Optical ring cavities as tailored four-level systems: An application of the group U(2,2)

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We report on the experimental realization and theoretical analysis of four-level systems produced in passive optical ring cavities by introducing various intracavity elements. Birefringent elements couple waves with different polarizations and thus lift the polarization degeneracy of the cavity modes. Reflecting elements couple waves propagating in different directions and lift the propagation degeneracy. A combination of these elements leads, in general, to a splitting of every longitudinal cavity mode into a four-mode system. Under the restriction that the optical components are loss-free, the description of these ring cavities in a transmission-matrix formalism leads in a natural way to the study of the Lie group U(2,2) and its Lie algebra u(2,2). We associate each of the 16 generators of this algebra with a specific type of optical element, some of which are standard components, others not. From the commutation relations of the generators, we derive a recipe for the construction of the nonstandard components as a sequence of standard ones. It follows that the entire U(2,2) group can actually be realized. The number of independent parameters—16 for a general U(2,2) element—is shown to be reduced substantially if the optical components are selected out of a restricted number of types, provided that the corresponding generators define a subalgebra of u(2,2). In such cases, a subgroup of U(2,2) is realized and the number of independent parameters of the optical system is given by the number of generators of the subalgebra. A connection is established between the subalgebras and symmetry properties of the optical components in the cavity. We consider the influence of symmetries on the mode structure and consider our experimental results from this viewpoint. In particular, we discuss time-reversal invariance, leading to the symplectic group USp(2,2), and we identify antiunitary symmetries leading to Kramers's degeneracy in the mode spectrum. We propose the application of the group U(2,2) in optics as a tool yielding direct practical results.

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I. INTRODUCTION

It has recently been demonstrated that optical ring cavities allow convenient experimental implementation of two-level systems, including the possibility to drive these systems and study their dynamical behavior [1–6]. In these experiments two levels were created by lifting either the propagation or the polarization degeneracy of a single longitudinal mode of the optical ring cavity. The resulting two normal modes of the cavity—the two levels—were still twofold degenerate: when the polarization degeneracy was lifted, the propagation degeneracy remained and vice versa. In this paper, we shall be concerned with the four-level structure that results if both degeneracies are removed. A practical example of such a four-level structure is found in the so-called multioscillator ring-laser gyroscope [7,8].

In Sec. II we demonstrate experimentally that a large variety of four-level systems may be designed by introducing birefringence and backscattering into a ring cavity. The "energy-level diagrams" can be adjusted by turning knobs in the experiment. When one thinks of extending the experiments reported for the special case of a driven two-level system ("optical atom") [3,4], this creates numerous possibilities for dynamical experiments in tailored four-level schemes, i.e., in a four-dimensional state space. This large variety of four-level systems led us to the theoretical study of two problems: First, the exploration of the full range of possibilities that can be achieved by inserting optical elements into the cavity; second, the application of symmetry arguments in order to make general statements with respect to the eigenmode structure, such as the occurrence of degeneracies.

As a basis for these theoretical considerations we adopt the transmission-matrix formalism introduced by Lenstra and Geurten [9], which we summarize in Sec. III A. Restriction to loss-free optical elements leads naturally to the study of the Lie group U(2,2). In Sec IV we introduce the corresponding Lie algebra u(2,2) and its 16 generators and we associate every generator with a specific type of optical element. In Sec. V, the group-theoretical approach is used as a starting point for "optical engineering." Using the commutation relations, we show how certain "nonstandard" optical elements can be constructed as a sequence of standard elements. We also show that in this way the entire group U(2,2) can be realized and discuss how and to what extent the Sagnac effect (in rotating ring cavities) can be mimicked in nonrotating cavities. In Sec. VI we consider situations where the optical system is restricted to a subgroup of U(2,2). Such situations occur when the optical components are restricted to a few types, in such a way that their generators have closed commutation relations, i.e., generate a subalgebra of u(2,2). In Sec. VII a method for finding the mode spectrum of an optical ring cavity is formulated. This method is then applied in Sec. VIII, where we discuss the role played by symmetries in determining the mode structure. It turns out that such symmetries give rise to subalgebras as studied in Sec. VI. We also investigate what symmetries are present in the experiments of Sec. II. Finally, conclusions are drawn in Sec. IX.

The present paper thus gives an application of the Lie group U(2,2) which, to our knowledge, is the first time this group is applied in optics. Two of its subgroups, SU(2) and SU(1,1), have already been applied extensively in optics. The group SU(2) is well known in polarization optics as the group of transformations of the Poincaré sphere [10] and has also been used in a description of the lossless beamsplitter [11,12]. The group SU(1,1) was studied in connection with squeezing [11,13,14]. Both groups, SU(2) as well as SU(1,1), have furthermore been studied in connection with Berry's geometrical phase in optics [10,13,15–18]. We stress that the optical ring cavities studied in this paper contain only *passive* optical elements, in contrast to most of the optical applications of the noncompact group SU(1,1) so far [11,13,14], which require *active* elements. Nevertheless the group U(2,2) appears in quite a natural way; in fact the work got its inspiration from the experimental effort. Hence in most cases the mathematical statements go hand in hand with their optical interpretation.

