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Harmonic oscillator with time-dependent mass and frequency and a perturbative potential

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A general treatment of the quantized harmonic oscillator with time-dependent mass and frequency is presented. The treatment is also applied to the time-dependent oscillator under the action of a time-dependent perturbative potential. The treatment is based on the use of some time-dependent transformations and in the method of invariants of Lewis and Riesenfeld [J. Math. Phys. **10**, 1458 (1969)]. Exact coherent states for such systems are also constructed.

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I. INTRODUCTION

The study of problems involving harmonic oscillators with time-dependent frequencies or with time-dependent masses (or both simultaneously) or with a time-dependent perturbative potential has attracted considerable interest in the past few years [1–17,23,27,28]. Apart from its intrinsic mathematical interest, these problems have invoked much attention because of their connections with many other problems belonging to different areas of physics, such as plasma physics, gravitation, quantum optics, etc. For example, Colegrave and Abdalla [18] studied the harmonic oscillator with a constant frequency and a time-dependent mass in order to describe the electromagnetic field intensities in a Fabry-Pérot cavity. Lemos and Natividade [19] studied a harmonic oscillator with a time-dependent frequency and a constant mass in an expanding universe. Also, Khandekar and Lawande [27] have solved the harmonic oscillator under the action of a particular time-dependent perturbative potential.

In this paper, we present an alternative treatment of the quantal harmonic oscillator with time-dependent mass and frequency. The treatment is also applied to the time-dependent oscillator under the action of a general perturbative potential. The treatment is based on the use of some time-dependent transformations and in the method of invariants of Lewis and Riesenfeld [20]. Exact coherent states for such systems are also constructed. This paper is organized in the following manner. In Secs. II–IV we outline our treatment by considering the harmonic oscillator with time-dependent mass and frequency. In Sec. V we construct exact coherent states and calculate the uncertainty relations. In Sec. VI we apply the treatment for constructing coherent states for the oscillator acted on by a general time-dependent perturbative force. Finally, some concluding remarks are added in Sec. VII.

II. TIME-DEPENDENT HARMONIC OSCILLATOR

Consider the time-dependent harmonic-oscillator Hamiltonian

$$H(t)=\frac{p^2}{2M(t)}+\tfrac{1}{2}M(t)\omega^2(t)q^2 , \tag{2.1}$$

where q and p are canonically conjugate with $[q,p]=i\hbar$ and $M(t)$ and $\omega(t)$ are, respectively, the mass and frequency associated with the oscillator, and which are arbitrary real functions of time. From (2.1) we obtain the equation of motion

$$\ddot{q}+\gamma(t)\dot{q}+\omega^2(t)q=0 , \tag{2.2}$$

where

$$\gamma(t)=\frac{d}{dt}[\ln M(t)] . \tag{2.3}$$

In Ref. [21], it was shown that the Hamiltonian (2.1) can be transformed to $H_1(t)$,

$$H_1(t)=\frac{P^2}{2m}+\frac{m\Omega^2(t)}{2}Q^2 , \tag{2.4}$$

where

$$\Omega^2(t)=\omega^2(t)-\left[\frac{\gamma^2(t)}{4}+\frac{\dot{\gamma}(t)}{2}\right] , \tag{2.5}$$

by means of the new canonical variables

$$Q=\left[\frac{m}{M(t)}\right]^{1/2}q , \tag{2.6a}$$

$$P=\left[\frac{m}{M(t)}\right]^{1/2}p+[mM(t)]^{1/2}\frac{\gamma(t)}{2}q , \tag{2.6b}$$

that can be achieved by the following generating function:

$$F(q,P,t)=\tfrac{1}{2}(qP+Pq)\left[\frac{m}{M(t)}\right]^{-1/2}-\frac{M(t)\gamma(t)}{4}q^2 , \tag{2.7}$$

where m is a constant mass. Note that $[Q,P]=[q,p]$, which implies that the commutation relations remain the same in both coordinates. Also, observe that the Hamiltonian (2.4) is of the form of that considered by Lewis and Riesenfeld [20]. Here, let us recall that these authors have developed a general theory of explicitly time-dependent invariants for quantum systems characterized by explicitly time-dependent Hamiltonians. They have derived a simple relation between eigenstates of such an invariant and solutions of the corresponding Schrödinger equation. In the next section we briefly review the theory of Lewis and Riesenfeld for the system characterized by the transformed Hamiltonian (2.4).

