Nondemolition observation of a free quantum particle

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A stochastic model of a continuous nondemolition observation of a quantum system is presented. The nonlinear stochastic wave equation describing the posterior dynamics of the observed quantum system is solved for a free particle of mass $m > 0$. It is shown that the dispersion of the Gaussian wave packet does not increase to infinity as for a free unobserved particle, but tends to the finite limit $\tau_\infty^2 = (\hbar/2\lambda m)^{1/2}$, where λ is the accuracy coefficient of an indirect nondemolition measurement of the particle's position.

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I. INTRODUCTION

Continuously-improving experimental technique is approaching the point in which such quantum systems as single atoms (ions) can be observed. The development of a theoretical apparatus appropriate for the dynamical description of a quantum measurement should go in a parallel manner. Although the problem of measurement theory is as old as quantum mechanics itself, dynamical models of the measurement process have appeared only recently.

The Schrödinger equation describes the time development of the wave function of a quantum system only for the time intervals between the succeeding instants of measurements. At the instant of measurement of some observable with a discrete spectrum Z, the quantum system makes an immediate transition (jump) from the $\psi(t)$ state to the eigenstate $\psi_z(t)$ corresponding to the eigenvalue z of Z with probability $|\langle \psi(t)|\psi_z(t)\rangle|^2$. Such a stochastic time behavior of the system at the instant of measurement assures the repeatability of the results of the measurements. If a second measurement were taken immediately after the first one, then for a discrete observable Z the measurement would again give z [1]. It is intuitively obvious that if one were to perform measurements with a high frequency—in a limit continuously in time—the quantum system would show a stochastic irreversible behavior for the whole period of observation. Therefore the time development of a continuously observed quantum system cannot be governed by the deterministic Schrödinger equation describing the reversible motion. This statement remains true also in the case of the measurements of an observable with a continuous spectrum, though for observables with continuous spectra, the repeatability hypothesis is not assumed [1–4] as, in general, there are nonzero (*a priori*) probabilities of the results of such a measurement belonging to disjoint Borel sets.

The irreversible and stochastic behavior of the continuously observed quantum system expressed by the so-called collapse or reduction of the wave function has no analog in classical deterministic mechanics. The Hamilton equations do not depend on whether the dynamical object is observed during its motion along its trajectory. That difference in the behavior of classical and quantum observed objects cannot be ignored.

The question of the time development of a continuously observed quantum system is not easy to answer. A seemingly promising approach to this problem—via the standard projection postulate—leads to paradoxical results called the quantum Zeno paradox [5] (cf. also Refs. [6–11]). The essence of the quantum Zeno paradox is that the continuous observation freezes the state of a quantum system. For example, "...An unstable particle observed continuously whether it has decayed or not will never be found to decay..." [5].

Such paradoxes can be avoided only by consistent investigation of the disturbed stochastic dynamics of the quantum system undergoing an observation. It is quite natural to discuss the problem in the framework of stochastic quantum mechanics of open systems [12,13] (cf. also Refs. [14–16]) on the basis of the theory of nondemolition measurements developed recently [17–20].

The principle of a nondemolition continuous observation of a quantum system can be formulated as follows [20].

(i) There exist observables $\hat{Q}(r)$, $r \le t$, that commute, for any t, with all Heisenberg operators $\hat{Z}(t)$ of the system represented in the Hilbert space corresponding to "the system plus measuring apparatus."

(ii) According to the causality principle one does not impose any conditions on the future observables $\hat{Q}(s)$, $s > t$, with respect to the past observables of the system $\hat{Z}(r)$, $r \le t$.

A nontrivial nondemolition observation in the above-mentioned sense is provided by indirect measurements that can be only realized by considering the observed quantum system as an open one.

To avoid a misunderstanding we would like to emphasize that the above idea of a continuous nondemolition observation in a quantum system differs from that of Braginsky, Vorontzov, and Halili [21] (cf. also Refs. [22–24]). Their approach concerns a measurement in a closed quantum system (the interaction with a measuring apparatus is not taken into account). It requires a family of systematic observables $\{O_t\}$ such that the corresponding Heisenberg observables $O_t(t)=U^{\dagger}_{\rm sys}(t)O_tU_{\rm sys}(t)$ are mutually compatible, i.e., they satisfy $[O_s(s),O_t(t)]=0$ for all instants s and t. The measurement of these operators is a nondemolition one—the only possible reduction of a state of the quantum system occurs at the beginning of the measurement.

From the experimental point of view it is natural to consider indirect measurements because any measurement is taken with the help of some experimental device. Some theoretical models illustrating this approach [quantum system in contact with a bath (apparatus)] were developed in Refs. [25–28]. The indirect measurements allow one to describe the state changes resulting from the measurements of observables with continuous spectra [4], which are assumed to be nonideal. The necessity to use indirect measurements for the existence of the continual limit (with $\Delta t \to 0$) for successive instantaneous measurements taken at instants separated by Δt is proved in Ref. [29].

In this paper we shall illustrate the approach of the continuous quantum-nondemolition measurement for the example of resolving the quantum Zeno paradox for a three-dimensional free particle undergoing an observation modeling the measurement of a trajectory of a quantum particle in a bubble chamber as was briefly reported by us in Ref. [30] for the one-dimensional case.

Section II is of a preparatory character: we present the stochastic model of a continuous nondemolition observation of a quantum system interacting with an M-dimensional Bose-field reservoir representing the measuring device. By assuming the singular-reservoir limit [31–33] ($\tau_R=0$, where τ_R is the decay time of the correlation functions of the reservoir) one can consider the measuring apparatus as a macroscopic device [32]. The condition $\tau_R=0$ is assured [31] by taking the fields of the flat spectra (singular fields [33]) prepared initially in the vacuum state.

In Sec. III we derive the filtering equation—the stochastic nonlinear differential equation describing the time development of the wave function of the quantum system observed by means of the vector "field coordinate" process. This equation was recently obtained with the help of a quantum-filtration method [34,35]. The present derivation—via a stochastic instrument in the sense of Davies and Lewis [2,3]—generalizes the result of Ref. [36] to the case of a multidimensional observation. Our derivation utilizes the method of the generating map of the instrument, which is essentially due to Barchielli and co-workers [14–16,29].

Nonlinear stochastic differential equations describing a dynamical collapse (reduction) of a wave function of the observed quantum system were considered by Pearle [37–40], Gisin [41–43], Ghirardi and co-workers [44,45] and Diósi [46–49]. We would like to stress that in contrast to those authors we do not postulate the equation but derive it within the model of a quantum system interacting with a measuring device represented by the Bose field. However, in this paper we deal with the "diffusion" observation; we would like to mention that an analogous stochastic differential equation describing the time development of the wave function of the quantum system observed by means of the continuous photon-counting measurement can be found [50–52]. From the latter, by a limiting procedure, the filtering equation corresponding to the diffusion observation can be obtained [52].

In Sec. IV we solve the filtering equation for the three-dimensional free quantum particle undergoing the continuous nondemolition observation of its position. We prove that the dispersion of the Gaussian wave packet does not spread out in time but tends to the finite limit $\lim_{t\to\infty}\tau^2(t)=(\hbar/2\lambda m)^{1/2}$, where $m>0$ is the mass of the observed particle and λ stands for the accuracy coefficient of the indirect nondemolition measurement of the particle's position. We call this result the watchdog effect (observation effect): the continuous observation prevents the Gaussian wave packet from spreading out. (In some papers, cf., for instance, Refs. [8] and [10], the term "watchdog effect" appears in the context of the continuous observation of a quantum system based on the standard projection postulate and can be replaced with "Zeno paradox"). The same asymptotic behavior of the dispersion of the Gaussian wave packet was obtained by Diósi [48,49] and Caves and Milburn [53]. Nevertheless, there are significant differences between our approach and theirs. These problems will be discussed at the end of Sec. IV.

