PHYSICAL REVIEW A VOLUME 45, NUMBER 2 15 JANUARY 1992

Intermittently chaotic oscillations for a differential-delay equation with Gaussian nonlinearity

Ian Hamilton
Department of Chemistry, The University of Ottawa, Ottawa, Canada K1N 9B4
(Received 1 July 1991)

For a differential-delay equation the time dependence of the variable is a function of the variable at a previous time. We consider a differential-delay equation with Gaussian nonlinearity that displays intermittent chaos. Although not the first example of a differential-delay equation that displays such behavior, for this example the intermittency is classified as type III, and the origin of the intermittent chaos may be qualitatively understood from the limiting forms of the equation for large and small variable magnitudes.

PACS number(s): 05.45.+b, 03.20.+i

Differential-delay equations [1] model the dynamics of systems for which the time dependence of the variable is a function of the variable at $t-\tau$, where τ is the delay time. Differential-delay equations have been used for systems in which the variable is coupled to other variables for which the time dependence is unknown (because, for example, these variables are transients). For these cases there is therefore a loss of information, and only in favorable instances is it possible to determine the form of the differential-delay equation.

Differential-delay equations are classified as linear or nonlinear depending on the form of the function of $t-\tau$. For linear differential-delay equations there may be monotonic decay or periodic oscillations [1] and, for some choices of parameter values, there are periodic oscillations superimposed on monotonic decay. In general, monotonic decay and periodic oscillations superimposed on monotonic decay correspond to fixed-point behavior, while periodic oscillations correspond to limit-cycle behavior [2], and both types of regular dynamics are readily observed. Linear differential-delay equations may be integrated analytically using a variety of techniques [3], although these may be impractical for large τ. For nonlinear differential-delay equations there is, in general, complicated behavior with irregular dynamics. Analytical integration is not feasible, but numerical integration is efficient and accurate. Various nonlinear differential-delay equations have been used to model a variety of kinetic phenomena [4] in areas ranging from biology [5] to optics [6]. Of particular relevance here, LeBerre, Ressayre, and Tallet [7] recently considered a differential-delay equation with exponential nonlinearity that displays intermittent chaos [2,8]. Intermittent chaos, which has been observed experimentally [9], is of interest since it is one stage in a standard model for the transition to turbulence [10]. For the differential-delay equation considered here, the origin of the intermittent chaos may be qualitatively understood from the limiting forms of the equation for large and small variable magnitudes.

We consider a differential-delay equation that is linear when the magnitude of the variable is far from zero but nonlinear when the magnitude of the variable is close to zero. This could model a system in which the dynamics of the variable is affected by a local (attractive) force, the position of which defines the zero of the variable. Specifically, we consider a differential-delay equation with Gaussian nonlinearity,

$$x'(t)=-Px(t), \quad t\le\tau \tag{1a}$$

$$x'(t)=-Px(t)-Qx(t-\tau)\times\{1-\exp[-x(t-\tau)/R]^2\}/R, \quad t>\tau , \tag{1b}$$

where P, Q, and R are positive and the value of R determines the nonlinearity. We consider high nonlinearity (small R) only. For $\tau=0$ there is monotonic decay of $x(t)$, but for $\tau>0$ there is, in general, complicated behavior. For Eq. (1) the limiting forms of $x'(t)$ for $|x(t-\tau)|\gg 0$ and $|x(t-\tau)|\approx 0$ are easily seen to be

$$x'(t)=-Px(t)-Qx(t-\tau)/R, \quad |x(t-\tau)|\gg 0 \tag{2a}$$

$$x'(t)=-Px(t)-Qx^3(t-\tau)/R^3, \quad |x(t-\tau)|\approx 0 . \tag{2b}$$

Equation (2a) is a linear differential-delay equation and, for some choices of parameter values, there are periodic oscillations superimposed on monotonic decay. However, Eq. (2b) is a nonlinear differential-delay equation and, for small R, the coefficient of the term in $x^3(t-\tau)$ is large. Consequently, there is rapid change in $x(t)$ when $x(t-\tau)$ passes through zero. Qualitatively, the coupling of this rapid change to the periodic oscillations is the mechanism that causes irregular dynamics when the magnitude of the variable is small. In this respect, Eq. (1) is analogous to the equation for the periodically kicked pendulum [11], a well-studied system for which the dynamics is chaotic. Below, we perform a linear stability analysis by considering the maximal Floquet multiplier [8], which is calculated from the linearized form of Eq. (1).

