PHYSICAL REVIEW A VOLUME 45, NUMBER 3 1 FEBRUARY 1992

Reduced Schrödinger-Coulomb Green's function for excited states

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(Received 9 August 1991)

The momentum-space representation of the reduced Schrödinger-Coulomb Green's function is obtained in closed form for all states n. Explicit forms are given for $n=1$ and 2.

PACS number(s): 03.65.Ge

I. INTRODUCTION

The Green's function for the Schrödinger equation is an extremely useful object in quantum mechanics. The Green's function can be written as a sum over states

$$G_E=\sum_m \frac{|m\rangle\langle m|}{(E-E_m)} \tag{1}$$

in Dirac notation. It satisfies the inhomogeneous time-independent Schrödinger equation

$$(E-H)G_E=1\ . \tag{2}$$

The Green's function contains complete information about the energy levels and wave functions for the Hamiltonian H.

The reduced Green's function $\hat{G}_n$ is obtained from G_E by subtracting out the E_n pole term and taking the limit $E\rightarrow E_n$. The reduced Green's function can be written as

$$\hat{G}_n=\sum_{m,m\neq n} \frac{|m\rangle\langle m|}{(E_n-E_m)}\ . \tag{3}$$

The formulas of bound-state perturbation theory can be expressed in terms of $\hat{G}_n$. For example, the energy-level shift in the presence of a perturbing Hamiltonian H' is [1]

$$\begin{aligned}\Delta E_n &= \langle n|H'|n\rangle+\sum_{m,m\neq n}\frac{\langle n|H'|m\rangle\langle m|H'|n\rangle}{(E_n-E_m)}+O(H'^3)\\ &= \langle n|H'|n\rangle+\langle n|H'\hat{G}_nH'|n\rangle+O(H'^3)\ .\end{aligned} \tag{4}$$

Of particular interest are the Green's function and reduced Green's function for a particle in the Coulomb potential [2] $V(r)=-Z\alpha/r$. The Schrödinger-Coulomb Green's function was worked out by Hostler [3–5], Schwinger [6], and others [7,8] in various forms. The reduced Schrödinger-Coulomb Green's function in the coordinate representation has been considered by several authors [9–13]. In the momentum representation, Hostler [14] has given forms for the excited-state reduced Green's functions in one-dimensional space along with a prescription for obtaining the three-dimensional results, and Douglas [15] has given forms containing one-parameter integrals for the $n=1$ and 2 reduced Green's functions.

One important use for the reduced Schrödinger-Coulomb Green's function is in bound-state QED [15–19]. Modern bound-state formalisms for QED are built around a soluble reference problem that describes the Coulombic binding. The solution to the reference problem is based on the known nonrelativistic solution with the four-dimensional structure of relativity added on. As an example, in positronium the $n=2$ energy splittings have been measured [20] with an accuracy of a few megahertz. A theoretical calculation to that level of accuracy will require corrections of order $m\alpha^6$. These corrections involve the reduced Green's function, which will be needed for $n=2$. The bound-state perturbation scheme is usually set up in the momentum representation, so we will be interested in the momentum-space representation of the reduced Green's function $\hat{G}_n(\mathbf{p},\mathbf{p}')=\langle\mathbf{p}|\hat{G}_n|\mathbf{p}'\rangle$.

In this paper we develop a closed-form expression for the reduced Schrödinger-Coulomb Green's function $\hat{G}_n(\mathbf{p},\mathbf{p}')$ in three dimensions for all states n. We start with Schwinger's one-parameter integral representation of the full Green's function and perform the subtraction and limit in a straightforward way. The form obtained still contains a one-parameter integral that we evaluate in terms of Gegenbauer and Chebyshev polynomials. The reduced Green's functions for the most useful cases $n=1$ and 2 are written explicitly. In Appendix A we show how the momentum-space Coulomb bound-state wave functions can be obtained from the Green's function. In Appendix B we show that the three-dimensional reduced Green's function derived from Hostler's one-dimensional form [14] is equivalent to our result.