II. STOCHASTIC MODEL OF A CONTINUOUS MULTIDIMENSIONAL-DIFFUSION OBSERVATION OF A QUANTUM SYSTEM

Let us assume that a quantum system $\mathcal{S}$ with the Hamiltonian H existing in the Hilbert space $\mathcal{H}$ is coupled at instant $t=0$ to the reservoir (measuring device) R consisting of M independent Bose fields in the vacuum state. The fields are described by vector operators $\mathbf{b}(\omega)=[b_j(\omega)]_{j=1}^{M}$, $\mathbf{b}^{\dagger}(\omega)=[b_j^{\dagger}(\omega)]_{j=1}^{M}$ acting in $\mathcal{F}=\mathcal{F}_{\rm sym}(\mathbb{C}^M\otimes\mathcal{L}^2(\mathbb{R}))$, the symmetric Fock space over $\mathbb{C}^M\otimes\mathcal{L}^2(\mathbb{R})$. The components of $\mathbf{b}(\omega)$ and $\mathbf{b}^{\dagger}(\omega)$ satisfy the canonical commutation relations (CCR's)

$$[b_j(\omega),b_k(\omega')]=0\ ,\quad [b_j(\omega),b_k^{\dagger}(\omega')]=\delta_{jk}\delta(\omega-\omega')\qquad (j,k=1,\ldots,M)\ . \tag{2.1}$$

Under the following assumptions:

Assumption 1: the coupling is linear in the field operators,

Assumption 2: the rotating-wave approximation is made, the Hamiltonian H_{S+R} of the compound system has the form

$$H_{S+R}=H+H_R+H_I\ , \tag{2.2}$$

$$H_R=\hbar\sum_{j=1}^{M}\int_{-\infty}^{+\infty}d\omega\, e_j(\omega)b_j^\dagger(\omega)b_j(\omega)\ , \tag{2.3}$$

$$H_I=\frac{i\hbar}{(2\pi)^{1/2}}\sum_{j=1}^{M}\int_{-\infty}^{+\infty}d\omega\, k_j(\omega)[L_jb_j^\dagger(\omega)-L_j^\dagger b_j(\omega)]\ . \tag{2.4}$$

In the interaction Hamiltonian, L_j are system operators and $k_j(\omega)$ are the so-called coupling constants assumed to be real.

In the sequel we shall need two more assumptions [13].

Assumption 3: The spectrum of the bath (measuring device) is flat.

This means that, for each j in (2.3), $e_j(\omega)=\omega$. Such reservoirs are called singular because their energy is unbounded from above and from below. Singular reservoirs model external macroscopic devices driving open quantum systems to asymptotic states far from equilibrium [31–33].

Assumption 4: "Coupling constants" $k_j(\omega)$ do not depend on ω.

Including a scalar factor that is responsible for the strength of each coupling into systematic operators L_j, we insert $k_j(\omega)=1$ for each j in (2.4).

As usual, we assume the unitary evolution of the compound system $\mathcal{S}+\mathcal{R}$, generated by the Hamiltonian H_{S+R}. By going to the interaction picture with respect to the free dynamics of $\mathcal{R}$ [generated by H_R given by (2.4) modified with the help of assumption 3] one gets the unitary evolution operator $U(t)$ in the form of a chronologically ordered exponential function

$$U(t)=\overleftarrow{T}\exp\left[-\frac{i}{\hbar}\int_0^t(H+H_I(s))ds\right]\ . \tag{2.5}$$

In this formula

$$H_I(t)=i\hbar\sum_{j=1}^{M}[L_jb_j^\dagger(t)-L_j^\dagger b_j(t)] \tag{2.6}$$

with

$$b_j(t)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}d\omega\, e^{-i\omega t}b_j(\omega)\ . \tag{2.7}$$

Equation (2.7) combined with CCR's (2.1) yields the following CCR's for the time-dependent field operators

$$[b_j(t),b_k(s)]=0\ ,\quad [b_j(t),b_k^\dagger(s)]=\delta_{jk}\delta(t-s)\qquad (j,k=1,\ldots,M)\ . \tag{2.8}$$

The reservoir is assumed to be initially prepared in the vacuum state; therefore

$$\langle b_k(t)\rangle_v=\langle b_k^\dagger(t)\rangle_v=\langle b_k^\dagger(t)b_j(s)\rangle_v=0\ , \tag{2.9}$$

$$\langle b_k(t)b_j^\dagger(s)\rangle_v=\delta_{kj}\delta(t-s)\ . \tag{2.10}$$

With the interaction (2.6) the latter means that we have arrived at the so-called singular-reservoir limit [31–33], in which the decay time of the two-point time-correlation functions of the reservoir is zero, $\tau_R=0$. The real and imaginary parts of $\mathbf{b}(t)$ defined as $\mathrm{Re}\mathbf{b}(t)=\frac{1}{2}[\mathbf{b}(t)+\mathbf{b}^\dagger(t)]$, $\mathrm{Im}\mathbf{b}(t)=1/2i[\mathbf{b}(t)-\mathbf{b}^\dagger(t)]$ do not commute, but each of them has the statistical properties of (classical) standard M-dimensional white noise. Similarly, as in the classical case [54], the time evolution of the system interacting with the reservoir can be described in a mathematically rigorous way in terms of a stochastic differential equation [12,13]. A quantum-stochastic calculus (QSC) of the Ito type has been developed by Hudson and Parthasarathy [12]. Here we give the formal rules of the QSC, which will be needed in our paper.

Let us define annihilation and creation processes,

$$B_j(t)=\int_0^t b_j(s)ds\ ,\quad B_j^\dagger(t)=\int_0^t b_j^\dagger(s)ds\ , \tag{2.11}$$

which satisfy [due to (2.8)] the following commutation relations:

$$[B_j(t),B_k(s)]=0\ ,\quad [B_j(t),B_k^\dagger(s)]=\delta_{jk}\min(t,s)\qquad (j,k=1,\ldots,M)\ . \tag{2.12}$$

The pair $\mathbf{B}(t)=[B_j(t)]_{j=1}^M$, $\mathbf{B}^\dagger(t)=[B_j^\dagger(t)]_{j=1}^M$ is the quantum analog of the standard M-dimensional Wiener diffusion process. The stochastic differentials of the processes in (2.11),

$$\begin{aligned} dB_j(t)&\equiv B_j(t+dt)-B_j(t)=\int_t^{t+dt}b_j(s)ds\ ,\\ dB_j^\dagger(t)&\equiv B_j^\dagger(t+dt)-B_j^\dagger(t)=\int_t^{t+dt}b_j^\dagger(s)ds\ , \end{aligned} \tag{2.13}$$

satisfy the multiplication rules

$$dB_j(t)dB_k^\dagger(t)=\delta_{jk}dt \tag{2.14}$$

and all other products involving $dB_j(t)$, $dB_j^\dagger(t)$, and dt are equal to zero [12].