For type-I intermittency, destabilization occurs via a saddle node (or reverse tangent) bifurcation at critical parameter values [8]. This destabilization process, for a differential-delay equation with exponential nonlinearity, is considered by LeBerre, Ressayre, and Tallet [7], but the destabilization process for Eq. (1) is not considered here. Rather, for Eq. (1), we consider intermediate pa-

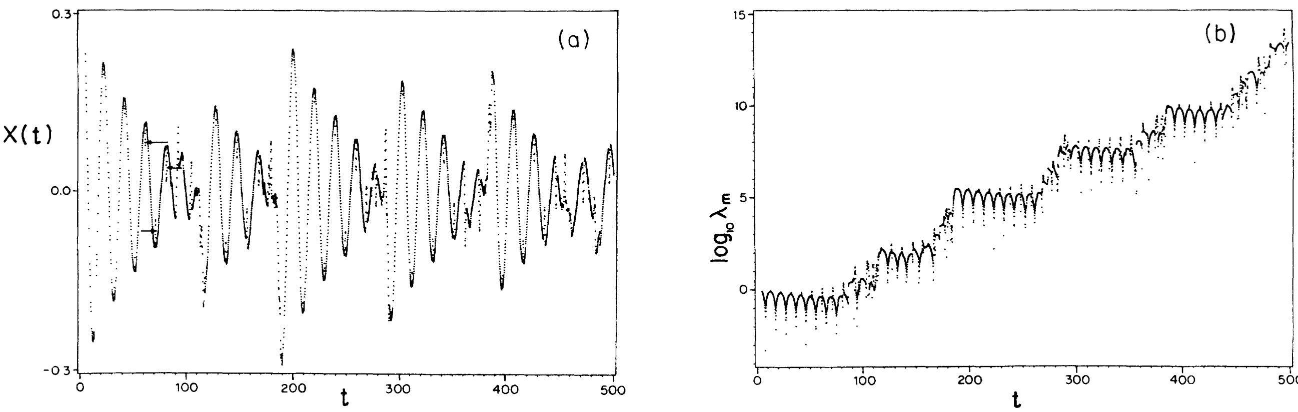

FIG. 1. (a) $x(t)$ (arrows are explained in text) and (b) $\log_{10}\lambda_m(t)$, for $t=0$–500 for Eq. (1) with $P=0.003$, $Q=0.03$, $R=0.01$, $\tau=5$, and $x(0)=0.3$.

rameter values where this equation displays intermittent chaos, and we examine the phase-space structure of the trajectory in the linear and nonlinear regimes.

Equation (1) was numerically integrated using the well-tested method of Gear [12], with fixed step size Δ (here we choose $\Delta=0.01$). We first obtained the solution of Eq. (1a) for $0<t\leq\tau$ and stored the τ/Δ values of $x(t)$. These values were then updated at each step of the solution of Eq. (1b). For certain parameter values, Eq. (1) models the observed irregular oscillations in the condensation kinetics of CCl_3F [13]. Here we chose $P=0.03$, $Q=0.3$, $R=0.01$, and $\tau=5$, with initial condition $x(0)=0.3$, which is chosen as being a typical large fluctuation for these parameter values. For this R value Eq. (1) is highly nonlinear and might be realistic only under extreme conditions of temperature or pressure. Figure 1(a) shows $x(t)$ for $t=0$–500. It may be seen that there are periodic oscillations superimposed on monotonic decay

