$
$$\langle P_y\rangle^{1-2}(t_2+\tau) = c\sin[\phi_2(t_1+T) - \phi_1(t_1) + \alpha]\sin\alpha . \tag{5.5}$$

The phase α can be selected so as to separate the echo pulse spatially from the applied pulses. In particular, with $\alpha = \pi/2$, the 1-2 polarization pulse is along the y direction. This pulse contains the information about the phase change $\phi_2(t_1+T) - \phi_1(t_1)$ undergone by level 2 during the passage of the soliton pulse.

VI. SUMMARY

The phenomenon of self-induced transparency is ideally suited for studying various aspects of geometric phases. In this paper we have focused on the Abelian quantum geometric phase developed in two-level atoms as a consequence of their interaction with a classical electric field which is shaped by the atoms while propagating through the optical medium. This system is somewhat distinct from other systems considered in the literature on geometric phases in that the entire dynamics is self-consistently determined and not driven by the variation of external parameters. As a consequence, both the atomic variables and the electric field vary on the same time scale and the geometric phase is genuinely an Aharonov-Anandan phase. There is no limit under which it reduces to the Berry phase.

Under the assumption that all damping processes can be neglected, the system is exactly integrable. The shape of the propagating electric pulse depends crucially on the initial conditions chosen for the atomic state amplitudes. We considered two initial conditions. In one case where initially the atoms are in the ground state, the propagating electric field is a single pulse with a secant hyperbolic form. Cyclical evolution in the atomic system comes about naturally since this pulse returns the atom to their ground state following its passage. In the second case where the initial state is partially excited out of the ground state, continuous pulse trains are obtained as the analytical solutions for the propagating electric field. Here cyclical evolution must be brought about by choosing the parameters to take values lying on a particular curve in the parameter space. In both cases we showed that the atomic state acquires a geometric phase which depends on the detuning and the pulse width. An explicit calculation shows that this geometric phase is one-half the solid angle subtended on the Bloch sphere, thus justifying the apellation of *geometric*. In turn this relation is used to relate the geometric phase to the maximum value of the atomic inversion. Another physically meaningful quantity, the energy uncertainty, was related to the velocity along the curve on the Bloch sphere, in accordance with the result of Anandan and Aharonov [17]. We believe therefore that in cases where a solution of Schrödinger's equation is not available, study of the geometry of the curves on the projective Hilbert space by more formal methods would yield physically useful information. We outlined an experimental procedure involving photon echoes with three levels of the atom whereby the total phase change undergone by the state vector for a two-level subsystem can be measured. In the particular case of the sech pulse exactly on resonance with the atoms, the total phase change is entirely geometric.

There are several ways of extending the present work. One, analysis of the case when the electromagnetic field is also quantized, is in progress and will be the subject of a future publication. Another is the possibility of associating geometric phases with noncyclic paths—as done, for example, by Samuel and Bhandari [24]—and relating them to physical observables. Finally we mention that self-induced transparency in the presence of degenerate levels also allows for the appearance of the non-Abelian geometric phase.

ACKNOWLEDGMENTS

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APPENDIX A: PARAMETERS IN EQ. (4.26)

Here we list the symbols appearing in the expression for the geometric phase (4.26) in terms of $\Delta\tau$ and λ where

$$\Delta = \omega_0 - \omega$$

is the detuning, τ is a measure of the pulse width, expressed in terms of the other variables as

$$\tau = \frac{1}{\sqrt{\kappa(r_+ - r_-)}} ,$$

and λ is the initial angle made by the Bloch vector with the negative $\hat{e}_3$ axis. The various quantities are

$$\begin{aligned}
\kappa &= \frac{1}{\tau^2}[\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + (\Delta\tau)^2\cos\lambda] ,\\
q_a^2 &= \frac{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda + 1}{\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + 2(\Delta\tau)^2\sin^2\lambda/2 + 1} ,\\
q_b^2 &= -\frac{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda + 1}{\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + 2(\Delta\tau)^2\cos^2\lambda/2 - 1} ,\\
l &= \frac{1}{\sqrt{2}}\sqrt{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda + 1} ,\\
r_+ &= \frac{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda + 1}{2[\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + (\Delta\tau)^2\cos\lambda]} ,\\
r_- &= \frac{\cos\lambda\sqrt{1-(\Delta\tau)^4\sin^2\lambda} - (\Delta\tau)^2\sin^2\lambda - 1}{2[\sqrt{1-(\Delta\tau)^4\sin^2\lambda} + (\Delta\tau)^2\cos\lambda]} .
\end{aligned} \tag{A1}$$