II. EXPERIMENTAL REALIZATIONS OF FOUR-LEVEL SYSTEMS IN OPTICS

In this section, we show how four-level systems have been realized experimentally in an optical ring cavity. The four states were created all from the same longitudinal mode of the ring cavity. The heart of the experimental setup, as shown in Fig. 1(a), consists of a planar four-mirror ring cavity ($c/L \approx 100$ MHz) with various intracavity elements to shape the mode structure. The mirrors were high-reflectivity ($\lambda = 633$ nm) multilayer dielectric mirrors. In all experiments the cavity contained an electro-optic modulator (EOM) between mirrors $M1$ and $M2$. This modulator acted as a linear retarder with fast and slow axes parallel and perpendicular to the plane of the ring and a phase retardation proportional to the voltage V_{EOM} across its electrodes. This voltage was used as a control parameter: the mode structure of the cavity was measured as a function of V_{EOM}. A second optical element OE was positioned between mirrors $M3$ and $M4$. Depending on the choice of this element, the finesse of the ring cavity was typically 25–30. By choosing specific elements (or combinations thereof) for OE, different mode structures and thus different four-level systems could be obtained. At this point it is good to realize that every mirror must be treated as half-wave plate with its fast axis in the plane of incidence [7,8,17,19]. The mode structure can therefore be affected by the mirrors as well. In our configuration there are two mirrors between the optical elements EOM and OE, namely, either ($M2$,$M3$) or ($M1$,$M4$). Such pairs of mirrors can be left out of the analysis, since their reflection-induced phase retardations add up to 2π [20]. Note that in a ring cavity with an odd number of mirrors the effect of at least one unpaired mirror must be taken into account. The effect of the mirrors is also important in nonplanar cavities, where it gives rise to a topologically induced optical activity ("Berry's phase") [21]. This phenomenon is sometimes used for out-of-plane biasing of ring-laser gyros [7,8].

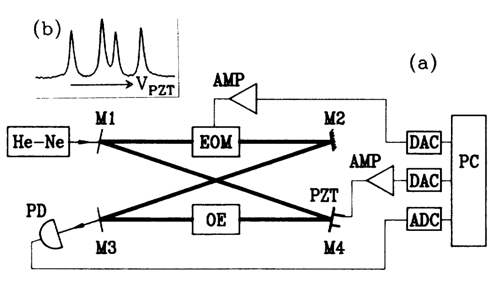

FIG. 1. Experimental setup (a); $M1$–$M4$: mirrors with radii of curvature $r = -1$ m for $M1$ and $M2$, $r = -5$ m for $M3$ and $r = -6$ m for $M4$; EOM: electro-optic modulator; OE: optical element (see text); PD: photodiode; PZT: piezoelectric transducer; AMP: amplifiers; DAC: digital-to-analog converters; ADC: analog-to-digital converter; PC: personal computer. The inset (b) shows a typical spectrum recorded with the photodiode as the cavity is scanned by ramping the voltage V_{PZT} applied to PZT.

In order to measure the mode structure we injected light from a fixed-frequency single-mode He-Ne laser ($\lambda = 633$ nm). The length of the ring cavity was scanned over a distance of order λ by means of a piezoelectric element (PZT) mounted to mirror $M4$ and the intensity of the light leaking out of the ring through mirror $M3$ was recorded using the photodiode (PD). This resulted in a spectrum with up to four resonance peaks per free spectral range of the cavity, see Fig. 1(b). The strength of the resonances depended on the polarization of the injected laser light and we adjusted this polarization such that all resonances were sufficiently excited. The photodiode signal was fed into a personal computer (PC), which determined the peak positions and controlled the experiment according to the following protocol. The EOM voltage V_{EOM} was increased stepwise and for each value of V_{EOM} the voltage V_{PZT} on the piezoelectric element was ramped. The positions of the resonances in the photodiode signal, expressed as the corresponding piezovoltage V_{PZT} were then plotted as a function of the EOM voltage V_{EOM}, yielding a mapping of the four-level system. The results are shown in Fig. 2. A complete measurement of one of the pictures in Fig. 2 takes typically a few seconds. During this time the cavity is sufficiently stable to allow absolute determination of the resonance positions. Results like those in Fig. 2, which are essentially cavity-mode spectra as a function of a control parameter (V_{EOM}), have been described previously as optical band structure [1–3,9] and we shall adhere to this nomenclature throughout this paper. The optical band structures described here have a vertical periodicity given by the free spectral range of the ring cavity and a horizontal periodicity given by the full-wave voltage ($V_{2\pi}$) of the EOM.

The results shown in Fig. 2 were obtained with various optical elements. In this section we give a qualitative explanation of the results. A method for quantitative description was introduced in Ref. [9] and will be extended in the following sections. If the cavity contains nothing but the ever-present electro-optic modulator, the mode structure is as shown in Fig. 2(a). The upward and downward sloping lines correspond to the x- and y-polarized modes. These are pulled apart by EOM by an amount proportional to the voltage $V_{\rm EOM}$.

For Fig. 2(b) we took as the OE a second EOM, with its fast and slow axes under $\theta=45°$ with respect to those