The Ito quantum-stochastic differential equation (QSDE) with respect to the M-dimensional Wiener diffusion process has the form [12,13]

$$dN(t)=\sum_{j=1}^{M}[P_j(t)dB_j^\dagger(t)+R_j(t)dB_j(t)]+S(t)dt\ , \tag{2.15}$$

where P_j, R_j, and S are adapted processes, i.e., they are operators on $\mathcal{H}\otimes\mathcal{F}$, which depend on $\mathbf{B}(s)$ and $\mathbf{B}^\dagger(s)$ only for times $s\le t$. Note that the adapted processes commute with the stochastic differentials $dB_j^\dagger$, dB_j directed to the future [Eq. (2.13)]. No further commutativity assumption concerning processes appearing in the QSDE of the type (2.15) is made.

The Hudson-Parthasarathy differentiation formula [12] for the product $M(t)N(t)$ of the adapted processes reads

$$d(M(t)N(t))=dM(t)N(t)+M(t)dN(t)+dM(t)dN(t)\ . \tag{2.16}$$

The unitary (adapted) evolution operator $U(t)$ in $\mathcal{H}\otimes\mathcal{F}$ for the system $\mathcal{S}$ coupled to the Bose reservoir is assumed to satisfy the Ito QSDE in the form [12,13]

$$dU(t)=\left[-Kdt+\sum_{j=1}^{M}[L_j dB_j^\dagger(t)-L_j^\dagger dB_j(t)]\right]U(t)\ ,\quad U(0)=I \tag{2.17}$$

where

$$K=\frac{i}{\hbar}H+\tfrac{1}{2}\sum_j L_j^\dagger L_j\ . \tag{2.18}$$

In these formulas H stands for the Hamiltonian of $\mathcal{S}$, $i\hbar\sum_j[L_j dB_j^\dagger(t)-L_j^\dagger dB_j(t)]$ describes the interaction between $\mathcal{S}$ and the fields (more precisely $i\hbar\sum_j[L_j dB_j^\dagger(t)-L_j^\dagger dB_j(t)]/dt=i\hbar\sum_j[L_j b_j^\dagger(t)-L_j^\dagger b_j(t)]$ is the interaction Hamiltonian, cf. (2.6)), and $-\frac{1}{2}\sum_j L_j^\dagger L_j$ is the Ito correction term. (If one applied, instead of (2.17), a QSDE based on the quantum Stratonovich integral [13], this term would disappear.)

It is easy to check [14] that the solution $U(t)$ of Eq. (2.17) is unitary. It can be readily seen from the formal solution of (2.17):

$$U(t)=\overleftarrow{\mathrm{T}}\exp\left[-\frac{i}{\hbar}\int_0^t\{H\,ds+i\hbar\sum_j[L_j dB_j^\dagger(s)-L_j^\dagger dB_j(s)]\}\right]. \tag{2.19}$$

To verify the last statement one has to calculate $dU(t)=U(t+dt)-U(t)$. Equation (2.19) yields

$$U(t+dt)=\exp\left[-\frac{i}{\hbar}H\,dt+\sum_j[L_j dB_j^\dagger(t)-L_j^\dagger(t)dB_j(t)]\right]U(t)\ .$$

Next, the result follows by expanding the exponential function in the last formula and making use of the multiplication rules (2.14). The Ito correction term appearing in (2.17) results from the second-order term of the expansion.

With the help of (2.17) the Heisenberg equation of motion for any observable of $\mathcal{S}$ can be easily obtained. By applying to the product

$$\hat{Z}(t)=U^\dagger(t)ZU(t)\ , \tag{2.20}$$

the quantum Ito formula (2.16), Eq. (2.17) and its adjoint equation, one can check with the help of (2.14) and (2.18) that the Heisenberg observable $\hat{Z}(t)$ satisfies the following QSDE:

$$d\hat{Z}+\left[\hat{K}^\dagger\hat{Z}+\hat{Z}\hat{K}-\sum_j\hat{L}_j^\dagger\hat{Z}\hat{L}_j\right]dt=\sum_j([\hat{Z},\hat{L}_j]dB_j^\dagger+[\hat{L}_j^\dagger,\hat{Z}]dB_j)\ , \tag{2.21}$$

where we have employed the simplified notation $\hat{Z}$ for $\hat{Z}(t)$, etc.

Equation (2.17) or (2.21) describes the distorted dynamics (in the Hilbert space $\mathcal{H}\otimes\mathcal{F}$) of the initially closed quantum system $\mathcal{S}$ under the stochastic interaction with Bose fields. The fields, however, not only disturb the system, they also give some possibility of a continuous (in time) observation of $\mathcal{S}$. Let us first pay attention to their time development. In the Heisenberg picture, the processes

$$\hat{B}_j(t)=U^\dagger(t)B_j(t)U(t) \tag{2.22}$$

remain unchanged for all times $s\geq t$ [14,15], i.e.,

$$\hat{B}_j(t)=U^\dagger(s)B_j(t)U(s)\ ,\quad s\geq t\ . \tag{2.23}$$

Obviously, the same holds for the creation process $\mathbf{B}^\dagger(t)$. The property (2.23) results, essentially, from two facts: Eq. (2.17) is written in the interaction picture with respect to the free dynamics of the fields and the coupling between $\mathcal{S}$ and the fields is singular. The annihilation and creation processes $\mathbf{B}(t)$ and $\mathbf{B}^\dagger(t)$ are called input (annihilation, creation) processes while $\hat{\mathbf{B}}(t),\hat{\mathbf{B}}^\dagger(t)$ are called output processes [13]. The input processes describe Bose fields before their interaction with $\mathcal{S}$, the output ones—after the interaction. Note that due to (2.23) the output processes satisfy the nondemolition conditions [20]

$$\begin{aligned}&[\hat{\mathbf{B}}(s),\hat{Z}(t)]=U^\dagger(t)[\mathbf{B}(s),Z]U(t)=0\quad\forall\ s\leq t\\&[\hat{\mathbf{B}}^\dagger(s),\hat{Z}(t)]=U^\dagger(t)[\mathbf{B}^\dagger(s),Z]U(t)=0\quad\forall\ s\leq t\ .\end{aligned} \tag{2.24}$$

Let us consider the continuous measurement of the output vector "field coordinate" ("diffusion") process

$$\hat{\mathbf{Q}}(t)=\hat{\mathbf{B}}(t)+\hat{\mathbf{B}}^\dagger(t)=U^\dagger(t)\mathbf{Q}(t)U(t)\ , \tag{2.25}$$

where $\mathbf{Q}(t)=\mathbf{B}(t)+\mathbf{B}^\dagger(t)$ is the input Wiener process. From (2.12) it follows that

$$[\hat{\mathbf{Q}}(t),\hat{\mathbf{Q}}(t')]=0\quad\forall\ t,t'\geq 0\ , \tag{2.26}$$

i.e., the output Hermitian process $\hat{\mathbf{Q}}$ is self-nondemolition. Due to (2.24) and (2.25) the measurement of $\hat{\mathbf{Q}}$ is a nondemolition one [19,20] with respect to the time evolution of the system: for any Z,

$$[\hat{\mathbf{Q}}(s),\hat{Z}(t)]=0\quad\forall\ s\leq t\ . \tag{2.27}$$

This means that the measurement of $\hat{\mathbf{Q}}$ disturbs neither the present nor the future state of the system $\mathcal{S}$. Note that due to its Hermiticity and self-commutativity the output nondemolition process $\hat{\mathbf{Q}}(t)$ can be treated (in the representation in which it is diagonal) as a classical one. Let us observe that $\hat{\mathbf{Q}}(t)$ describes a continuous imperfect measurement of the quantum observable $2\,\mathrm{Re}\hat{\mathbf{L}}(t)$. The latter can be easily seen from the QSDE for $\hat{\mathbf{Q}}(t)$ [obtained in a way quite analogous to Eq. (2.21)]

$$d\hat{\mathbf{Q}}(t)=[\hat{\mathbf{L}}(t)+\hat{\mathbf{L}}^\dagger(t)]dt+d\mathbf{Q}(t)\ . \tag{2.28}$$

Equation (2.17) does not include any observation, it describes the perturbed dynamics of the unobserved system $\mathcal{S}$ (represented in $\mathcal{H}\otimes\mathcal{F}$). Following Refs. [19,20] we shall call it the prior dynamics. Similarly, for any initial systematic observable Z, Eq. (2.21) is the equation for the time development of the unobserved process $\hat{Z}(t)$. But for each Z we have the possibility of considering Eq. (2.21) together with Eq. (2.28); consequently, for any initial Z, $\hat{Z}(t)$ becomes partially observed. As is proved in Ref. [34], Eq. (2.27) gives the possibility of defining the posterior (observed) mean values of $\hat{Z}(t)$ under the condition of observation of any nonanticipating function of $\hat{\mathbf{Q}}$ up to the moment t.