APPENDIX B: SOLID ANGLE SUBTENDED IN THE CASE OF CONTINUOUS PULSES

Here we will show that the geometric phase calculated in Sec. IV is related to the solid angle subtended by the closed loop on the Bloch sphere. The initial components of the Bloch vector s are

$$u(\xi_0) = \sin\lambda\cos\phi_{ba}(\xi_0) , \tag{B1}$$

$$v(\xi_0) = \sin\lambda\sin\phi_{ba}(\xi_0) , \tag{B2}$$

$$w(\xi_0) = -\cos\lambda . \tag{B3}$$

This shows that at the initial instant the Bloch vector makes an angle of λ with the negative $\hat{e}_3$ axis. The third conservation law (2.29) can be written as

$$\frac{p}{\hbar}\mathcal{E}u = 2\Delta\left(Q_a^2 - \sin^2\frac{\lambda}{2}\right) . \tag{B4}$$

This is used to determine the component u at all times except when the electric field $\mathcal{E}$ and $(Q_a^2 - \sin^2\lambda/2)$ both vanish as indeed they do at ξ_0 . At times after the initial instant the components of s are given by

$$u = 2\Delta\sqrt{\frac{r_+}{\kappa}}\mathrm{cn}(U,l) \equiv \sin\mu\cos\nu , \tag{B5}$$

$$v = 2\sqrt{r_+(r_+ - r_-)}\mathrm{sn}(U,l)\mathrm{dn}(U,l) \equiv \sin\mu\sin\nu , \tag{B6}$$

$$w = 2r_+\mathrm{cn}^2(U,l) - \cos\lambda \equiv \cos\mu , \tag{B7}$$

where $\mathrm{sn}(U,l)$, $\mathrm{cn}(U,l)$, and $\mathrm{dn}(U,l)$ are the Jacobian elliptic functions and μ and ν are the polar and azimuthal angles, respectively. The explicit expressions for the angles as functions of ξ are

$$\mu(\xi) = \arccos[2r_+\mathrm{cn}^2(U,l) - \cos\lambda] , \tag{B8}$$

$$\begin{aligned}
\nu(\xi) &= \phi_{ba}\\
&= \Delta\tau[(1-q_b^2)\Pi(\mathrm{am}\ U, q_b^2, l)\\
&\qquad +(1-q_a^2)\Pi(\mathrm{am}\ U, q_a^2, l)]_{K(l)}^{U}\\
&\quad -\Delta(\xi - \xi_0) + \phi_{ba}(\xi_0) .
\end{aligned} \tag{B9}$$

Provided the condition (4.20) for cyclical evolution holds, the Bloch vector returns to its initial position at the instant $\xi = \xi_P$, having traced out a closed loop on the unit sphere. The solid angle Ω subtended by the closed loop at the center of the unit sphere is found by integrating (for $n = 1$)

$$\Omega = \int_{\phi_{ba}(\xi_0)}^{\phi_{ba}(\xi_P)} d\nu \int_{\pi-\lambda}^{\mu(\nu)} \sin\mu\, d\mu . \tag{B10}$$

We obtain

$$\begin{aligned}
\Omega = 4\Delta\tau\Bigg(&\frac{\cos^4\frac{\lambda}{2}}{\cos^2\frac{\lambda}{2} - r_+}\Pi(q_b^2, l) + \frac{\sin^4\frac{\lambda}{2}}{\sin^2\frac{\lambda}{2} + r_+}\Pi(q_a^2, l)\\
&+(r_+ - r_-)\mathfrak{E}(l) + (\cos\lambda + r_-)K(l)\Bigg) .
\end{aligned} \tag{B11}$$

Using the explicit form of the condition for cyclical evolution (4.20), this reduces to

$$\begin{aligned}
\Omega &= 4\Delta\tau\left[(1-q_a^2)\Pi(q_a^2, l) + (r_+ - r_-)\mathfrak{E}(l) + \left(r_- - \sin^2\frac{\lambda}{2}\right)K(l)\right]\\
&= -2\beta .
\end{aligned} \tag{B12}$$

Hence we have shown by an explicit calculation that the geometric phase is minus one-half the solid angle subtended by the closed loop traced on the Bloch sphere. All two-level systems which develop a geometric phase must in fact exhibit this relationship.