II. DERIVATION OF THE REDUCED GREEN'S FUNCTION

The Schrödinger-Coulomb Green's function has the form [4,6,7]

$$G_E(\mathbf{p},\mathbf{p}')=(-2m)\left[\frac{(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')}{D}+\frac{8\pi\gamma}{DRD'}+\frac{32\pi\gamma^2p_0}{DD'}\int_0^1 dx\, x^{-\xi}\frac{1}{H(\mathbf{p},\mathbf{p}')}\right], \tag{5}$$

where

$$\gamma = mZ\alpha,\quad p_0=\sqrt{-2mE}\,,\quad \xi=\gamma/p_0\,,$$
$$D=\mathbf{p}^2+p_0^2,\quad D'=\mathbf{p}'^2+p_0^2,\quad R=(\mathbf{p}-\mathbf{p}')^2\,, \tag{6}$$
$$H=4xp_0^2R+(1-x)^2DD'\,.$$

It is the sum of a zero-potential (or free) term that is of order α^0, a one-potential term that is of order α^1, and a many-potential term that, because of the $\gamma^2 x^{-\gamma/p_0}$, is a sum of terms having order α^2 and higher. The Green's function has poles at the bound-state energies

$$E_n=-\frac{\gamma_n^2}{2m},\quad \gamma_n=\frac{\gamma}{n}\,, \tag{7}$$

with the form

$$G_E(\mathbf{p},\mathbf{p}')\to\sum_{l,m}\frac{\psi_{nlm}(\mathbf{p})\psi^*_{nlm}(\mathbf{p}')}{(E-E_n)}\quad \text{as } E\to E_n\,. \tag{8}$$

The reduced Green's function is

$$\hat{G}_n(\mathbf{p},\mathbf{p}')=\lim_{E\to E_n}\left[G_E(\mathbf{p},\mathbf{p}')-\sum_{l,m}\frac{\psi_{nlm}(\mathbf{p})\psi^*_{nlm}(\mathbf{p}')}{(E-E_n)}\right], \tag{9}$$

where all states of energy E_n are subtracted out. The bound-state poles are found in the x integral in the many-potential term of $G_E(\mathbf{p},\mathbf{p}')$. The pole at energy $E=E_n$ (which corresponds to $\xi=n$) can be isolated by making subtractions on $1/H$ [21]

$$\int_0^1 dx\, x^{-\xi}\frac{1}{H}=\int_0^1 dx\, x^{-\xi}\left[\frac{1}{H}-\sum_{m=0}^{n-1}\frac{x^m}{m!}\left[\frac{1}{H}\right]^{(m)}\right]+\sum_{m=0}^{n-1}\frac{1}{m!}\left[\frac{1}{H}\right]^{(m)}\int_0^1 dx\, x^{-\xi+m}$$
$$=\int_0^1 dx\, x^{-\xi}\left[\frac{1}{H}-\sum_{m=0}^{n-1}\frac{x^m}{m!}\left[\frac{1}{H}\right]^{(m)}\right]+\sum_{m=0}^{n-2}\frac{1}{m!}\left[\frac{1}{H}\right]^{(m)}\frac{1}{-\xi+m+1}+\frac{1}{(n-1)!}\left[\frac{1}{H}\right]^{(n-1)}\frac{1}{-\xi+n}\,, \tag{10}$$

where we use the notation

$$f^{(m)}=\left[\frac{d}{dx}\right]^m f\bigg|_{x=0}\,. \tag{11}$$

The integrals are evaluated using the assumption that $E>0$ (so ξ is imaginary). Negative values of E (positive values of ξ) can be obtained by analytic continuation. It is clear that the pole at $\xi=n$ resides only in the last term, which can be written as

$$\frac{1}{(n-1)!}\left[\frac{1}{H}\right]^{(n-1)}\frac{1}{-\xi+n}=\frac{-1}{n!}\left[\frac{1}{H}\right]^{(n-1)}\frac{p_0(p_0+\gamma_n)}{2m}\frac{1}{(E-E_n)}\,. \tag{12}$$