The reduced Schrödinger dynamics of $\mathcal{S}$ (the prior dynamics of $\mathcal{S}$ in $\mathcal{H}$) is a Gaussian dynamical semigroup for the density matrix $\rho(t)$:

$$\begin{aligned} &\frac{d}{dt}\rho(t)=L\rho(t) , \\ &L\rho=-(\rho K^{\dagger}+K\rho)+\sum_j L_j\rho L_j^{\dagger} . \end{aligned} \tag{2.29}$$

Equation (2.29) is obtained in a standard way by averaging both sides of Eq. (2.21) with the vacuum state for the fields. That yields the semigroup (nonstochastic) evolution of $\mathcal{S}$ in the Heisenberg picture as the vacuum expectation of the right-hand side of Eq. (2.21) is 0. Then by going to the Schrödinger picture one finds L. The master equation (2.29) can be also obtained in the manner of the stochastic averaging of selective evolutions corresponding to the trajectories of the output observation process (cf. Sec. III). Therefore Eq. (2.29) describes the nonselective evolution [29,14–16,52,53] of $\mathcal{S}$ coupled to the measuring device (when the results of the measurement are not read out).

III. QUANTUM FILTERING EQUATION

In this section we shall derive the quantum filtering equation—the QSDE that describes the time development of the posterior state of the quantum system $\mathcal{S}$ undergoing the M-dimensional diffusion observation of $\hat{\mathbf{Q}}$ [Eq. (2.25)]. It shall be done by solving the differential equation for the generating map of the corresponding instrument [2,3]. For $M=1$ this approach was applied by one of us (V.P.B.) in Ref. [36].

Let us denote by $\nu=\otimes_{j=1}^{M}\nu_j$ the standard product Wiener probability measure on the space Ω of continuous trajectories $\mathbf{q}=\{\mathbf{q}(t)|t>0\}$ of the observed process $\hat{\mathbf{Q}}$ restricted to the space Ω^t of the trajectories that are stopped at t: $\mathbf{q}^t=\{\mathbf{q}(r)|r\le t\}$. Consider the instrument $\mathcal{I}^t$ on the algebra of operators Z of the observed quantum system $\mathcal{S}$ as a function of the observed event $d\mathbf{q}$ up to the instant t. Then $\mathcal{I}^t$, by its definition, defines the time evolution $\rho\to\rho^t(d\mathbf{q})$ of an initial-state functional ρ: $Z\to\rho[Z]$ of $\mathcal{S}$ to the state $\rho^t(d\mathbf{q})=\rho\circ\mathcal{I}^t(d\mathbf{q})$ normalized to the probability $\mu^t(d\mathbf{q})=\rho[\mathcal{I}^t(d\mathbf{q})[I]]$.

Define the generating map of $\mathcal{I}^t$ in the following way (cf. also Refs. [15,16])

$$\Gamma(1,t)[Z]=\int_{\Omega^t}\exp\left[\int_0^t \mathbf{l}(r)d\mathbf{q}(r)\right]\mathcal{I}^t(d\mathbf{q})[Z] , \tag{3.1}$$

where $\mathbf{l}(t)=[l_j(t)]_{j=1}^{M}$, with the components l_j being integrable c-valued functions. The generating map can also be defined by the condition

$$\langle\psi|\Gamma(1,t)[Z]\psi\rangle=\langle\hat{Y}(1,t)\hat{Z}(t)\rangle , \tag{3.2}$$

where

$$\begin{aligned}\hat{Y}(1,t)&=\exp\left[\sum_{j=1}^{M}\int_0^t l_j(r)d\hat{Q}_j(r)\right] \\ &=\prod_{j=1}^{M}\exp\left[\int_0^t l_j(r)d\hat{Q}_j(r)\right] .\end{aligned} \tag{3.3}$$

The mean value on the right-hand side of (3.2) is taken with respect to $\psi\otimes\upsilon$ with $\psi\in\mathcal{H}$ being an (arbitrary) initial pure state of $\mathcal{S}$ and $\upsilon\in\mathcal{F}$ the vacuum-state vector for the fields. Note that the M-exponential output process $\hat{Y}(1,t)$ given by (3.3) is a nondemolition and a self-nondemolition one.

Let us now find the differential equation for the generating map $\Gamma(1,t)$ of the instrument $\mathcal{I}^t$. According to (3.2) it can be done by finding the differential equation for the mean value $\langle\hat{Y}(1,t)\hat{Z}(t)\rangle$. First we obtain the stochastic differential equation for $\hat{G}(t)=\hat{Y}(t)\hat{Z}(t)$. Let us write $\hat{G}(t)$ in the form $\hat{G}(t)=U^{\dagger}(t)G(t)U(t)=U^{\dagger}(t)Y(t)ZU(t)$, where $Y(t)$ is the input process corresponding to (3.3):

$$Y(1,t)=\exp\left[\sum_{j=1}^{M}\int_0^t l_j(r)dQ_j(r)\right] . \tag{3.4}$$

Then from Ito's formula (2.16) applied to the product $\hat{G}=U^{\dagger}GU$ we get

$$\begin{aligned} d\hat{G}&=dU^{\dagger}GU+U^{\dagger}dGU+U^{\dagger}GdU+dU^{\dagger}dGU+dU^{\dagger}GdU+U^{\dagger}dGdU+dU^{\dagger}dGdU \\ &=U^{\dagger}\left[\sum(\tfrac{1}{2}l_j^2G+L_j^{\dagger}Gl_j+l_jGL_j+L_j^{\dagger}GL_j)-K^{\dagger}G-GK\right]Udt \\ &\quad+U^{\dagger}\left[\sum_j[L_j^{\dagger}G+G(l_j-L_j^{\dagger})]dB_j\right]U+U^{\dagger}\left[\sum_j[GL_j+(l_j-L_j)G]dB_j^{\dagger}\right]U , \end{aligned} \tag{3.5}$$

where we have used (2.17), multiplication rules (2.14), and the stochastic differential of G, $dG=dY\,Z$ with

$$dY(1,t)=\sum_j[l_j(t)dQ_j(t)+\tfrac{1}{2}l_j^2(t)dt]Y(1,t) , \tag{3.6}$$

which can be obtained from (3.4) by the classical Ito formula [54].