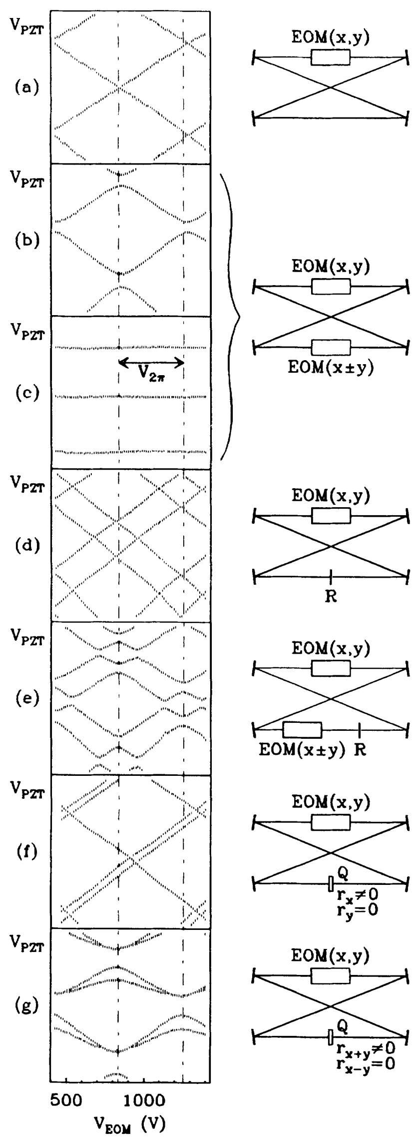

FIG. 2. Experimental results of optical band structure for various choices for the optical element OE in Fig. 1. The cavity-mode frequencies, expressed as the piezovoltage $V_{\rm PZT}$ (arb. units) for which the cavity was resonant with the injected light, are shown as a function of the voltage $V_{\rm EOM}$ applied to the permanent electro-optic modulator (EOM) in Fig. 1. The vertical periodicity is given by the free spectral range of the cavity, the horizontal periodicity by the full-wave voltage $V_{2\pi}$ of the EOM. In (d) and (e) R is an isotropic partial reflector; in (f) and (g) Q is a quarter-wave plate employed as a polarization-selective reflector (see text).

of the first EOM. The phase retardation of this second EOM couples the x and y polarizations and therefore transforms every crossing in Fig. 2(a) into an anticrossing with a minimum separation determined by the coupling strength, which is governed by the voltage across the second EOM. If this voltage is increased far enough, the coupling can be made so strong that the band structure is completely flattened out, as shown in Fig. 2(c). We also tried configurations with $\theta\neq45°$ and found that they yielded essentially the same result, although the anticrossings occurred for different values of $V_{\rm EOM}$. Note that in Figs. 2(a)–2(c) the propagation degeneracy [clockwise (CW), counterclockwise (CCW)] is still present, although only one propagation direction was excited by the injection laser. This degeneracy can be removed by coupling the CW and CCW modes using a partially reflecting element. Starting again from Fig. 2(a) this reflector then splits every upward and downward sloping line in two parallel lines [Fig. 2(d)]. The modes, no longer degenerate, are now standing waves: two for each polarization. For each polarization the two standing waves are complementary, the nodes of one coinciding with the antinodes of the other. Introducing an additional EOM with axes under $\theta=45°$ with the first, we lift all degeneracies in the crossing points of Fig. 2(d), which results in the mode structure of Fig. 2(e), a manifold of "double-well potentials." The double well could also be made asymmetric by taking $\theta\neq45°$.

It is also possible to lift the CW-CCW degeneracy selectively for only one polarization, as shown in Fig. 2(f). Here a partially reflecting element was used, selectively reflecting the x polarization. There was practically no reflection for the y polarization. The reflector employed here was a quarter-wave plate made of quartz, with fast and slow axes along x,y. It was aligned perpendicular to the beam axis, so that the two dielectric interfaces air-quartz and quartz-air yielded interfering contributions to the reflected wave. This interference was destructive for the y polarization and constructive for the x polarization. (The interference condition can be controlled by thermal tuning of the optical path length in the quarter-wave plate.) Such a polarization-selective reflector is the subject of Sec. V B. When the axes of the polarization-selective reflector were rotated by $\theta=45°$, Fig. 2(g) was obtained. In this case the qualitative description is somewhat more complicated. The transmission of the quarter-wave plate brings us from Fig. 2(a) to a structure like in Fig. 2(b), still containing the CW-CCW degeneracy. The reflection of the quarter-wave plate will now be most effective at the anticrossings, since there one of the eigenmode polarizations matches the polarization selected by reflection.

It is remarkable that so many different band structures occur in Fig. 2 using only fairly simple optical configurations. Note also that the electro-optic modulator does not really play a preferential role, contrary to what might be suggested by the above discussion. For example, optical band structure very similar to that in Fig. 2(b) has been reported, where the same role was played by a magnetic field inside a Faraday rotator [2,3]. All this raises the question of how far our freedom in designing such mode structures extends. In the following sections we try to answer this question.

III. TRANSMISSION-MATRIX FORMALISM

In this section we adopt as a basic tool the formalism introduced by Lenstra and Geurten [9] for describing the polarization-mode structure in optical ring resonators and give some additional definitions. The formalism is a combination of the M-matrix description employed in one-dimensional scattering problems in quantum mechanics [22] and Jones calculus employed in polarization optics [23]. We consider a generalized ring cavity [Fig. 3(a)] containing several optical elements, such as partially reflecting elements, birefringent elements, sections of free space, etc. The optical field at the reference point P is decomposed in its left- and right-traveling components and each of these components is written as a complex two-component Jones vector to describe its polarization. Choosing $z=0$ in P, the electric field near P is given by

$$\mathbf{E}(z,t)=(\mathbf{A}e^{ikz}+\mathbf{B}e^{-ikz})e^{-i\omega t}+\text{c.c.}\ , \tag{1}$$

where the Jones vectors $\mathbf{A}$ and $\mathbf{B}$ represent the waves traveling in the positive (or CCW) and negative (or CW) z direction, respectively, and c.c. stands for complex conjugate. We combine the two Jones vectors into one single vector so that we have a four-component complex vector that fully characterizes the optical field in P.