The pole can be isolated using

$$\frac{A(E)}{(E-E_n)}+B(E)=\frac{A(E_n)}{(E-E_n)}+A'(E_n)+B(E_n)+O(E-E_n)\,. \tag{13}$$

The pole term contains the Coulomb wave functions as discussed in Appendix A. The remainder, after subtracting the pole term and taking the $E\to E_n$ limit, is

$$\hat{G}_n(\mathbf{p},\mathbf{p}')=(-2m)\left[\frac{(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')}{D_n}+\frac{8\pi\gamma}{D_nRD_n'}+\hat{I}_n(\mathbf{p},\mathbf{p}')\right], \tag{14}$$

$$\hat{I}_n(\mathbf{p},\mathbf{p}')=\frac{32\pi n^2\gamma_n^3}{(D_nD_n')^2}\left\{(\tfrac{5}{2}-4B_n)\frac{1}{n!}\left[\frac{1}{h}\right]^{(n-1)}-\frac{2(1-B_n)}{A_n}\frac{1}{n!}\left[\frac{x}{h^2}\right]^{(n-1)}\right.$$
$$\left.+\sum_{m=0}^{n-2}\frac{1}{(m-n+1)m!}\left[\frac{1}{h}\right]^{(m)}+\int_0^1\frac{dx}{x^n}\left[\frac{1}{h}-\sum_{m=0}^{n-1}\frac{x^m}{m!}\left[\frac{1}{h}\right]^{(m)}\right]\right\}, \tag{15}$$

where

$$D_n=\mathbf{p}^2+\gamma_n^2,\quad D_n'=\mathbf{p}'^2+\gamma_n^2\,,$$
$$A_n=\frac{D_nD_n'}{4\gamma_n^2R},\quad B_n=\frac{\gamma_n^2(D_n+D_n')}{D_nD_n'}\,, \tag{16}$$

and

$$h=\frac{H|_{E=E_n}}{D_nD_n'}=1-2\beta x+x^2,\quad \beta=1-\frac{1}{2A_n}\,. \tag{17}$$

The reduced Green's function can be expressed in closed form in terms of Gegenbauer and Chebyshev polynomials. The polynomials are defined by the generating functions

$$\frac{1}{(1-2\beta x+x^2)^\lambda}=\sum_{m=0}^{\infty} x^m C_m^\lambda(\beta) \;, \tag{18a}$$

$$-\tfrac{1}{2}\ln(1-2\beta x+x^2)=\sum_{m=1}^{\infty}\frac{x^m}{m}T_m(\beta) \;. \tag{18b}$$

It follows that [22]

$$\frac{1}{(n-1)!}\left[\frac{1}{h}\right]^{(n-1)}=C_{n-1}^1(\beta)\;, \qquad \frac{1}{(n-1)!}\left[\frac{x}{h^2}\right]^{(n-1)}=C_{n-2}^2(\beta)\;. \tag{19}$$

The integral in $\hat{I}_n$ has integrand $N_n(x)/h$ where the numerator

$$\begin{aligned} N_n(x)&=\frac{h}{x^n}\left[\frac{1}{h}-\sum_{m=0}^{n-1}\frac{x^m}{m!}\left[\frac{1}{h}\right]^{(m)}\right] \\ &=\frac{h}{x^n}\left[\sum_{m=0}^{\infty}x^m C_m^1(\beta)-\sum_{m=0}^{n-1}x^m C_m^1(\beta)\right] \\ &=\frac{h}{x^n}\sum_{m=n}^{\infty}x^m C_m^1(\beta) \end{aligned} \tag{20}$$

is a power series with no negative powers of x. Written as

$$N_n(x)=\frac{1}{x^n}\left[1-h\sum_{m=0}^{n-1}x^m C_m^1(\beta)\right]\;, \tag{21}$$

it is clear that $N_n(x)$ has maximum power x^1. Taking the x^0 term from (20) and the x^1 term from (21), one has