Equation (3.5) yields the following differential equation for the mean value of $\hat{G}(t)=\hat{Y}(1,t)\hat{Z}(t)$:

$$\langle d\hat{G}\rangle=\left\langle\hat{\eta}(t)\left|\sum_j[\tfrac{1}{2}l_j^2G+l_j(L_j^\dagger G+GL_j)+L_j^\dagger GL_j]-(K^\dagger G+GK)\right|\hat{\eta}(t)\right\rangle dt\ , \tag{3.7}$$

with $\hat{\eta}(t)=U(t)\eta$, $\eta=\psi\otimes\omega$. Note that the mean values of terms containing dB_j and $dB_j^\dagger$ in (3.5) do not appear in (3.7); they are equal to zero, because for each j,

$$dB_j(t)U(t)\eta=U(t)dB_j(t)\eta=0\ . \tag{3.8}$$

From (3.2) and (3.7) one can easily get the forward differential equation for the generating map Γ:

$$\frac{d}{dt}\Gamma[Z]=\Gamma\left[\sum_j[\tfrac{1}{2}l_j^2Z+l_j(L_j^\dagger Z+ZL_j)+L_j^\dagger ZL_j]-K^\dagger Z-Zk\right] \tag{3.9}$$

with the initial condition $\Gamma(1,0)[Z]=Z$.

We shall prove that the solution of (3.9) has the form

$$\Gamma(1,t)[Z]=\int_{\Omega^t}Y(1,\mathbf{q}^t)V^\dagger(\mathbf{q}^t)ZV(\mathbf{q}^t)d\nu(\mathbf{q}^t) \tag{3.10}$$

with the stochastic propagator $V(t)$ being the solution of a QSDE in the form

$$dV(t)=-KV(t)dt+\sum_j L_jV(t)dQ_j(t),\ V(0)=I\ . \tag{3.11}$$

Let us define the stochastic map $\Phi(t)$ from the algebra of observables of $\mathcal{S}$ into itself,

$$\Phi(t)[Z]=V^\dagger(t)ZV(t)\ . \tag{3.12}$$

Then from Ito's formula (2.16) applied to the product appearing in (3.12) we get

$$d(\Phi(t)[Z])=dV^\dagger(t)ZV(t)+V^\dagger(t)ZdV(t)+dV^\dagger(t)ZdV(t)\ .$$

By making use of (3.11) we obtain the recursive filtering equation for the stochastic map $\Phi(t)$

$$d(\Phi(t)[Z])=\Phi(t)\left[\sum_j L_j^\dagger ZL_j-K^\dagger Z-ZK\right]dt+\sum_j\Phi(t)(L_j^\dagger Z+ZL_j)dQ_j(t)\ ,$$

$$\Phi(0)[Z]=Z\ .$$

The stochastic map (3.12) defines for any trajectory $\mathbf{q}$ the selective instrument $\Phi(t)(\mathbf{q})[Z]=\Phi(\mathbf{q}^t)[Z]=V^\dagger(\mathbf{q}^t)ZV(\mathbf{q}^t)$. Taking into account that

$$\begin{aligned}d(Y(1,t)\Phi(t)[Z])&=dY(1,t)\Phi(t)[Z]+Y(1,t)d\Phi(t)[Z]+dY(1,t)d\Phi(t)[Z]\\&=Y(1,t)\sum_j\Phi(t)[l_j(t)Z+L_j^\dagger Z+ZL_j]dQ_j(t)\\&\quad+Y(1,t)\Phi(t)\left[\sum_j[\tfrac{1}{2}l_j^2(t)Z+l_j(t)(L_j^\dagger Z+ZL_j)+L_j^\dagger ZL_j]-K^\dagger Z-ZK\right]dt\end{aligned}$$

and averaging it with respect to the standard product Wiener measure, one obtains (3.9) for the mean value (3.10) of the product $Y(1,\mathbf{q}^t)\Phi(\mathbf{q}^t)[Z]$.

So, the wave function $\hat{\chi}(t)=V(t)\psi$ of the system $\mathcal{S}$ under the continuous nondemolition diffusion observation $\hat{\mathbf{Q}}$ satisfies the stochastic dissipative differential equation

$$d\hat{\chi}(t)+\left[\frac{i}{\hbar}H+\tfrac{1}{2}\sum_j L_j^\dagger L_j\right]\hat{\chi}(t)dt=\sum_j L_j\hat{\chi}(t)dQ_j(t)\ ,\quad \hat{\chi}(0)=\psi\ . \tag{3.13}$$

Equation (3.13) plays an analogous role to the Schrödinger equation for the unobserved quantum system. [In (3.13) $d\mathbf{Q}$ can be replaced with $d\hat{\mathbf{Q}}$ because in the Schrödinger picture $\mathbf{Q}$ and $\hat{\mathbf{Q}}$ coincide.] As was mentioned earlier, the process $\mathbf{Q}$ can be diagonalized. Starting from Eq. (3.13) $\mathbf{Q}$ is considered as the classical M-dimensional diffusion process. The posterior wave function $\hat{\chi}(t)$ is normalized to the probability density

$$p(\mathbf{q}^t)=\langle V(\mathbf{q}^t)\psi|V(\mathbf{q}^t)\psi\rangle\equiv\hat{p}(t)(\mathbf{q}) \tag{3.14}$$

of the observed process $\hat{\mathbf{Q}}$ with respect to the standard product Wiener measure of the input process $\mathbf{Q}$. It follows from the integral representation of (3.2) that

$$\begin{aligned}\langle\hat{Y}(1,t)\hat{Z}(t)\rangle&=\int_{\Omega^t}Y(1,\mathbf{q}^t)\langle V(\mathbf{q}^t)\psi|ZV(\mathbf{q}^t)\psi\rangle d\nu(\mathbf{q}^t)\\&=\int_{\Omega^t}Y(1,\mathbf{q}^t)p(\mathbf{q}^t)\langle Z\rangle(\mathbf{q}^t)d\nu(\mathbf{q}^t)\ ,\end{aligned} \tag{3.15}$$

giving for $Z=I$ the mean value of the output process (3.3) as the generating function of the output probability measure

$$d\mu(\mathbf{q}^t)=p(\mathbf{q}^t)d\nu(\mathbf{q}^t)\ . \tag{3.16}$$

The formula (3.15) defines the posterior mean value $\langle Z\rangle(\mathbf{q}^t)$ as

$$\langle Z\rangle(\mathbf{q}^t)=\langle\psi(\mathbf{q}^t)|Z\psi(\mathbf{q}^t)\rangle\equiv\hat{z}(t)(\mathbf{q}) \tag{3.17}$$

in terms of the normalized posterior wave function $\hat{\psi}(t)(\mathbf{q})=\psi(\mathbf{q}^t)$, $\psi(\mathbf{q}^t)=\chi(\mathbf{q}^t)/p(\mathbf{q}^t)^{1/2}$.

The normalized posterior wave function $\hat{\psi}(t)$ satisfies the nonlinear stochastic wave equation

$$d\hat{\psi}(t)+\left[\frac{i}{\hbar}\tilde{H}(t)+\tfrac{1}{2}\sum_j\tilde{L}_j^\dagger(t)\tilde{L}_j(t)\right]\hat{\psi}(t)dt=\sum_j\tilde{L}_j(t)d\tilde{Q}_j(t)\hat{\psi}(t)\ , \tag{3.18}$$

where $\hat{l}_j(t)$ is given by (3.17) with $Z=L_j$,

$$\tilde{L}_j(t)=L_j-\mathrm{Re}\hat{l}_j(t)\ , \tag{3.19}$$

$$\tilde{H}(t)=H-\hbar\sum_j \mathrm{Re}\hat{l}_j(t)\mathrm{Im}L_j\ , \tag{3.20}$$

and

$$d\tilde{Q}_j(t)=dQ_j(t)-2\,\mathrm{Re}\hat{l}_j(t)dt \tag{3.21}$$

is the Ito differential of the observed commutative Wiener innovating process.