APPENDIX C: DERIVATION OF EQ. (5.5)

In this appendix we obtain the polarization of the 1-2 subsystem in a three-level atom, following the sequence of pulses mentioned in Sec. V. The components of the

transverse polarization or the coherence between any two levels can be obtained from the elements of the density matrix. We will assume that our system is in a pure state at all times. In that case the density matrix $\rho(t)$ at any time t may be calculated from the state vector $|\psi(t)\rangle$ as

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| \ .$$

Since we already have obtained the state vector for the two-level subsystem by solving Schrödinger's equation in Secs. III and IV it is more convenient to use this expression for the density matrix rather than solve the Liouville equation for the density matrix.

Let $|1\rangle$, $|2\rangle$, and $|3\rangle$ be the unperturbed state vectors of the three levels with energies E_1, E_2, and E_3, respectively. We consider the cascade configuration of the atom where the allowed transitions are between the ground state $|1\rangle$ and the intermediate state $|2\rangle$ and between $|2\rangle$ and $|3\rangle$, the highest excited state. The resonance frequencies of the transitions $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$ are

$$\omega_1 = \frac{E_2 - E_1}{\hbar} \ , \quad \omega_2 = \frac{E_3 - E_2}{\hbar} \ ,$$

respectively.

With the initial population all in the lowest-energy state, the density matrix at time $t = 0$ is

$$\rho(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{C1}$$

where ρ_{11} represents the population in the highest state $|3\rangle$, ρ_{22} that in the intermediate state $|2\rangle$, and ρ_{33} the population in the lowest state $|1\rangle$. We now inject a pulse of constant amplitude with area $\pi/2$ and resonant with the 1-2 transition. This pulse equalizes the populations of levels 1 and 2 and induces a coherence between them. The state vector at some time t_1 after the passage of this pulse is

$$|\Psi\rangle(t_1) = \frac{1}{\sqrt{2}}(e^{i\phi_1(t_1)}|1\rangle + |2\rangle) \tag{C2}$$

to which corresponds the density matrix

$$\rho(t_1) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & e^{-i\phi_1(t_1)} \\ 0 & e^{i\phi_1(t_1)} & 1 \end{pmatrix} \ . \tag{C3}$$

Next this three-level system is subjected to a propagating pulse nearly resonant with the 2-3 transition. Assuming that this pulse is circularly polarized, the electric field with carrier frequency ω can be written as

$$\mathbf{E} = \mathcal{E}(z,t)(\hat{\mathbf{x}}\cos\Phi + \hat{\mathbf{y}}\sin\Phi) \ , \tag{C4}$$

where $\mathcal{E}$ and Φ are the amplitude and phase, respectively, of the field. The Hamiltonian in the presence of this field is

$$H = \begin{pmatrix} E_3 & -p_2\mathcal{E}e^{-i\Phi} & 0 \\ -p_2\mathcal{E}e^{i\Phi} & E_2 & 0 \\ 0 & 0 & E_1 \end{pmatrix} \ , \tag{C5}$$

where p_2 is the dipole moment between states $|2\rangle$ and $|3\rangle$. We transform to a frame rotating with the field by using the unitary matrix

$$U(t) = \exp\left[i \begin{pmatrix} (\frac{E_2 t}{\hbar} + \Phi) & 0 & 0 \\ 0 & \frac{E_2 t}{\hbar} & 0 \\ 0 & 0 & \frac{E_1 t}{\hbar} \end{pmatrix} \right] \ . \tag{C6}$$

In the rotating frame the transformed state vector is

$$|\Psi\rangle_{\rm tr} = U(t)|\Psi\rangle \tag{C7}$$

which evolves according to the transformed Hamiltonian

$$H_{\rm tr} = \begin{pmatrix} \hbar\Delta_2 & -p_2\mathcal{E} & 0 \\ -p_2\mathcal{E} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{C8}$$

where we have defined the detuning parameter $\Delta_2 = \omega_2 - \omega$.

Our aim is to obtain the density matrix after the passage of the pulse. We proceed by first solving the Schrödinger equation for the state vector. We write the state vector as a superposition of the unperturbed energy eigenstates

$$|\Psi\rangle_{\rm tr}(t) = C_1(t)|1\rangle + C_2(t)|2\rangle + C_3(t)|3\rangle \ , \tag{C9}$$

where C_1, C_2, and C_3 are complex coefficients. Schrödinger's equation then reduces to the system of equations

$$i\hbar\frac{\partial C_1}{\partial t} = 0 \ , \tag{C10}$$

$$i\hbar\frac{\partial C_2}{\partial t} = -p_2\mathcal{E}C_3 \ , \tag{C11}$$

$$i\hbar\frac{\partial C_3}{\partial t} = -p_2\mathcal{E}C_2 + \hbar\Delta_2 C_3 \ . \tag{C12}$$