$$N_n(x)=C_n^1(\beta)-C_{n-1}^1(\beta)x\;. \tag{22}$$

It follows that the integral in $\hat{I}_n$ is

$$\begin{aligned} \int_0^1\frac{dx}{x^n}&\left[\frac{1}{h}-\sum_{m=0}^{n-1}\frac{x^m}{m!}\left[\frac{1}{h}\right]^{(m)}\right] \\ &=\int_0^1\frac{dx}{h}[C_n^1(\beta)-C_{n-1}^1(\beta)x] \\ &=\frac{1}{(1-\beta^2)^{1/2}}T_n(\beta)\arctan\left[\frac{1+\beta}{1-\beta}\right]^{1/2} \\ &\quad-\tfrac{1}{2}C_{n-1}^1(\beta)\ln[2(1-\beta)]\;, \end{aligned} \tag{23}$$

since

$$\int_0^1 dx\frac{a+bx}{h}=\frac{(a+\beta b)}{(1-\beta^2)^{1/2}}\arctan\left[\frac{1+\beta}{1-\beta}\right]^{1/2}+\frac{b}{2}\ln[2(1-\beta)] \tag{24}$$

(where $1-\beta^2\geq 0$) and [23]

$$C_n^1(\beta)-\beta C_{n-1}^1(\beta)=T_n(\beta)\;. \tag{25}$$

One can replace β in the result for the integral with A_n according to

$$(1-\beta^2)^{1/2}=\frac{(4A_n-1)^{1/2}}{2A_n}\;, \qquad \left[\frac{1+\beta}{1-\beta}\right]^{1/2}=(4A_n-1)^{1/2}\;, \qquad 2(1-\beta)=\frac{1}{A_n}\;. \tag{26}$$

Then a closed form for $\hat{I}_n$ is [21]

$$\begin{aligned} \hat{I}_n(\mathbf{p},\mathbf{p}')=\frac{32\pi n^2\gamma_n^3}{(D_nD_n')^2}&\left[(\tfrac{5}{2}-4B_n)\frac{1}{n}C_{n-1}^1(\beta)-\frac{2}{n}\frac{(1-B_n)}{A_n}C_{n-2}^2(\beta)+\sum_{m=0}^{n-2}\frac{1}{(m-n+1)}C_m^1(\beta)\right. \\ &\left.+\frac{2A_n}{(4A_n-1)^{1/2}}T_n(\beta)\arctan[(4A_n-1)^{1/2}]+\tfrac{1}{2}C_{n-1}^1(\beta)\ln(A_n)\right]\;. \end{aligned} \tag{27}$$

This, combined with (14), gives our closed form for the reduced Green's function $\hat{G}_n$.

III. EXPLICIT FORMS FOR $n=1$ AND 2

Explicit forms for the reduced Green's function $\hat{G}_n$ for any value of n can now be written. For the most useful cases $n=1$ and 2 we will need

$$C_0^\lambda(\beta)=1,\quad C_1^\lambda(\beta)=2\lambda\beta,\quad T_1(\beta)=\beta,\quad T_2(\beta)=2\beta^2-1\;. \tag{28}$$

Then the reduced Green's functions $\hat{G}_1$ and $\hat{G}_2$ are given by (14) with

$$\hat{I}_1(\mathbf{p},\mathbf{p}')=\frac{32\pi\gamma^3}{(D_1D_1')^2}\left[\frac{5}{2}-4B_1+\frac{2A_1-1}{\sqrt{4A_1-1}}\arctan(\sqrt{4A_1-1})+\tfrac{1}{2}\ln(A_1)\right]\;, \tag{29}$$

and

$$\hat{I}_2(\mathbf{p},\mathbf{p}')=\frac{128\pi\gamma_2^3}{(D_2D_2')^2}\left[\frac{3}{2}-\frac{9}{4A_2}-4B_2+\frac{3B_2}{A_2}+\frac{2A_2-4+\frac{1}{A_2}}{\sqrt{4A_2-1}}\arctan(\sqrt{4A_2-1})+\left[1-\frac{1}{2A_2}\right]\ln(A_2)\right] . \quad (30)$$