An optical element is now represented by a 4×4 complex matrix M relating the two Jones vectors on its left to those on its right [Fig. 3(b)],

$$\begin{bmatrix}\mathbf{C}\\ \mathbf{D}\end{bmatrix}=M\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}=\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\begin{bmatrix}\mathbf{A}\\ \mathbf{B}\end{bmatrix}, \tag{2}$$

where the m_{ij} are 2×2 submatrices. For a *nonreflecting* element the off-diagonal submatrices vanish: $m_{12}=m_{21}=0$ and m_{11} and m_{22} reduce to the ordinary Jones matrices. Another intuitively clear case occurs when the element does not influence the polarization. We shall call these elements *isotropic.* For an isotropic element the submatrices m_{jj} have the form of a complex number times the 2×2 unit matrix I_2 so that we can treat the submatrices as numbers and the M matrix as a 2×2 matrix. An example of an element that is isotropic as well as nonreflecting is a section of free space. In this case the matrix M contains only phase factors:

$$M=\begin{bmatrix}e^{ikL}I_2 & 0\\ 0 & e^{-ikL}I_2\end{bmatrix}, \tag{3}$$

where $k=\omega/c$ is the wave vector, L the length of the section, and I_2 the 2×2 unit matrix. A more extensive list of matrices for simple optical elements can be found in Ref. [9].

Note that, as an alternative, we could have chosen to represent an optical element by a scattering matrix S relating the outgoing fields $(\mathbf{C},\mathbf{B})$ to the incoming ones $(\mathbf{A},\mathbf{D})$; see Fig. 3(b). The advantage of using transmission matrices M as introduced in Eq. (2) lies in the possibility of representing a sequence of optical elements by the product of their individual transmission matrices. In particular, for a ring cavity we may form the transmission matrix M_{RT} for one round-trip through the entire system. We discuss in Sec. VII how one calculates the eigenmode spectrum of the cavity from the matrix M_{RT}.

Throughout this paper we shall impose the condition that the optical elements are loss-free by requiring that the sum of the input intensities equal the sum of the output intensities: $|\mathbf{A}|^2+|\mathbf{D}|^2=|\mathbf{B}|^2+|\mathbf{C}|^2$. However, rather than relating output to input fields, the M matrix of an optical element relates the fields on one side to those on the other side. It is therefore convenient to rewrite this condition as the flux conservation relation [9]: $|\mathbf{A}|^2-|\mathbf{B}|^2=|\mathbf{C}|^2-|\mathbf{D}|^2$. This can be written more compactly if we introduce the metric

$$J=\begin{bmatrix}I_2 & 0\\ 0 & -I_2\end{bmatrix}, \tag{4}$$

and write the complex four-vector in Dirac notation: $|\Psi\rangle=(\mathbf{A},\mathbf{B})$. The flux conservation relation then simply states that M conserves the quantity $\langle\Psi|J|\Psi\rangle$. The condition imposed on M then takes the form

$$M^{\dagger}JM=J\ , \tag{5}$$

with the dagger denoting Hermitian conjugation. Matrices M with this property are called *pseudounitary* and since the signature of the metric J is $(++--)$, the matrices M are by definition elements of the Lie group U(2,2) [24,25]. Note that the corresponding scattering matrix S [see Fig. 3(b)] is unitary, i.e., it obeys a condition similar to Eq. (5): $S^{\dagger}J'S=J'$, where the metric J' is now the 4×4 unit matrix I_4. In contrast to earlier applications of the pseudounitary group SU(1,1) in optics [11,13,14], which *active* optical elements were necessary, here we consider only *passive* optical elements. It is the coupling of counterpropagating waves (reflections) that introduces the pseudounitary character of the M matrices. If we disregard the polarization degree of freedom and consider the light as a scalar wave, the Jones vectors are replaced by complex numbers and we have a 2×2 matrix formalism. In that case the signature of the metric would be $(+-)$, leading to the Lie group U(1,1).

It is easily verified that the pseudounitary matrices M defined by Eq. (5) indeed form a group. The closure of

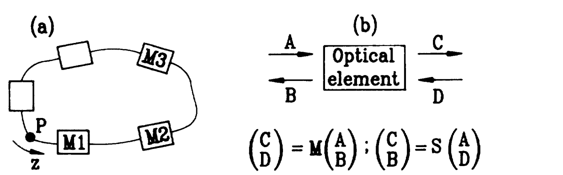

FIG. 3. Generalized optical ring cavity (a), with optical elements M_i and a reference point P. Each optical element (b) is represented by a transmission matrix M relating the Jones vectors $(\mathbf{A},\mathbf{B})$ on its left to those on its right $(\mathbf{C},\mathbf{D})$. Alternatively one may relate the outgoing $(\mathbf{C},\mathbf{B})$ to the incoming fields $(\mathbf{A},\mathbf{D})$ by a scattering matrix S.

the group [the requirement that the product of two matrices M obeying Eq. (5) must itself obey it] is physically obvious, since a sequence of two loss-free optical elements must itself be loss-free. Mathematically, we use Eq. (5) to obtain the identity

$$(M_2M_1)^\dagger J(M_2M_1)=M_1^\dagger(M_2^\dagger J M_2)M_1=M_1^\dagger J M_1=J \ . \tag{6}$$

The unit element of the group is simply the 4×4 unit matrix I_4. By left multiplication of Eq. (5) by J we find that the inverse of a group element M is given by $M^{-1}=J M^\dagger J$. The verification of the group structure is completed by noting that matrix multiplication is associative. The formalism thus represents every loss-free optical element by an element of the Lie group U(2,2). This raises the question whether it is also true that every element of U(2,2) can be realized in optics. It will be shown in Sec. V that the full group can indeed be realized with standard optical components.