Equation (3.18) can be obtained from Eq. (3.13) in the following way. Writing $\hat{\psi}(t)$ in the form $\hat{\psi}(t)=\hat{\chi}(t)[\hat{\chi}^\dagger(t)\hat{\chi}(t)]^{-1/2}$ we get

$$d\hat{\psi}=d\hat{\chi}(\hat{\chi}^\dagger\hat{\chi})^{-1/2}+\hat{\chi}d((\hat{\chi}^\dagger\hat{\chi})^{-1/2})+d\hat{\chi}d((\hat{\chi}^\dagger\hat{\chi})^{-1/2})\ . \tag{3.22}$$

For $\hat{\chi}$ satisfying Eq. (3.13) one easily finds

$$d(\hat{\chi}^\dagger\hat{\chi})=2\sum_j\hat{\chi}^\dagger(\mathrm{Re}L_j)\hat{\chi}dQ_j\ ,$$

and by the classical Ito formula,

$$d((\hat{\chi}^\dagger\hat{\chi})^{-1/2})=(\hat{\chi}^\dagger\hat{\chi})^{-1/2}\left[-\sum_j\mathrm{Re}\hat{l}_j(t)dQ_j+\tfrac{3}{2}\sum_j[\mathrm{Re}\hat{l}_j(t)]^2dt\right]\ . \tag{3.23}$$

Finally, combining (3.22), (3.23), and (3.13) yields Eq. (3.18).

If the initial state $\hat{\sigma}(0)=\rho$ is a mixed one (density matrix) then its linear posterior time development [which does not preserve the normalization of $\hat{\sigma}(t)$] is given by the QSDE of the form

$$d\hat{\sigma}(t)=\left[-\frac{i}{\hbar}[H,\hat{\sigma}(t)]+\tfrac{1}{2}\sum_j\{[L_j\hat{\sigma}(t),L_j^\dagger]+[L_j,\hat{\sigma}(t)L_j^\dagger]\}\right]dt+\sum_j[L_j\hat{\sigma}(t)+\hat{\sigma}(t)L_j^\dagger]dQ_j(t)\ . \tag{3.24}$$

The latter equation follows easily from (3.13). The normalized posterior density matrix $\hat{\rho}(t)$ satisfies the following QSDE:

$$d\hat{\rho}(t)=\left[-\frac{i}{\hbar}[\tilde{H}(t),\hat{\rho}(t)]+\tfrac{1}{2}\sum_j\{[\tilde{L}_j(t)\hat{\rho}(t),\tilde{L}_j^\dagger(t)]+[\tilde{L}_j(t),\hat{\rho}(t)\tilde{L}_j^\dagger(t)]\}\right]dt+\sum_j[\tilde{L}_j(t)\hat{\rho}(t)+\hat{\rho}(t)\tilde{L}_j^\dagger(t)]d\tilde{Q}_j(t)\ , \tag{3.25}$$

which is derived from Eq. (3.24) in an analogous way to Eq. (3.18) from (3.13). In the formulas defining the quantities marked with tildes [(3.19)–(3.21)] $\hat{l}_j(t)$ now stands for the posterior mean value of $\hat{L}_j$ with respect to the mixed posterior state $\hat{\rho}(t)$: $\hat{l}_j(t)=\mathrm{Tr}[\hat{\rho}(t)L_j]$. It can be easily verified that all the tildes appearing in the term against dt in (3.25) can be omitted.

The prior dynamics (2.29) of the system $\mathcal{S}$ can be also obtained by performing the stochastic average of both sides of Eq. (3.25). To show that let us first observe that the mean value of the process $d\mathbf{Q}(t)$ conditioned by the trajectory $\mathbf{q}^t$ up to time t is given by

$$\langle d\mathbf{Q}(t)\rangle(\mathbf{q}^t)=2\,\mathrm{Re}\hat{\mathbf{l}}(t)dt\ . \tag{3.26}$$

The stochastic average of $d\mathbf{Q}(t)$ can be first obtained with the help of (2.28) as a mean value of $d\hat{\mathbf{Q}}(t)$ (the operator in $\mathcal{H}\otimes\mathcal{F}$) in the mixed initial state $\rho\otimes|\omega\rangle\langle\omega|$. Next, similarly as before, this mean can be reexpressed as the mean of the differential of the classical Wiener diffusion process. By (3.15) specified for $\mathbf{l}=\mathbf{0}$ and $Z=2\,\mathrm{Re}L_j$ and (3.16) one gets

$$\langle d\mathbf{Q}(t)\rangle_{\mathrm{st}}=\langle d\hat{\mathbf{Q}}(t)\rangle=2\langle\mathrm{Re}\hat{\mathbf{L}}(t)\rangle dt=2\,dt\int_{\Omega^t}\langle\mathrm{Re}\mathbf{L}\rangle(\mathbf{q}^t)d\mu(\mathbf{q}^t)\ , \tag{3.27}$$

hence (3.26) holds. (Here $\langle(\ )\rangle=\mathrm{Tr}[(\ )(\rho\otimes|\omega\rangle\langle\omega|)]$.)

To perform the stochastic mean of Eq. (3.25) we first take the mean of the only term directed to the future (the last one). According to (3.27) and (3.21) that mean vanishes. The averaging of the remaining terms with respect to the measure $\mu(\mathbf{q}^t)$ (up to t) yields (2.29) with

$$\rho(t)=\langle\hat{\rho}(t)\rangle_{\mathrm{st}}=\int_{\Omega^t}\rho(\mathbf{q}^t)d\mu(\mathbf{q}^t)\ .$$

IV. WATCHDOG EFFECT

The Schrödinger equation for a free particle

$$\dot{\psi}-\frac{i\hbar}{2m}\Delta\psi=0 \tag{4.1}$$

describes the effect of the spreading out of the wave packet. The probability of detection of the quantum particle in any finite coordinate region tends to zero as time increases.

Experimental data on observed quantum particles show their well-localized paths (for instance, in bubble-chamber experiments). This phenomenon does not agree with predictions of Eq. (4.1), but it should not be surprising. The typical observations in quantum systems are indirect (in the bubble chamber the path of an ionizing particle is made by a string of vapor bubbles); moreover, one has to consider the interaction with the measuring device, hence the observed quantum object should be considered as an open quantum system. The mentioned difficulty of the orthodox quantum mechanics can be resolved in the framework of the posterior quantum dynamics.

The aim of this section is to demonstrate the watchdog effect that occurs for a free quantum particle coupled to the three-dimensional Bose field in the vacuum state (measuring device), the position of which is continuously observed. We shall consider an indirect measurement of the particle position $\mathbf{X}=[X_1,X_2,X_3]$; therefore we choose the coupling operator $\mathbf{L}$ [cf. (2.17) and (2.28)] to be proportional to $\mathbf{X}$,

$$\mathbf{L}=\left[\frac{\lambda}{2}\right]^{1/2}\mathbf{X} . \tag{4.2}$$

(Throughout this section we employ the notation $\mathbf{X}\psi_{\mathbf{x}}=\mathbf{x}\psi_{\mathbf{x}}$.) With such a choice of $\mathbf{L}$ we get the QSDE's describing the perturbed dynamics of the particle in the Heisenberg picture by putting for Z in Eq. (2.21) the position and momentum components

$$\begin{aligned} d\hat{\mathbf{X}}(t)&=\frac{1}{m}\hat{\mathbf{P}}(t)dt , \\ d\hat{\mathbf{P}}(t)&=(2\lambda)^{1/2}\hbar d(\mathrm{Im}\mathbf{B}^{\dagger}(t)) . \end{aligned}\tag{4.3}$$

Equations (4.3) describe the motion of the particle upon the stochastic (Langevin) force $f(t)=(2\lambda)^{1/2}\hbar\,\mathrm{Im}\mathbf{b}^{\dagger}(t)=-(2\lambda)^{1/2}\hbar\,\mathrm{Im}\mathbf{b}(t)$ from the Bose reservoir.