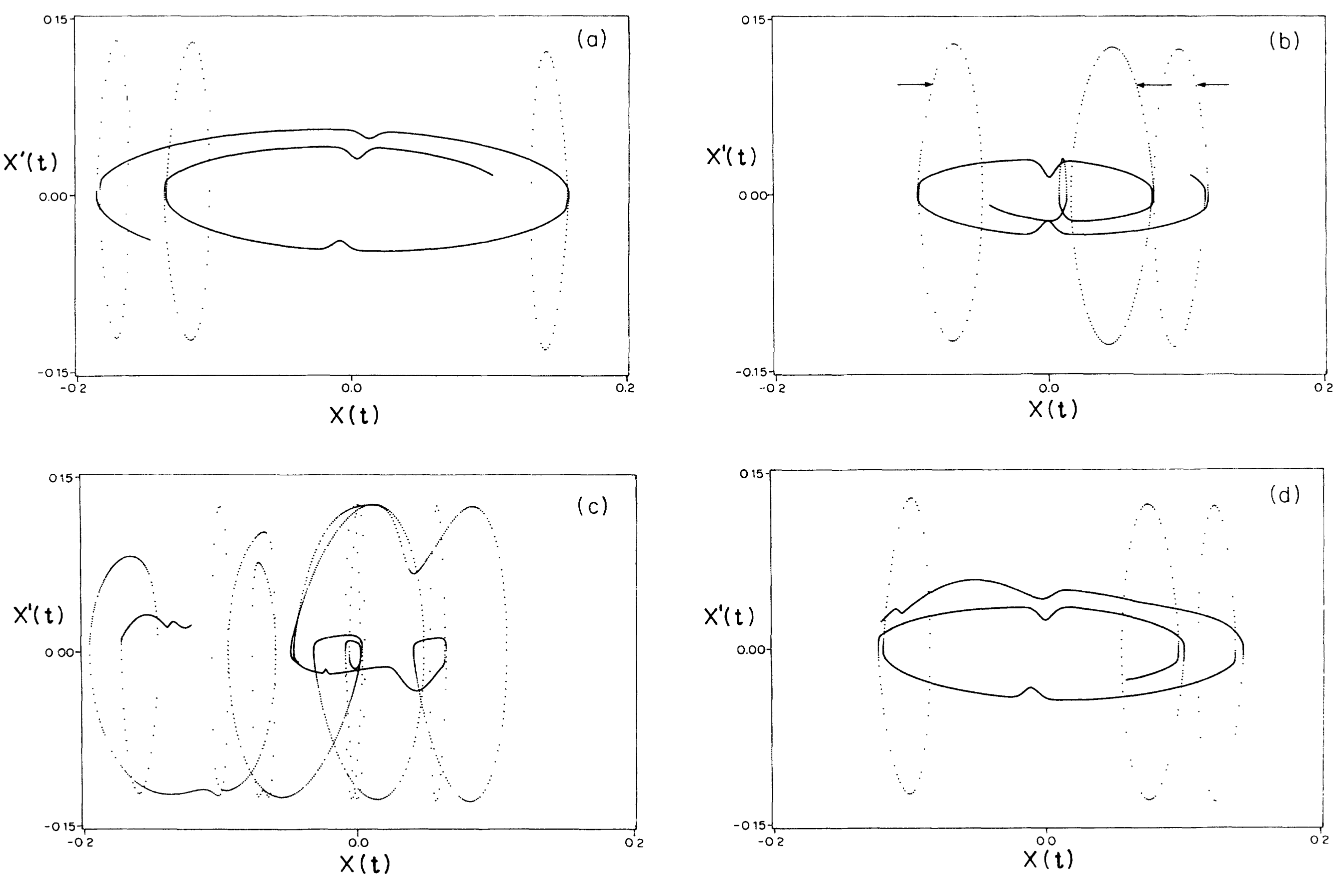

FIG. 2. Phase plots of $x'(t)$ vs $x(t)$ for the trajectory of Fig. 1 for (a) $t=30$–60, (b) $t=60$–90 (arrows are explained in text), (c) $t=90$–120, and (d) $t=120$–150.