We end this section by giving a definition of the time-reversal operator T which will play an important role in the following sections. We find this operator by replacing t by $-t$ in Eq. (1). After a rearrangement we obtain

$$\mathbf{E}(z,-t)=(\mathbf{B}^* e^{ikz}+\mathbf{A}^* e^{-ikz})e^{-i\omega t}+\text{c.c.} \ , \tag{7}$$

where the asterisk denotes complex conjugation. The time-reversed version of $(\mathbf{A},\mathbf{B})$ is thus $T(\mathbf{A},\mathbf{B})=(\mathbf{B}^*,\mathbf{A}^*)$. The time-reversal operator is antiunitary [26], i.e., it can be written as the product of a unitary matrix and the complex conjugation operator $\mathcal{C}$:

$$T=\begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}\mathcal{C} \ . \tag{8}$$

Note that $T^2=I_4$ as it should be, since photons are bosons (for fermions we would expect $T^2=-I_4$) [26–28]. We now define time-reversal symmetry of an optical element by the requirement that M commute with the time-reversal operator T:

$$[T,M]=0 \ . \tag{9}$$

This definition is shown to be equivalent with a more familiar definition in terms of unitary matrices in Sec. VIII. Using the explicit form of the time-reversal operator, Eq. (8), we see that a time-reversal symmetric M matrix is quickly recognized by the following relations between its submatrices: $m_{11}=m_{22}^*$ and $m_{12}=m_{21}^*$.

IV. THE LIE ALGEBRA u(2,2)

A. Definition of the generators

The study of a Lie group is in many cases facilitated by the study of the corresponding Lie algebra. The algebra corresponding to the group U(2,2) is denoted u(2,2) and consists of the matrices K for which $\exp(i\phi K)$ is an element of U(2,2) for all real values of ϕ [29]:

$$M=e^{i\phi K}\in \mathrm{U}(2,2) \ \text{ for all real } \phi \ . \tag{10}$$

If we substitute $M=\exp(i\phi K)$ into Eq. (5) and take the derivative with respect to ϕ we find that the Lie algebra can be defined by the following condition for K:

$$J K = K^\dagger J \ . \tag{11}$$

Note that this equation states that $J K$ is Hermitian. The Lie algebra u(2,2) can be considered as a linear vector space with an additional structure, viz., the definition of the commutator $[K,K']$. These commutation relations will be studied in Sec. IV C. Here we introduce a basis for the linear vector space. As a consequence of the fact that U(2,2) is a 16-parameter group, the vector space has 16 dimensions [24,25]. The basis vectors are called generators. A convenient choice for the generators of u(2,2) is the set of tensor products of the generators of the smaller Lie algebras u(1,1) and u(2). For the algebra u(2) we take the following generators, denoted by σ_s ($s=0,\ldots,3$):

$$\sigma_0=I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2=\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{12}$$

For the algebra u(1,1) we take the following generators, denoted by κ_k ($k=0,\ldots,3$):

$$\kappa_0=\sigma_0=I_2, \quad \kappa_1=i\sigma_1, \quad \kappa_2=i\sigma_2, \quad \kappa_3=\sigma_3 \ . \tag{13}$$

Thus the generators of u(2,2) are

$$K_{ks}=\kappa_k\otimes\sigma_s \quad (k,s=0,\ldots,3) \ . \tag{14}$$

For example,

$$K_{12}=\kappa_1\otimes\sigma_2=\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\otimes\sigma_2=\begin{bmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{bmatrix} . \tag{15}$$

Note that JK_{ks} is Hermitian for every generator.

B. Optical identification of the generators

An advantage of our choice of generators is that every generator can be associated with a specific type of optical element. The association is made by recognizing the corresponding group element $M_{ks}(\phi)\equiv\exp(i\phi K_{ks})$, where ϕ may be considered as a "strength" parameter of the optical element. By noting that $K_{0s}^2=-K_{1s}^2=-K_{2s}^2=K_{3s}^2=I_4$, we can simplify the power-series expansion of the exponential function:

$$M_{ks}(\phi)=e^{i\phi K_{ks}}=\begin{cases}(\cos\phi)I_4+i(\sin\phi)K_{ks} & \text{if } k=0 \text{ or } 3\\ (\cosh\phi)I_4+i(\sinh\phi)K_{ks} & \text{if } k=1 \text{ or } 2 \ .\end{cases} \tag{16}$$

TABLE I. The generators of u(2,2) and their optical identification.

Generator	Identification	Eigenpolarization	Symmetry under time reversal
K_{00}	Sagnac effect	isotropic	$-$
K_{10},K_{20}	partial reflector	isotropic	$+$
K_{30}	free propagation	isotropic	$+$
K_{01}	retarder	$x+y,x-y$	$-$
K_{11},K_{21}	partial reflector	$x+y,x-y$	$+$
K_{31}	retarder	$x+y,x-y$	$+$
K_{02}	optical activity	σ^+,σ^-	$+$
K_{12},K_{22}	partial reflector	σ^+,σ^-	$-$
K_{32}	Faraday effect	σ^+,σ^-	$-$
K_{03}	retarder	x,y	$-$
K_{13},K_{23}	partial reflector	x,y	$+$
K_{33}	retarder	x,y	$+$

For example, the generator $K_{30}=J$ is associated with free propagation because $M_{30}(\phi)=\exp(i\phi J)$ is the transmission matrix for a section of free space [see Eq. (3)]. Not all generators are identified as easily as K_{30}, and we shall now discuss some properties according to which they can be categorized. The results are listed in Table I.