This form for $\hat{I}_1$ agrees with that obtained previously by Caswell and Lepage [16]. As a check of these results we verified that

$$\frac{1}{(2\pi)^3}\int d^3p'\hat{G}_n(\mathbf{p},\mathbf{p}')\psi_{rlm}(\mathbf{p}')=\frac{\delta_{n\neq r}}{(E_n-E_r)}\psi_{rlm}(\mathbf{p}) \quad (31)$$

(where the δ function is one for $n\neq r$ and zero for $n=r$) holds for $r=1,2$ and all l,m.

IV. CONCLUSION

We have obtained a closed-form result for the momentum-space representation of the reduced Coulomb Green's function $\hat{G}_n(\mathbf{p},\mathbf{p}')$ in three dimensions for all n. This expression should prove useful for calculations of properties of Coulombic systems in atomic physics and bound-state QED.

ACKNOWLEDGMENTS

We gratefully acknowledge the support of the NSF through Grant No. PHY90-08449 and of the Franklin and Marshall College through the Hackman Scholars Program.

APPENDIX A: DERIVATION OF THE COULOMB WAVE FUNCTIONS FROM THE GREEN'S FUNCTION

The Coulomb wave functions $\psi_{nlm}(\mathbf{p})$ can be obtained from the Green's function $G_E(\mathbf{p},\mathbf{p}')$ by factoring the residue of the pole at $E=E_n$ into a term dependent on $\mathbf{p}$ and a term dependent on $\mathbf{p}'$. The pole of G_E is contained in the x integral in the many-potential part of G_E. From (10), (12), and (13) it is clear that the residue of the pole is

$$(-2m)\left[\frac{32\pi\gamma^2\gamma_n}{D_nD_n'}\frac{-1}{n!}\left[\frac{1}{H}\right]^{(n-1)}\bigg|_{E=E_n}\frac{\gamma_n(2\gamma_n)}{2m}\right] =\frac{64\pi n\gamma_n^5}{(D_nD_n')^2}C_{n-1}^1(\beta) . \quad (A1)$$

The Gegenbauer polynomial $C_{n-1}^1(\beta)$ can be expanded with the help of the addition theorem for Gegenbauer polynomials [24]

$$C_{n-1}^{\lambda}(\beta)=\frac{\Gamma(2\lambda-1)}{[\Gamma(\lambda)]^2}\sum_{l=0}^{n-1}(-1)^l4^l\Gamma(n-l)[\Gamma(l+\lambda)]^2\frac{(2l+2\lambda-1)}{\Gamma(n+l+2\lambda-1)} \times\left[\frac{2p\gamma_n}{D_n}\right]^l\left[\frac{2p'\gamma_n}{D_n'}\right]^l C_{n-l-1}^{l+\lambda}\left[\frac{\bar{D}_n}{D_n}\right]C_{n-l-1}^{l+\lambda}\left[\frac{\bar{D}_n'}{D_n'}\right]C_l^{\lambda-1/2}(-u) , \quad (A2)$$

where $\bar{D}_n=\mathbf{p}^2-\gamma_n^2$, $\bar{D}_n'=\mathbf{p}'^2-\gamma_n^2$, and $u=\hat{p}\cdot\hat{p}'$. In our case $\lambda=1$ and we use

$$C_l^{1/2}(-u)=P_l(-u)=(-1)^lP_l(u) \quad (A3)$$

and the addition theorem for spherical harmonics

$$P_l(u)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\hat{p})Y_{lm}^*(\hat{p}') \quad (A4)$$

to write the residue as

$$\frac{64\pi n\gamma_n^5}{(D_nD_n')^2}\sum_{l=0}^{n-1}\sum_{m=-l}^{l}4^l\Gamma(n-l)[\Gamma(l+1)]^2\frac{4\pi}{\Gamma(n+l+1)} \times\left[\frac{2p\gamma_n}{D_n}\right]^l\left[\frac{2p'\gamma_n}{D_n'}\right]^l C_{n-l-1}^{l+1}\left[\frac{\bar{D}_n}{D_n}\right] \times C_{n-l-1}^{l+1}\left[\frac{\bar{D}_n'}{D_n'}\right]Y_{lm}(\hat{p})Y_{lm}^*(\hat{p}') . \quad (A5)$$