(for example, from $t=0$ to 90) that are followed by chaotic bursts (for example, from $t=90$ to 120). The spikes in Fig. 1(a) (indicated for arrows for $t=60-90$, as explained below), which are evident before each chaotic burst when $x(t)$ reaches an extremum, are considered below in conjunction with the phase-space structure of the trajectory.

For a differential-delay equation the number of degrees of freedom is infinite, but, in practice (as indicated above), the delay interval is discretized into a finite number of intervals. This results in a corresponding number of Floquet multipliers and, in principle, all of these are required for a stability analysis [7]. Here we simply consider the maximal Floquet multiplier, $\lambda_m(t)$, which is calculated from the linearized form of Eq. (1), and which is assumed to be much greater than the other Floquet multipliers. Figure 1(b) shows $\log_{10}\lambda_m(t)$ for $t=0-500$. It may be seen that there are oscillatory decreases in $\log_{10}\lambda_m(t)$ when there are periodic oscillations superimposed on monotonic decay in Fig. 1(a). However, there are sharp increases in $\log_{10}\lambda_m(t)$ when there are chaotic bursts in Fig. 1(a). The overall rate of increase of $\log_{10}\lambda_m(t)$ [the limit as t tends to infinity of $(1/t)\log_{10}\lambda_m(t)$] is directly proportional to the maximal Floquet exponent [8]. From Fig. 1(b) it is clear that the maximal Floquet exponent is a (finite) positive number, as expected when the dynamics is chaotic.

We now consider the phase-space structure of a segment of the above trajectory. Useful information may be obtained from the $x(t)-x(t-\tau)$ and related phase plots, and from the corresponding surfaces of section, but here we simply consider the $x'(t)-x(t)$ phase plot. Figures 2(a) and 2(b) show the $x'(t)-x(t)$ phase plot for $t=30-60$ and 60–90 respectively. Points are plotted at a time interval of 0.0075, and the phase-space structures appear as solid lines only when the phase-space velocity is small. It may be seen that the phase-space trajectory spirals, with slow velocity, towards $(x(t),x'(t))=(0,0)$, although there are loops, with fast velocity, when $x(t)$ reaches an extremum. The loops in Fig. 2 [indicated by arrows in Fig. 2(b)], which become more prominent as the spiral approaches $(x(t),x'(t))=(0,0)$, correspond to the spikes in Fig. 1(a) [indicated by arrows for the time interval of Fig. 2(b)]. When a loop has completed, the phase-space trajectory has returned to the spiral at approximately the same phase-space point. Consequently, a loop results in a phase-space delay of the trajectory and manifests a period doubling of the variable, as there is then a second frequency for the variable. Because of this period doubling, and the fixed point-limit cycle behavior for regular dynamics, the intermittency is classified as type III.

Figure 2(c) shows the $x'(t)-x(t)$ phase plot for $t=90-120$, and it may be seen that, as anticipated from the limiting form of Eq. (1) for small variable magnitudes, the phase-space structure is irregular. The phase-space structure remains irregular while the trajectory remains in the nonlinear regime, but towards the end of this time interval there is a large (negative) fluctuation in $x(t)$ that "randomly" reinjects the variable into the linear regime. Figure 2(d) shows the $x'(t)-x(t)$ phase plot for $t=120-150$, and it may be seen that the phase-space structure is regular and is similar to that for Fig. 2(a). This sequence of phase-space structures is repeated for subsequent segments of the above trajectory. Similar results are obtained for other choices of $x(0)$, although there is a long relaxation time if $x(0)$ is large. For smaller R values, the average time interval for which the phase-space structure remains regular is longer primarily because the fluctuations in $x(t)$ are larger and the variable is "randomly" reinjected deeper into the linear regime. Thus, for the differential-delay equation with Gaussian nonlinearity, the limiting forms of the equation for large and small variable magnitudes provide a qualitative understanding of the origin of the intermittent chaos.

I wish to thank NSERC (Canada) for partial funding of this work and the University of Ottawa for a grant of computer time.

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