1. *Compactness: retarders versus reflectors*

A categorization of the generators that is suggested already by Eq. (16) is according to the first index k. For $k=0$ or 3 the generator is Hermitian so that the corresponding group element $M_{ks}(\phi)$ is unitary. The generator and its group element are block diagonal, which implies that the corresponding optical element is nonreflecting, i.e., it is a *retarder* (note that in this terminology free space is also considered as a retarder.) The identification with a physical device is possible by comparing $M_{ks}(\phi)$ with the matrices given in Ref. [9]. From Eq. (16) we see that for a retarder the matrix elements of $M_{ks}(\phi)$ are bounded: their modulus cannot exceed unity. The generators K_{0s} and K_{3s} are often called *compact* [25] and we shall use the terms "compact" and "nonreflecting" generators as synonyms here. In contrast with the Hermitian or compact generators, those for $k=1$ or 2 are anti-Hermitian and do not yield a unitary $M_{ks}(\phi)$. In this case the generator is off diagonal so that the group element $M_{ks}(\phi)$ also has nonvanishing off-diagonal blocks. This means that the corresponding optical element is a *partial reflector.* The matrix elements of $M_{ks}(\phi)$ are not bounded and the generators are called *noncompact.*

2. *Eigenpolarization*

The categorization according to eigenpolarization is governed by the second index, s. We define the eigenpolarizations by the Jones vectors that are eigenvectors of σ_s. In the symbolic notation of Table I, x and y stand for the Jones vectors (1,0) and (0,1), $x\pm y$ for the linear polarizations $(1,\pm 1)$, and $\sigma^{\pm}$ for the circular polarizations $(1,\pm i)$. We speak about circular or linear retarders, depending on their eigenpolarization. Hence Faraday rotators and optically active elements are circular retarders and a quarter-wave plate is a linear retarder. The eigenpolarizations are simply those which are unaltered by the optical element. For example, a quarter-wave plate with fast and slow axes along x,y is described by $M_{33}(\pi/4)$ and does not change the polarization of a light wave if it is x or y polarized. For the unit matrix σ_0 every Jones vector is an eigenvector so that the corresponding group element does not influence the polarization. The group element is then isotropic as defined in Sec. III. The generators K_{k0} will also be called *isotropic.*

3. *Symmetry under time reversal*

Time-reversal invariance for a U(2,2) element was defined in Eq. (9) as commutation with the time-reversal operator T. We call a generator K_{ks} time-reversal symmetric, or *T symmetric,* if $M_{ks}(\phi)$ is time-reversal invariant. The generator then anticommutes with T (see also Sec. VIII); the T-symmetric generators are indicated by a plus in the fourth column of Table I. In contrast, we call the generators that commute with the time-reversal operator *T antisymmetric,* indicated by a minus in Table I. Summarized:

$$\begin{aligned} &T \text{ symmetry } (+):\ \{K_{ks},T\}=0 \Longleftrightarrow [M_{ks}(\phi),T]=0\ , \\ &T \text{ antisymmetry } (-):\ [K_{ks},T]=0 \Longleftrightarrow M_{ks}(\phi)T=T\,M_{ks}^{-1}(\phi)\ . \end{aligned} \tag{17}$$

The optical implications of T symmetry and T antisymmetry are illustrated by a few examples. The quarter-wave plate mentioned in Sec. IV B 2 was T symmetric. It is the standard optical component one normally uses in experiments. The T-antisymmetric counterpart of such a quarter-wave plate is described by $M_{03}(\pi/4)$. Antisymmetry with respect to time reversal in this case means that the fast and slow axes are interchanged if the propagation direction of the light is reversed. Such an antisymmetric quarter-wave plate is not commercially available, but we shall see in Sec. V how it may be constructed.

As another example of symmetric and antisymmetric counterparts we compare optical activity (T symmetric) with Faraday rotation (T antisymmetric) [9]. Optical activity occurs, for example, in a quartz crystal, if the light propagates along the optical axis. Such a crystal, described by $M_{02}(\phi)$, rotates the plane of polarization of linearly polarized light over an angle ϕ. If the propagation direction of the light is reversed, the sense of rotation is also reversed so that a light beam that is reflected back through the crystal retrieves its original polarization. Faraday rotation, described by $M_{32}(\phi)$, is induced by applying a magnetic field to a medium (and thus breaking time-reversal symmetry.) In this case the sense of rotation is unchanged if the propagation direction of the light is reversed. Thus, if the light beam is reflected back through the Faraday rotator, the rotation angle of the plane of polarization is doubled.

4. *Partial reflectors*

Partial reflectors are generated by noncompact generators ($k=1$ or 2). For example, the U(2,2) element corresponding to the isotropic generator K_{10} is

$$M_{10}(\phi)=\begin{bmatrix} (\cosh\phi)I_2 & -(\sinh\phi)I_2 \\ -(\sinh\phi)I_2 & (\cosh\phi)I_2 \end{bmatrix} . \tag{18}$$

The matrix is easily recognized, with the help of Ref. [9], as representing an isotropic partial reflector with amplitude reflection coefficient $r=-\tanh\phi$.