This has the form

$$\sum_{l=0}^{n-1}\sum_{m=-l}^{l}\psi_{nlm}(\mathbf{p})\psi_{nlm}^*(\mathbf{p}') , \quad (A6)$$

where the wave functions are

$$\psi_{nlm}(\mathbf{p})=\phi_nN_{nl}\frac{p^l\gamma_n^{l+1}}{D_n^{l+2}}C_{n-l-1}^{l+1}\left[\frac{\bar{D}_n}{D_n}\right]Y_{lm}(\hat{p}) . \quad (A7)$$

The constants in (A7) are the coordinate-space wave functions at contact

$$\phi_n=\left[\frac{\gamma_n^3}{\pi}\right]^{1/2}=\psi_{n00}(\mathbf{x}=0) \quad (A8)$$

and

$$N_{nl}=2^{2l+3}\pi l!\left[\frac{4\pi n(n-l-1)!}{(n+l)!}\right]^{1/2} . \quad (A9)$$

The normalization condition is [25]

$$\frac{1}{(2\pi)^3}\int d^3p\, \psi^*_{nlm}(\mathbf{p})\psi_{n'l'm'}(\mathbf{p})=\delta_{n,n'}\delta_{l,l'}\delta_{m,m'} . \quad \text{(A10)}$$

These momentum-space Coulomb wave functions were obtained previously by Podolsky and Pauling [25] through Fourier transformation of the coordinate-space results, by Fock [26], who solved the momentum-space Schrödinger equation directly, and by Schwinger from the momentum-space Green's function [6,8]. They are discussed more thoroughly in the book by Bethe and Salpeter [27].

APPENDIX B: DERIVATION OF THE THREE-DIMENSIONAL REDUCED GREEN'S FUNCTION FROM HOSTLER'S ONE-DIMENSIONAL RESULT

In this Appendix we show that the three-dimensional reduced Green's function derived from Hostler's one-dimensional form is identical to our result. Hostler's normalization differs from ours by a factor of $2m$: he defines the Green's function by $-2m(H-E)G_E=1$ [see Ref. [14], Eq. (2)] while we use $(E-H)G_E=1$. Hostler's one-dimensional reduced Green's function [Ref. [14], Eq. (71)], times -1, written in our notation ($k_1\to p$, $k_2\to p'$, $na_1\to 1/\gamma_n$, $r\to\beta$, and with $q=p'-p$) is

$$-K_1(E_n)=\frac{2\pi\delta(q)}{D_n}-\frac{2\gamma_n}{D_nD_n'}+\sum_{l=0}^{n-1}\frac{4\gamma_n}{D_nD_n'}\frac{nT_l(\beta)}{(l-n)}+\frac{2(3-4B_n)\gamma_n}{D_nD_n'}T_n(\beta)-\frac{16n(1-B_n)\gamma_n^3q^2}{(D_nD_n')^2}C^1_{n-1}(\beta)$$
$$-\frac{4(1-\beta^2)^{1/2}\gamma_n}{D_nD_n'}nC^1_{n-1}(\beta)\arctan\left[\left[\frac{1+\beta}{1-\beta}\right]^{1/2}\right]-\frac{4\gamma_n}{D_nD_n'}\frac{n}{2}T_n(\beta)\ln[2(1-\beta)] . \quad \text{(B1)}$$