III. TIME-DEPENDENT INVARIANTS AND THE SCHRÖDINGER EQUATION

It is well known that an exact invariant for (2.4) is given by [1,21]

$$I(t)=\frac{1}{2m}[m^2Q^2\rho^{-2}+(P\rho-m\dot{\rho}Q)^2]\ , \tag{3.1}$$

where $Q(t)$ satisfies the equation

$$\ddot{Q}+\Omega^2(t)Q=0\ , \tag{3.2}$$

and $\rho(t)$ is a c-number quantity satisfying the auxiliary equation

$$\ddot{\rho}+\Omega^2(t)\rho=1/\rho^3\ . \tag{3.3}$$

The invariant $I(t)$ satisfies the equation [20,22]

$$\frac{dI}{dt}=\frac{\partial I}{\partial t}+\frac{1}{i\hbar}[I,H_1]=0,\quad I^\dagger=I\ . \tag{3.4}$$

In order to make $I(t)$ Hermitian, we choose only the real solutions of (3.3). Further, the eigenfunctions $\phi_n(Q,t)$ of $I(t)$ are assumed to form a complete orthonormal set corresponding to the time-independent eigenvalue λ_n. Thus

$$I\phi_n(t)=\lambda_n\phi_n(Q,t)\ , \tag{3.5}$$

$$(\phi_{n'},\phi_n)=\delta_{n'n}\ . \tag{3.6}$$

Now consider the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=H_1(t)\psi\ , \tag{3.7}$$

with

$$H_1(t)=-\frac{\hbar}{2m}\frac{\partial^2}{\partial Q^2}+\frac{m\Omega^2(t)}{2}Q^2\ , \tag{3.8}$$

where $P=-i\hbar\partial/\partial Q$ has been used. The solutions $\psi_n(Q,t)$ of the Schrödinger equation (3.7) are related to $\phi_n(Q,t)$ by the relation [20]

$$\psi_n(Q,t)=e^{i\alpha_n(t)}\phi_n(Q,t)\ , \tag{3.9}$$

where the phase functions $\alpha_n(t)$ satisfy the equation

$$\hbar\frac{d\alpha_n(t)}{dt}=\left\langle\phi_n\left|i\hbar\frac{\partial}{\partial t}-H_1(t)\right|\phi_n\right\rangle\ . \tag{3.10}$$

Then, since each ψ_n satisfies the Schrödinger equation, the general solution of (3.7) may be written as

$$\psi(Q,t)=\sum_n c_n e^{i\alpha_n(t)}\phi_n(Q,t)\ , \tag{3.11}$$

where the c_n are time-independent coefficients.

IV. SOLUTION OF THE SCHRÖDINGER EQUATION

In this section we are interested in solving the Schrödinger equation (3.7). To this end, we follow the procedure developed in Refs. [23] and [24]. Consider the unitary transformation

$$U=e^{-im\dot{\rho}Q^2/(2\hbar\rho)}\ . \tag{4.1}$$

Under this unitary transformation the eigenvalue equation (3.5) becomes

$$I'\phi'_n(\sigma)=\lambda_n\phi'_n(\sigma)\ , \tag{4.2}$$

with

$$I'=UIU^\dagger=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\sigma^2}+\frac{m\sigma^2}{2}\ , \tag{4.3}$$

and

$$\phi'_n=\rho^{1/2}U\phi_n\ ,\quad \sigma=Q/\rho\ . \tag{4.4}$$

Now (4.2) is an ordinary one-dimensional Schrödinger equation whose solution is given by

$$\phi'_n(Q,t)=\left[\frac{m^{1/2}}{\pi^{1/2}\hbar^{1/2}n!2^n}\right]^{1/2}\times e^{-m/2\hbar(Q/\rho)^2}H_n\left[\left[\frac{m}{\hbar}\right]^{1/2}\frac{Q}{\rho}\right]\ , \tag{4.5}$$