An example of a nonisotropic reflector results from the generator K_{13}. The group element is given by

$$M_{13}(\phi)=\begin{bmatrix} \cosh\phi & 0 & -\sinh\phi & 0 \\ 0 & \cosh\phi & 0 & \sinh\phi \\ -\sinh\phi & 0 & \cosh\phi & 0 \\ 0 & \sinh\phi & 0 & \cosh\phi \end{bmatrix} . \tag{19}$$

This reflector has an amplitude reflection coefficient $r_x=-\tanh\phi$ for the x-polarized component and $r_y=-r_x=\tanh\phi$ for the y-polarized component. Note that since $s=3$ the x and y polarizations are precisely the eigenpolarizations for this generator. The difference in sign in the reflection coefficients for the two polarizations may seem unimportant at this point, but it is essential if the reflected waves interfere, e.g., with other reflections. The minus sign indicates that, for the two polarizations, the reflected waves are in antiphase so that one polarization may experience constructive interference and the other destructive interference.

The reflecting generators in Table I are grouped in pairs (K_{1s},K_{2s}), because the only difference between K_{1s} and K_{2s} is a $\pi/2$ phase difference of the reflected waves. They are transformed into each other by a displacement of the reflector along the propagation direction over $\lambda/8$, with λ the wavelength of the light. Such a displacement can also be considered as an exchange of the reflector with a section of free space or, in other words, the removal of free space on one side and its insertion on the other side. Mathematically, K_{1s} and K_{2s} are obtained from one another by taking the commutator with the generator for free propagation K_{30}.

C. Commutation relations

Commutation relations determine the structure of a Lie algebra. They reflect the structure of the group multiplication on the level of the algebra. If a group is Abelian, i.e., the group multiplication is commutative, all commutators between elements of the algebra vanish. If not, the products $\exp(iK)\exp(iK')$ and $\exp(iK')\exp(iK)$ are generally different and the commutator $[K,K']$ essentially measures the extent to which they differ. The commutation relations are at the basis of all our applications.

The commutator of two u(2,2) generators is easily calculated with the rule $K_{ks}K_{lt}=(\kappa_k\kappa_l)\otimes(\sigma_s\sigma_t)$ when using the well-known products of the Pauli matrices:

$$\sigma_1\sigma_2=-\sigma_2\sigma_1=i\sigma_3 , \tag{20a}$$

$$\sigma_0\sigma_s=\sigma_s\sigma_0=\sigma_s , \tag{20b}$$

where the indices (1,2,3) may be permuted cyclically. The resulting commutation relations for u(2,2) are given in Table II.

We make a few remarks concerning Table II here. First of all, the commutator of two generators tells us whether it makes any difference if the corresponding optical elements are interchanged. If two generators do not commute, their commutator tells us, roughly speaking, what kind of effect may be produced by combining the optical elements associated with the two generators. It is found that the triple formed by the two generators and their commutator have the commutation relations of either the su(1,1) generators κ_1, κ_2, and κ_3 or those of the su(2) generators σ_1, σ_2, and σ_3 [11,14]:

$$\begin{aligned} &[\kappa_1,\kappa_2]=-2i\kappa_3, \quad [\sigma_1,\sigma_2]=2i\sigma_3 , \\ &[\kappa_2,\kappa_3]=2i\kappa_1, \quad [\sigma_2,\sigma_3]=2i\sigma_1 , \\ &[\kappa_3,\kappa_1]=2i\kappa_2, \quad [\sigma_3,\sigma_1]=2i\sigma_2 . \end{aligned} \tag{21}$$

The generator for the Sagnac effect in rotating ring cavities, $K_{00}=I_4$, commutes with every other generator and cannot be obtained as the commutator of other generators. This reveals that the Lie algebra u(2,2) can be written as a direct sum: u(2,2)=u(1)$\oplus$su(2,2), where the u(1) component is generated by K_{00} and the su(2,2) component by the remaining 15 generators. Correspondingly, the group U(2,2) is a direct product; U(2,2)=U(1)$\otimes$SU(2,2). This means simply that every element of U(2,2) has the form of a phase factor $\exp(i\alpha)$ times an SU(2,2) matrix [a U(2,2) matrix with determinant unity]. The phase factor describes the Sagnac phase difference between the counterpropagating waves. The 15 generators that remain in the absence of rotation have vanishing trace, so that the determinant of the M matrix equals unity and the group U(2,2) is reduced to SU(2,2). In the rest of this paper we shall mainly concentrate on SU(2,2).

The first four generators are the isotropic generators

TABLE II. Commutation relations for u(2,2); reading example: for the commutator $[K_{12},K_{20}]$ we look in the row labeled "*12*" and the column labeled "20" and find "$-$*32*", which is to be interpreted as $-2iK_{32}$. Entries in italic type indicate that the generator is T antisymmetric.

	00	10	20	30	*01*	11	21	31	02	*12*	*22*	*32*	*03*	13	23	33
00																
10			−30	−20			−31	−21			−*32*	−*22*			−33	−23
20		30		10		31		11		*32*		*12*		33		13
30		20	−10			21	−11			*22*	−*12*			23	−13	
01									*03*	13	23	33	−02	−*12*	−*22*	−*32*
11			−31	−21			−30	−20	13	−*03*			−*12*	02		
21		31		11		30		10	23		−*03*		−*22*		02	
31		21	−11			20	−10		33			*03*	−*32*			−02
02					−*03*	−13	−23	−33					*01*	11	21	31
12			−*32*	−*22*	−13	*03*					−30	−20	11	−*01*		
22		*32*		*12*	−23		*03*			30		10	21		−*01*	
32		*22*	−*12*		−33			−*03*		20	−10		31			*01*
03					02	*12*	*22*	*32*	−*01*	−11	−21	−31				
13			−33	−23	*12*	−02			−11	*01*					−30	−20
23		33		13	*22*		−02		−21		*01*			30		10
33		23	−13		*32*			02	−31			−*01*		20	−10	