The observed nondemolition field coordinate process $\hat{\mathbf{Q}}(t)$ [Eq. (2.25)] satisfies, due to (2.28) and (4.2),the QSDE in the form

$$d\hat{\mathbf{Q}}(t)=(2\lambda)^{1/2}\hat{\mathbf{X}}(t)dt+d\mathbf{Q}(t) . \tag{4.4}$$

Equation (4.4) describes the indirect (and imperfect) measurement of the particle position. Note that in terms of generalized derivatives of the processes $\hat{\mathbf{Q}}$ and $\mathbf{Q}$ Eq. (4.4) can be written as

$$\begin{aligned} \dot{\hat{\mathbf{Q}}}(t)&=(2\lambda)^{1/2}\hat{\mathbf{X}}(t)+2\,\mathrm{Re}\dot{\mathbf{B}}(t) \\ &=(2\lambda)^{1/2}\hat{\mathbf{X}}(t)+2\,\mathrm{Re}\mathbf{b}(t) ; \end{aligned}$$

therefore the (generalized) stochastic process $\dot{\hat{\mathbf{Q}}}(t)$ describes the measurement of $\hat{\mathbf{X}}(t)$ together with a random error given by the standard vector white noise $2\,\mathrm{Re}\mathbf{b}(t)$. From the last formula one can see that the positive constant λ can be interpreted as the measurement accuracy coefficient.

Let us denote by $\hat{\mathbf{q}}(t)=[\hat{\mathbf{q}}_j(t)]_{j=1}^3$ and $\hat{\mathbf{p}}(t)=[\hat{\mathbf{p}}_j(t)]_{j=1}^3$ the posterior mean values of position and momentum of the observed particle. We have

$$\begin{aligned} \hat{\mathbf{q}}(t)&=\int\hat{\psi}^*(t,\mathbf{x})\mathbf{x}\hat{\psi}(t,\mathbf{x})d\mathbf{x} , \\ \hat{\mathbf{p}}(t)&=\int\hat{\psi}^*(t,\mathbf{x})\frac{\hbar}{i}\nabla\hat{\psi}(t,\mathbf{x})d\mathbf{x} . \end{aligned}\tag{4.5}$$

According to (3.18) the posterior (normalized) wave function satisfies in the considered case the stochastic wave equation, which in the coordinate representation has the form

$$d\hat{\psi}-\left[\frac{i\hbar}{2m}\Delta\hat{\psi}-\frac{\lambda}{4}(\mathbf{x}-\hat{\mathbf{q}})^2\hat{\psi}\right]dt=\hat{\psi}\left[\frac{\lambda}{2}\right]^{1/2}(\mathbf{x}-\hat{\mathbf{q}})d\tilde{Q} ,$$

$$\hat{\psi}(0)=\psi \tag{4.6}$$

with $d\tilde{\mathbf{Q}}(t)=d\mathbf{Q}(t)-2\hat{\mathbf{q}}(t)dt$.

Let us now discuss the time development of the posterior wave function, assuming that the initial state ψ has the form of the Gaussian wave packet,

$$\psi(\mathbf{x})=(2\sigma_q^2\pi)^{-3/4}\exp\left[-\frac{1}{4\sigma_q^2}(\mathbf{x}-\mathbf{q})^2+\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}\right] , \tag{4.7}$$

where $\mathbf{p}$ and $\mathbf{q}$ denote the initial mean values of position and momentum of the particle and σ_q^2 stands for the initial dispersion of the wave packet. We shall prove that the solution of Eq. (4.6) corresponding to the initial condition (4.7) has the form of Gaussian packet

$$\hat{\psi}(t,\mathbf{x})=\hat{c}(t)\exp\left[-\tfrac{1}{2}\omega(t)[\mathbf{x}-\hat{\mathbf{q}}(t)]^2+\frac{i}{\hbar}\hat{\mathbf{p}}(t)\cdot\mathbf{x}\right] , \tag{4.8}$$

with posterior mean values $\hat{\mathbf{q}}(t),\hat{\mathbf{p}}(t)$, cf. (4.5), fulfilling linear filtration equations and $\omega(t)$ satisfying the Riccati differential equation. In Eq. (4.8), $\hat{c}(t)=(2\tau_q^2\pi)^{-3/4}$ up to the unessential stochastic phase factor and $\tau_q^2=\widehat{q^2}-\hat{\mathbf{q}}^2$ is the posterior position dispersion.

It is convenient to rewrite Eq. (4.6) in terms of the complex osmotic velocity. By introducing

$$T(t,\mathbf{x})=R(t,\mathbf{x})+iS(t,\mathbf{x})=\hbar\ln\psi(t,\mathbf{x}) ,$$

next by Ito's rule

$$dT(\hat{\psi})=T'(\hat{\psi})d\hat{\psi}+\tfrac{1}{2}T''(\hat{\psi})(d\hat{\psi})^2$$

applied to the function $T=\hbar\ln$ and by taking into account that

$$(d\hat{\psi})^2=\frac{\lambda}{2}(\mathbf{x}-\hat{\mathbf{q}})^2\hat{\psi}^2dt$$

we obtain Eq. (4.6) in terms of T. From this equation we get the following equation for the complex osmotic velocity $\mathbf{W}(t,\mathbf{x})=(1/m)\nabla T(t,\mathbf{x})=\mathbf{U}(t,\mathbf{x})+i\mathbf{V}(t,\mathbf{x})$

$$d\mathbf{W}+\left[\frac{\hbar\lambda}{m}(\mathbf{x}-\hat{\mathbf{q}})-\frac{i}{2}\left(\nabla\mathbf{W}^2+\frac{\hbar}{m}\Delta W\right)\right]dt=\left[\frac{\lambda}{2}\right]^{1/2}\frac{\hbar}{m}d\tilde{Q} . \tag{4.9}$$

We shall look for the solution of Eq. (4.9) corresponding to the initial condition

$$\mathbf{W}(0,\mathbf{x})=\frac{\hbar}{m}\nabla\ln\psi(\mathbf{x})=\frac{\hbar}{2m\sigma_q^2}(\mathbf{q}-\mathbf{x})+\frac{i}{m}\mathbf{p} . \tag{4.10}$$

in the linear form

$$\mathbf{W}(t,\mathbf{x})=\hat{\mathbf{w}}(t)-\frac{\hbar}{m}\omega(t)\mathbf{x} , \tag{4.11}$$

where in accordance with (4.8),

$$\hat{\mathbf{w}}(t)=\frac{\hbar}{m}\omega(t)\hat{\mathbf{q}}(t)+\frac{i}{m}\hat{\mathbf{p}}(t) . \tag{4.12}$$