where

$$\lambda_n=\hbar(n+\tfrac{1}{2}) \tag{4.6}$$

and H_n is the usual Hermitian polynomial of order n. Thus, by using (3.9), (4.1), (4.4), and (4.5) we find that the solution of the transformed Schrödinger equation (3.7) is given by

$$\psi_n(Q,t)=e^{i\alpha_n(t)}\left[\frac{m^{1/2}}{\pi^{1/2}\hbar^{1/2}n!2^n\rho}\right]^{1/2}\times\exp\left[\frac{im}{2\hbar}\left[\frac{\dot{\rho}}{\rho}+\frac{i}{\rho^2}\right]Q^2\right]H_n\left[\left[\frac{m}{\hbar}\right]^{1/2}\frac{Q}{\rho}\right]\ , \tag{4.7}$$

where the phase functions $\alpha_n(t)$ are given by [20,24]

$$\alpha_n(t)=-(n+\tfrac{1}{2})\int_0^t\frac{dt'}{\rho^2(t')}\ . \tag{4.8}$$

Here it is interesting to observe that the solutions (4.7) for Eq. (3.7) have also been obtained by Khandekar and Lawande [25] by using Feynman path integrals.

Let us now introduce the time-dependent transformation [21]

$$\rho(t)=\left[\frac{M(t)}{m}\right]^{1/2}x(t)\ , \tag{4.9}$$

where $x(t)$ is a real function of time that is to be determined. Then using (2.5), (2.6), and (4.9), the equation of motion (3.2) is converted into the original equation (2.2) and auxiliary equation (3.3) in the equation

$$\ddot{x}+\gamma(t)\dot{x}+\omega^2(t)x=\left[\frac{m}{M(t)}\right]\Big/x^3 \ . \quad (4.10)$$

The exact invariant (3.1) is transformed to the form

$$I(t)=\frac{1}{2m}[mq^2x^{-2}+(px-M\dot{x}q)^2] \ . \quad (4.11)$$

Also, in terms of the original variables, the eigenfunctions $\phi_n(q,t)$ of $I(t)$ are given by

$$\phi_n(q,t)=\left[\frac{1}{2^n n!\hbar^{1/2}}\left[\frac{m^2}{\pi M(t)x^2}\right]^{1/2}\right]^{1/2} \times\exp\left[\frac{iM(t)}{2\hbar}\left[\frac{\dot{x}}{x}+\frac{\gamma(t)}{2}+\frac{im}{M(t)x^2}\right]q^2\right] \times H_n\left[\left[\frac{m}{\hbar x^2}\right]^{1/2}q\right] \ . \quad (4.12)$$

Note that for $M(t)=me^{\gamma t}$ the above solution reduces to that obtained by Khandekar and Lawande [26]. Now the solutions $\psi_n(q,t)$ of the Schrödinger equation for the original system may be written as $\psi_n(q,t)=\exp[i\alpha_n(t)]\phi_n(q,t)$ where the phase functions $\alpha_n(t)$ are now given by

$$\alpha_n(t)=-(n+\tfrac{1}{2})\int_0^t\frac{m}{M(t')x^2(t')}dt' \ . \quad (4.13)$$

Note that when $M(t)\to m$, $\omega(t)\to\omega_0$, and $x(t)\to x_0=\text{const}=1/\omega_0^{1/2}$ [which is a particular solution of the auxiliary equation (4.10)], the solution (4.12) becomes the solution of the Schrödinger equation for the time-independent harmonic oscillator of mass m and frequency ω_0.