Hostler showed that the operator

$$\mathcal{H}=-2\pi\frac{1}{q}\frac{\partial}{\partial q} \quad \text{(B2)}$$

acts to increase by 2 the dimensionality d of Green's functions and reduced Green's functions:

$$\mathcal{H}\{-K_d(E_n)\}=-K_{d+2}(E_n) . \quad \text{(B3)}$$

We will write Hostler's operator as

$$\mathcal{H}=-4\pi\frac{\partial\beta}{\partial q^2}\frac{\partial}{\partial\beta}=\frac{8\pi\gamma_n^2}{D_nD_n'}\frac{\partial}{\partial\beta} \quad \text{(B4)}$$

and apply it to $-K_1(E_n)$. Some useful derivative formulas are

$$\frac{\partial}{\partial\beta}T_l(\beta)=lC^1_{l-1}(\beta) , \quad \text{(B5a)}$$

$$\frac{\partial}{\partial\beta}C^1_{n-1}(\beta)=2C^2_{n-2}(\beta) , \quad \text{(B5b)}$$

$$\frac{\partial}{\partial\beta}q^2=-\frac{D_nD_n'}{2\gamma_n^2} , \quad \text{(B5c)}$$

$$\frac{\partial}{\partial\beta}\left\{(1-\beta^2)^{1/2}C^1_{n-1}(\beta)\arctan\left[\left[\frac{1+\beta}{1-\beta}\right]^{1/2}\right]\right\}$$
$$=\tfrac{1}{2}C^1_{n-1}(\beta)-\frac{n}{(1-\beta^2)^{1/2}}T_n(\beta)\arctan\left[\left[\frac{1+\beta}{1-\beta}\right]^{1/2}\right] . \quad \text{(B5d)}$$

The identity

$$-C^1_{n-1}(\beta)+\frac{T_n(\beta)}{(1-\beta)}=\frac{1}{(1-\beta)}-2\sum_{l=1}^{n}C^1_{l-1}(\beta) , \quad \text{(B6)}$$

which can be proved by induction, will help in the reduction. We will identify

$$\frac{16\pi n\gamma_n^3}{(D_nD_n')^2}\frac{1}{(1-\beta)}=\frac{8\pi\gamma}{D_nRD_n'} \quad \text{(B7)}$$

as the one-potential term in (14). Also necessary is the curious formula

$$\mathcal{H}\{(2\pi)^d\delta^d(\mathbf{q})\}=(2\pi)^{d+2}\delta^{d+2}(\mathbf{q}) \quad \text{(B8)}$$

relating the d- and $(d+2)$-dimensional Dirac δ functions. It is now an easy exercise to show that the three-dimensional expression $-K_3(E_n)$ obtained from Hostler's one-dimensional result $-K_1(E_n)$ by action of $\mathcal{H}$ agrees with our result [(14) and (27)] for the reduced Green's function in three dimensions:

$$-K_3(E_n)=\mathcal{H}\{-K_1(E_n)\}=\frac{1}{(-2m)}\hat{G}_n(\mathbf{p},\mathbf{p}') . \quad \text{(B9)}$$

[1] For an application of this, see G. S. Adkins and R. F. Hood, Eur. J. Phys. **10**, 61 (1989).
[2] In Gaussian units, the fine-structure constant is $\alpha=e^2/\hbar c\approx(137)^{-1}$. The Coulomb potential for an electron in the field of a "nucleus" having charge Ze is thus $V(r)=-Z\alpha/r$, where we use the convention that $\hbar=c=1$.
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[21] The sum over m from 0 to $(n-2)$ is not present for $n=1$.
[22] We define $C_n^{\lambda}(\beta)=0$ for $n<0$.
[23] This follows from the generating functions. One has $\sum_{m=1}^{\infty} x^m T_m(\beta) = x(d/dx)(-1/2)\ln h = (1-\beta x)/h - 1$. The identity follows when $1/h$ is expanded in Gegenbauer polynomials and like powers of x are compared.
[24] This theorem, attributed to Gegenbauer, is found in A. Erdélyi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. I, p. 177, with the substitutions $n \to n-1$, $z \to \bar{D}_n/D_n$, and $z' \to \bar{D}'_n/D'_n$.
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