By putting $\nabla\mathbf{W}^2=-(2\hbar\omega/m)\mathbf{W}$, $\Delta\mathbf{W}=0$ into (4.9) we obtain the following system of equations for coefficients $\hat{\mathbf{w}}(t)$ and $\omega(t)$:

$$d\hat{\mathbf{w}}(t)+\frac{i\hbar}{m}\omega(t)\hat{\mathbf{w}}(t)dt=\left[\frac{\lambda}{2}\right]^{1/2}\frac{\hbar}{m}d\mathbf{Q}(t) , \tag{4.13}$$

$$\hat{\mathbf{w}}(0)=\frac{\hbar}{2m\sigma_q^2}\mathbf{q}+\frac{i}{m}\mathbf{p}\ ,$$

$$\frac{d}{dt}\omega(t)+\frac{i\hbar}{m}\omega(t)^2=\lambda\ ,\quad \omega(0)=\frac{1}{2\sigma_q^2}\ , \tag{4.14}$$

which define the solution of Eq. (4.9) in the form (4.11). From (4.12) we get $\hat{\mathbf{q}}(t)=m\,\mathrm{Re}\hat{\mathbf{w}}(t)/\hbar\,\mathrm{Re}\omega(t)$, which is the root of the equation $\nabla R(t,\mathbf{x})=m\mathbf{U}(t,\mathbf{x})=0$ for which the maximum of the posterior density

$$|\hat{\psi}(t,\mathbf{x})|^2=\exp\left[\frac{2}{\hbar}R(t,\mathbf{x})\right]$$

is attained. The posterior mean value of momentum $\hat{\mathbf{p}}(t)$ coincides with $m\mathbf{V}(t,\hat{\mathbf{q}}(t))=\nabla S(t,\mathbf{x})|_{\mathbf{x}=\hat{\mathbf{q}}(t)}$ and by (4.12) $\hat{\mathbf{p}}(t)=\mathrm{Im}[m\hat{\mathbf{w}}(t)-\hbar\omega(t)\hat{\mathbf{q}}(t)]$.

Equation (4.12) gives the time development of posterior mean values of position and momentum; with the help of (4.13) and (4.14) we obtain the Hamilton-Langevin equations

$$\begin{aligned} d\hat{\mathbf{q}}(t)-\frac{1}{m}\hat{\mathbf{p}}(t)dt&=\frac{(\lambda/2)^{1/2}}{\mathrm{Re}\omega(t)}d\tilde{\mathbf{Q}}(t)\ ,\quad \hat{\mathbf{q}}(0)=\mathbf{q}\\ d\hat{\mathbf{p}}(t)&=-\hbar\frac{(\lambda/2)^{1/2}\mathrm{Im}\omega(t)}{\mathrm{Re}\omega(t)}d\tilde{\mathbf{Q}}(t)\ ,\quad \hat{\mathbf{p}}(0)=\mathbf{p}\ . \end{aligned} \tag{4.15}$$

They are classical stochastic equations describing continuously and indirectly observed position and momentum of a free quantum particle disturbed by the measuring device [in the mean $\hat{\mathbf{p}}(t)$ and $\hat{\mathbf{q}}(t)$ coincide with $\mathbf{q}(t)=\mathbf{p}t/m$, $\mathbf{p}(t)=\mathbf{p}$].

One can check easily that for the posterior wave function in the form (4.8), posterior momentum and position dispersions are given by the formulas

$$\tau_q^2(t)=1/2\,\mathrm{Re}\omega(t)\ ,\quad \tau_p^2(t)=\hbar^2|\omega(t)|^2/2\,\mathrm{Re}\omega(t)\ , \tag{4.16}$$

with $\omega(t)$ being the solution of Eq. (4.14). These formulas yield the Heisenberg inequality $\tau_q^2\tau_p^2\geq\hbar^2/4$.

The general solution of Eq. (4.14) has the form

$$\omega(t)=\alpha\frac{\omega(0)+\alpha\tanh\left[\frac{\lambda}{\alpha}t\right]}{\omega(0)\tanh\left[\frac{\lambda}{\alpha}t\right]+\alpha}\ ,\quad \alpha=\left[\frac{\lambda m}{2\hbar}\right]^{1/2}(1-i)\ . \tag{4.17}$$

Obviously, $\lim_{t\to\infty}\omega(t)=\alpha$, i.e., α is the asymptotic stationary solution of Eq. (4.14). Consequently, the posterior dispersions of position and momentum tend to finite limits independent of its initial values

$$\tau_q^2(\infty)=(\hbar/2\lambda m)^{1/2}\ ,\quad \tau_p^2(\infty)=\hbar(\lambda m\hbar/2)^{1/2}\ , \tag{4.18}$$

giving the localization of the observed quantum particle. As follows from (4.18) the asymptotic localization of the particle in the coordinate representation in inversely proportional to its mass and the measurement accuracy λ. This means that the particle of mass zero cannot be localized by any measurement, and heavy particles ($m\to\infty$) can be localized at a point. Note that according to the dimension of λ, $[\lambda]=(m^2\mathrm{sec})^{-1}$, the measurement accuracy coefficient can be interpreted as inversely proportional to the scattering cross section and characteristic time of the transition process in a bubble chamber. The result of this section can be slightly strengthened by relaxing the assumption (4.7). It has been proved [55] that the Gaussian wave packet with the dispersion $\tau_q^2(\infty)$ given by (4.18) is the only (up to a stochastic phase factor) asymptotic solution of Eq. (4.6) for any initial square-integrable wave function. Some further results on the dynamics of the observed quantum particle in a quadratic potential (including the case of a free particle) can be found in Ref. [55]. In particular, it has been proved that Eq. (3.18) for the quantum particle with continuously observed (1) position, (2) momentum, and (3) position and momentum is uniquely relaxing. Any initial state of the particle given by a square-integrable function relaxes to the unique (up to a stochastic phase factor) Gaussian wave packet with a given dispersion (depending on the case of observation).

If the results of the measurement are not read out, i.e., they are averaged, the time development of the state (density matrix) of the free particle is governed by Eq. (2.29) with $K=(i/\hbar)H+\frac{1}{2}\mathbf{L}^2$ [cf. (2.18)], $\mathbf{L}=(\lambda/2)^{1/2}\mathbf{X}$, $H=-(\hbar^2/2m)\Delta$:

$$\dot{\rho}(t)=\frac{i\hbar}{2m}[\Delta,\rho(t)]-\frac{\lambda}{4}[\mathbf{X},[\mathbf{X},\rho(t)]]\ . \tag{4.19}$$

The spreading out of the wave packet is even faster than for the unobserved (isolated) particle: the dispersion of the particle position spreads out no longer as t^2 but as t^3. This should not be surprising: if the results of the measurement are not read out, the apparatus does not help the pure initial state to survive (by supplying it with the information contained in the measurement data). On the contrary, it only disturbs the system. For more details see Ref. [55].

The asymptotic localization of the free quantum particle (4.18) was also obtained by other authors. In the approach of Caves and Milburn this result was achieved [53] in a continuous limit of the succeeding instantaneous nonideal measurements of the particle position under the assumption of the Gaussian character of the initial wave function and the Gaussian character of the instrument [29]. Diósi [48,49] obtained (4.18) in the context of his phenomenological stochastic equation similar to Eq. (3.18). We would like to emphasize the substantial difference between his equation as well as other stochastic equations of type (3.18) or (3.25) supporting "dynamical theories of wave-function reduction" [37–49] and the ones presented by us. The term $d\tilde{Q}_j=dQ_j-2\,\mathrm{Re}\hat{l}_j dt$ in our equations is always replaced with the stochastic differential of the standard Wiener diffusion process. The reason for that is very simple: the phenomenological equations have to satisfy the only criterion—to yield (after stochastic averaging) the master equation of type (2.29). There is no reason to justify any other choice of $d\tilde{\mathbf{Q}}$ than the simplest one.

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