V. COHERENT STATES AND UNCERTAINTY RELATIONS

To obtain coherent states for harmonic oscillator with time-dependent mass and frequency we proceed as follows. Consider the operators A and $A^\dagger$ given by

$$A=\left[\frac{1}{2m\hbar}\right]^{1/2}\left[m\left[\frac{Q}{\rho}\right]+i\rho P\right] , \quad (5.1a)$$

$$A^\dagger=\left[\frac{1}{2m\hbar}\right]^{1/2}\left[m\left[\frac{Q}{\rho}\right]-i\rho P\right] , \quad (5.1b)$$

where $[A,A^\dagger]=1$. In terms of A and $A^\dagger$ the invariant I' [see (4.3)] can be written as

$$I'=\hbar(A^\dagger A+\tfrac{1}{2}) \ . \quad (5.2)$$

Now, Hartley and Ray [27] have shown that coherent states for I' have the form

$$\phi'_\alpha(\sigma,t)=e^{-|\alpha|^2/2}\sum_n\frac{\alpha^n}{(n!)^{1/2}}e^{i\alpha_n(t)}\phi'_n(\sigma) , \quad (5.3)$$

where $\alpha_n(t)$ is given by (4.7) and α is an arbitrary complex number. Note that when $\omega(t)\to\omega_0$ and $\rho(t)\to\rho_0=1/\omega_0^{1/2}$, the coherent states ϕ'_α become the correct coherent states for the usual time-independent harmonic oscillator.

The coherent states for the time-dependent oscillator (3.8) are obtained by the inverse transformation on $\phi'_\alpha(\sigma,t)$. They are given by

$$\phi_\alpha(Q,t)=\rho^{-1/2}e^{im\dot{\rho}Q^2/(2\hbar\rho)}\phi'_\alpha(\sigma,t) , \quad (5.4)$$

which, in terms of the original variables, can be rewritten as

$$\phi_\alpha(q,t)=\left[\left[\frac{m}{M(t)}\right]^{1/2}x\right]^{-1/2} \times\exp\left[-iM(t)\left[\dot{x}+\frac{\gamma(t)}{2}x\right]\left[\frac{q^2}{(2\hbar x)}\right]\right]\phi'_\alpha(\sigma,t) , \quad (5.5)$$

where $\sigma=q/x$. The states (5.5) are coherent states for the time-dependent system described by the Hamiltonian (2.1). These states satisfy the eigenvalue equation

$$a\phi_\alpha(q,t)=\alpha(t)\phi_\alpha(q,t) , \quad (5.6)$$

with

$$a=U^\dagger AU=\left[\frac{1}{2m\hbar}\right]^{1/2}\left[m\left[\frac{q}{x}\right]+i(xp-M\dot{x}q)\right] , \quad (5.7)$$

and

$$\alpha(t)=\alpha e^{2i\alpha_0(t)} , \quad (5.8)$$

$$\alpha_0(t)=-\tfrac{1}{2}\int_0^t\frac{m\,dt'}{M(t')x^2(t')} \ . \quad (5.9)$$

Note that, when $M(t)=m=cte$, the states reduce to the coherent states of the time-dependent harmonic oscillator where only the frequency is allowed to change with time [27].

In what follows we wish to calculate the uncertainty relation. After some calculation we find that the uncertainties in q and p in the state $\phi_\alpha(q,t)$ are

$$(\Delta q)^2=\frac{\hbar}{2m}x^2 , \quad (5.10a)$$

$$(\Delta p)^2=\frac{m\hbar}{2}\left[\frac{1}{x^2}+\left[\frac{M}{m}\right]^2x^2\dot{x}^2\right]^{1/2} . \quad (5.10b)$$

Thus the uncertainty product is expressed as

$$(\Delta q)(\Delta p)=\frac{\hbar}{2}\left[1+\frac{M^2(t)}{m^2}x^2\dot{x}^2\right]^{1/2} , \quad (5.11)$$

and, in general, does not attain its minimum value. However, for a time-dependent oscillator, we cannot expect to find strictly coherent states, i.e., $(\Delta q)(\Delta p)=\hbar/2$ for all time t. On the other hand, we have already shown [28,29] that the states $\phi_\alpha(q,t)$ are equivalent to well-known squeezed states whose characteristic feature is the squeezing. Now for $M(t)=m=cte$, the uncertainty product (5.11) reduces to that obtained in Ref. [27].