The evolution of the electric field is given by Maxwell's equations. The dynamics of the 2-3 transition is independent of level 1 and can be treated exactly as in Sec. III. Hence the state vector at any time is

$$|\Psi\rangle_{\rm tr}(t_1 + t) = C_1(t_1)|1\rangle + Q_2 e^{i\phi_2}|2\rangle + Q_3 e^{i\phi_3}|3\rangle \ , \tag{C13}$$

where Q_2, ϕ_2, Q_3, ϕ_3 are obtained by the method used in Sec. III. The soliton pulse takes levels 2 and 3 through a cyclical evolution with time period T. The amplitudes Q_2, Q_3 return to their initial values at time $t = t_1 + T$:

$$Q_2(t_1 + T) = \frac{1}{\sqrt{2}} \ , \quad Q_3(t_1 + T) = 0 \ .$$

At this time the state vector is

$$|\Psi\rangle_{\rm tr}(t_1 + T) = \frac{1}{\sqrt{2}}(e^{i\phi_1(t_1)}|1\rangle + e^{i\phi_2(t_1+T)}|2\rangle) \ , \tag{C14}$$

where $\phi_2(t_1 + T)$ is the total phase change undergone by state $|2\rangle$. Correspondingly the density matrix is

$$\rho_{\rm tr}(t_1+T)=\frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & e^{i[\phi_2(t_1+T)-\phi_1(t_1)]}\\ 0 & e^{-i[\phi_2(t_1+T)-\phi_1(t_1)]} & 1\end{pmatrix} . \tag{C15}$$

The components of the transverse polarization of the two-level subsystem 1-2 are obtained from the density matrix as

$$\langle P_x\rangle^{1-2}=c(\rho_{23}+\rho_{32}) , \quad \langle P_y\rangle^{1-2}=ic(\rho_{32}-\rho_{23}) , \tag{C16}$$

where c is a constant. From (C15) it follows that

$$\langle P_x\rangle^{1-2}(t_1+T)=c\cos[\phi_2(t_1+T)-\phi_1(t_1)] ,$$
$$\langle P_y\rangle^{1-2}(t_1+T)=c\sin[\phi_2(t_1+T)-\phi_1(t_1)] . \tag{C17}$$

At time $t_2\geq t_1+T$ we apply another $\pi/2$ pulse to the 1-2 transition to refocus the signal. This pulse is represented by the constant amplitude field

$$\mathbf{E}(t_2+t)=2\mathcal{E}_0\cos(\omega_1 t+\alpha)\hat{\mathbf{x}} \tag{C18}$$

which lasts for an interval τ and we have added a phase shift of α to the field. Applying the unitary transformation effected by $U(t)$ [given by Eq. (C6)] transforms the Hamiltonian in the presence of this field to

$$H_{\rm tr}(t_2+t)=\begin{pmatrix}\hbar\Delta_2 & 0 & 0\\ 0 & 0 & p_1\mathcal{E}_0e^{-i\alpha}\\ 0 & p_1\mathcal{E}_0e^{i\alpha} & 0\end{pmatrix} \quad 0\leq t\leq\tau \tag{C19}$$

where p_1 is the dipole matrix element between levels 1 and 2. Solving Schrödinger's equation for the state vector leads to the following density matrix at the end of the pulse:

$$\rho_{\rm tr}(t_2+\tau)=\frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1+\sin[\phi_2(t_1+T)-\phi_1(t_1)+\alpha] & e^{-i\alpha}\cos[\phi_2(t_1+T)-\phi_1(t_1)+\alpha]\\ 0 & e^{i\alpha}\cos[\phi_2(t_1+T)-\phi_1(t_1)+\alpha] & 1-\sin[\phi_2(t_1+T)-\phi_1(t_1)+\alpha]\end{pmatrix} . \tag{C20}$$

We find then that the echo signal following the pulse has the following polarization components:

$$\langle P_x\rangle^{1-2}(t_2+\tau)=c\cos[\phi_2(t_1+T)-\phi_1(t_1)+\alpha]\cos\alpha ,$$
$$\langle P_y\rangle^{1-2}(t_2+\tau)=c\sin[\phi_2(t_1+T)-\phi_1(t_1)+\alpha]\sin\alpha . \tag{C21}$$

This is the result (5.5) described in Sec. V.

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