Also, when $M(t)=m$ and $x(t)\to x_0=1/\omega_0^{1/2}$ the uncertainty relation (5.11) attains its minimum value. In this case, the operators a and $a^\dagger$ given in (5.7) reduce, respectively, to the usual annihilation and creation operators and the states $\phi_\alpha(q,t)$ become the correct coherent states for the time-independent harmonic oscillator.

VI. COHERENT STATES FOR THE OSCILLATOR WITH A PERTURBATIVE POTENTIAL

Consider the system described by the Hamiltonian

$$H(t)=\frac{p^2}{2M(t)}+\tfrac{1}{2}M(t)\omega^2(t)q^2+\frac{m^2}{M(t)x^2}g(q/x)\ , \qquad (6.1)$$

which possesses an invariant given by [21]

$$I(t)=\frac{1}{2m}[(px-M\dot{x}q)^2+m^2q^2x^{-2}+2m^2g(q/x)]\ , \qquad (6.2)$$

where $x(t)$ satisfies (4.10). Note that the Schrödinger equation for (6.1) does not depend on the auxiliary variable x when (i) $g=0$, the Lewis-Riesenfeld [20] problem, and (ii) $g=M(t)x^2/m^2q^2$, the problem treated by Khandekar and Lawande [25,26]. In all other cases the auxiliary function x appears in the potential energy of the Schrödinger equation associated with (6.1). It is then to be interpreted [23] as an external field whose time dependence is to be determined from the auxiliary equation (4.10). In the uncoupled systems $g=0$ and $M(t)x^2/m^2q^2$, x is just an auxiliary variable whose particular form drops out of any calculation of transition matrix elements [20,23]. However, in the coupled case, x is a physical field whose form determines the interaction of the system with the field through the interaction potential $m^2g(q/x)/Mx^2$.

Now, following the same steps of the precedent sections, we convert the Hamiltonian (6.1) in the form

$$H_2(t)=\frac{P^2}{2m}+\frac{m\Omega^2(t)}{2}Q^2+\frac{m^2}{\rho^2}g(Q/\rho)\ , \qquad (6.3)$$

where $\Omega(t)$ is given by (2.5) and $\rho(t)$ satisfies Eq. (3.3). The invariant (6.2) is converted into the form

$$I(t)=\frac{1}{2m}[(P\rho-m\rho\dot{Q})^2+m^2Q^2\rho^{-2}+2m^2g(Q/\rho)]\ . \qquad (6.4)$$

Now, following the same steps as those of Sec. IV, we have, in this case, I' given by

$$I'=UIU^\dagger=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\sigma^2}+\frac{m\sigma^2}{2}+mg(\sigma)\ . \qquad (6.5)$$

Then, the eigenvalue equation (4.2) becomes

$$\left[-\frac{\hbar}{2m}\frac{\partial^2}{\partial\sigma^2}+\frac{m\sigma^2}{2}+mg(\sigma)\right]\phi_n'=\lambda_n\phi_n'\ . \qquad (6.6)$$

So, the solution to the Schrödinger equation for (6.3) involves solving Eq. (6.6). Note that (6.6) is the time-independent Schrödinger equation for an arbitrary potential. Thus, we can use the invariant I' to construct coherent states $\phi_\alpha(\sigma,t)$ following Refs. [24] and [30]. For this purpose we define classical variables $X_c(\sigma)$ and $P_c(\sigma)$ associated with the classical motion of a Hamiltonian with "Hamiltonian" I'. Then, we have that [24,30]

$$X_c'(\sigma)=\frac{dX_c(\sigma)}{d\sigma}=\omega_c\left[\frac{m/2(A^2-X_c^2)^{1/2}}{I'-V(\sigma)}\right]\ , \qquad (6.7a)$$

$$P_c(\sigma)=m\dot{X}_c=p_\sigma X_c'\ , \qquad (6.7b)$$

where ω_c and A are constants and $p_\sigma=\dot{\sigma}$. So, the Nieto-Simmons operators X and P are given by

$$X(\sigma)\equiv X_c(\sigma)\ , \qquad (6.8a)$$

$$P\equiv\frac{\hbar}{2i}\left[\frac{\partial}{\partial\sigma}X'(\sigma)+X'(\sigma)\frac{\partial}{\partial\sigma}\right]\ , \qquad (6.8b)$$

with $[X,P]=i\hbar(X_c')^2$. Now, we use these operators to construct coherent states $\phi_\alpha'(\sigma)$. Following Refs. [24] and [30] we obtain

$$\frac{1}{2}\left[\frac{X}{\Delta X}+i\frac{P}{\Delta P}\right]\phi_\alpha'(\sigma)=\alpha\phi_\alpha'(\sigma)\ , \qquad (6.9)$$

where $\Delta M=(\langle M^2\rangle-\langle M\rangle^2)^{1/2}$ for any quantity M and

$$\alpha=\frac{1}{2}\left[\frac{\langle X\rangle}{\Delta X}+i\frac{\langle P\rangle}{\Delta P}\right]\ . \qquad (6.10)$$

The coherent states $\phi_\alpha'(\sigma)$ have the form

$$\phi_\alpha'(\sigma)=\sum c_n\phi_n'(\sigma)\ , \qquad (6.11)$$

where c_n are constants and $\phi_n'(\sigma)$ are eigenstates of I'. Next consider the unitary operator [24]

$$V=\exp\left[-i\frac{I'}{\hbar}\int\frac{dt'}{\rho^2}\right]\ . \qquad (6.12)$$

Then, the coherent states $\phi_\alpha'(\sigma,t)$ are now given by

$$\phi_\alpha'(\sigma,t)=\exp\left[-i\frac{I'}{\hbar}\int\frac{dt'}{\rho^2(t')}\right]\phi_\alpha'(\sigma)$$
$$=\sum_n c_n e^{i\alpha_n(t)}\phi_n'(\sigma)\ , \qquad (6.13)$$

and the new operators $\bar{X}$ and $\bar{P}$ by

$$\bar{X}=VXV^\dagger\ ,\quad \bar{P}=VPV^\dagger\ . \qquad (6.14)$$

Note that the operator V introduces the time-dependent phase factor $\exp[i\alpha_n(t)]$ into (6.13). For more details see Ref. [24]. Then, by using (4.1) we find the coherent states for the Hamiltonian (6.3)

$$\phi_\alpha(Q,t)=\rho^{-1/2}e^{im\dot{\rho}Q^2/(2\hbar\rho)}\phi_\alpha'(\sigma,t)\ . \qquad (6.15)$$

The new operators $\bar{X}'$ and $\bar{P}'$ are also given by

$$\bar{X}'=U^\dagger\bar{X}U\ ,\quad \bar{P}'=U^\dagger\bar{P}U\ . \qquad (6.16)$$

These states, in terms of the original variables, are coherent states for the time-dependent system described by the Hamiltonian (6.1).

VII. CONCLUDING REMARKS

In this paper we have presented an alternative treatment for the quantal harmonic oscillator with time-dependent mass and frequency and for the problem of the time-dependent oscillator under the action of a perturbative potential. The present treatment is based on the use of a time-dependent canonical transformation, two unitary transformations, an auxiliary time-dependent transformation, and on the method of invariants of Lewis and Riesenfeld. We also have used the procedures developed in Refs. [24], [27], and [30], to construct coherent states for two such systems. These coherent states have been expressed in terms of the eigenstates of the invariants I and are more general than those obtained in Refs. [24] and [27]. The physical interpretation of these states is discussed in Refs. [24] and [27]. For the time-dependent harmonic oscillator, they are states that are associated with the exact classical motion [24]. They should be useful in describing the radiation field of a single-mode laser as the laser is tuned. If the Nieto-Simmons coherent states have practical applications for molecule-laser interactions, then the time-dependent coherent states derived here should have similar applications involving such time-dependent systems.

Finally, we remark that Janussis and Bartzis [31] have also constructed coherent states for the harmonic oscillator with time-dependent mass and frequency. However, the approach used by these authors is considerably different from that presented in this paper. We also mention that it may be interesting to compare our treatment with those developed by Leach [9], Abdalla [13], and Colegrave and Abdalla [8].

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