K_{k0} and have closed commutation relations, i.e., the commutator of two isotropic generators is itself isotropic. This is physically obvious, since a sequence of isotropic elements can never produce polarization effects. Similarly, the nonreflecting or compact generators, K_{0s} and K_{3s}, also have closed commutation relations. Of course, we never expect a reflection from a sequence of nonreflecting elements. The compact generators are the generators of the maximal compact subgroup of U(2,2), which is U(2)⊗U(2). This is also the maximal unitary subgroup [25]. Subsets of generators with closed commutation relations are generally said to generate a subalgebra and will be studied in more detail in Sec. VI.

As a last remark we note that for a noncompact commutator we need one compact and one noncompact generator. A similar rule applies to time-reversal symmetry, where a commutator is antisymmetric only if it is produced from one symmetric and one antisymmetric generator. The signs in the last column of Table I can therefore simply be multiplied to obtain the symmetry of the commutator.

V. OPTICAL ENGINEERING WITH SU(2,2)

A. Sandwich construction

In Sec. IVB we associated with every generator an optical element without considering whether all those elements actually existed or not. We show here that all "nonstandard" components can be engineered employing a "sandwich construction" of standard (commercially available) components. The construction is based on the notion that the generators of su(2,2) occur in triples with the commutation relations of either su(2) or su(1,1), as discussed in the preceding section. The occurrence of such triples allows us to use identities that are known for the generators of su(2) and su(1,1). The sandwich construction we want to discuss is expressed in the identities

$$e^{i\phi\sigma_3}=e^{-i(\pi/4)\sigma_1}e^{i\phi\sigma_2}e^{i(\pi/4)\sigma_1}\,, \tag{22a}$$

$$e^{i\phi\kappa_2}=e^{-i(\pi/4)\kappa_3}e^{i\phi\kappa_1}e^{i(\pi/4)\kappa_3}\,, \tag{22b}$$

which are easily proven using Eqs. (16) and (21). Note that a cyclic permutation of indices is allowed in the first equation but not in the second: in the second equation κ_3 cannot be exchanged with a noncompact generator. (In both equations the "bread" of the sandwich is compact.) We may now replace the σ_s or κ_k by any set of generators with the same commutation relations. The sandwich construction can thus be employed to construct "nonstandard" optical components. For the construction of a nonstandard retarder we use the first equation (σ_3 is compact), for a nonstandard reflector we use the second (κ_2 is noncompact).

The only nonstandard retarders are the T-antisymmetric linear retarders, corresponding to the generators K_{01} and K_{03}. Suppose, for example, that we want to construct a T-antisymmetric retarder with axes along x,y (K_{03}). We then replace σ_1 by K_{31}, σ_2 by K_{32}, and σ_3 by K_{03}. The first factor on the right in Eq. (22a) is now the M matrix for a quarter-wave plate with its fast and slow axes aligned along $x\pm y$. The third factor describes the same quarter-wave plate with its fast and slow axes interchanged. The second factor is a Faraday rotator rotating the polarization over ϕ radians. This sandwich construction is shown schematically in Fig. 4(a). We see from Eq. (22a) that a construction of a Faraday rotator sandwiched between two mutually orthogonal quarter-wave plates yields the overall effect of the desired T-antisymmetric retarder. Moreover, the obtained phase retardation between the x and y components will be ϕ radians. As a reminder, antisymmetry under time reversal in this case means that the fast and slow axes of the element are interchanged when the propagation direction of the light is reversed. This construction was used in the experiments described in Refs. [2–4] to mimic the Sagnac effect as discussed in Sec. V C.

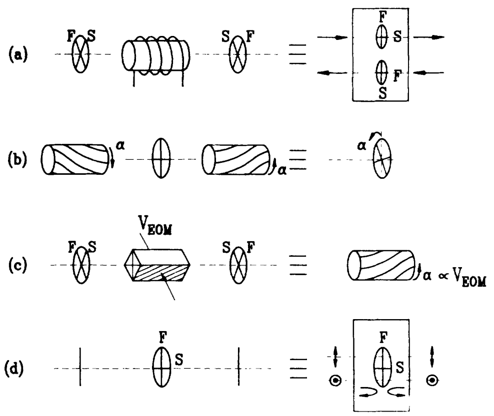

FIG. 4. Examples of engineering with the group SU(2,2). (a) A Faraday rotator sandwiched between two mutually orthogonal quarter-wave plates is effectively a T-antisymmetric linear retarder. (b) If a linear retarder is sandwiched between two compensating optically active elements, effectively its axes are rotated. (c) An adjustable *linear* retarder is transformed into an adjustable *circular* retarder by sandwiching it between two mutually orthogonal quarter-wave plates. (d) A quarter-wave plate between two isotropic reflectors can be made to reflect selectively only one polarization.

Among the noncompact generators, only the isotropic ones correspond to standard optical components, i.e., isotropic reflectors. The others can be realized by sandwiching an isotropic reflector (e.g., a dielectric interface) between retarders. For κ_1 we substitute K_{10} or K_{20} (both isotropic) and for κ_3 one of the retarders K_{31}, K_{32}, and K_{33}. The nonisotropic reflectors that are thus constructed are essentially elements that reflect two orthogonal polarizations with a π-phase difference as was shown in Sec. IV B 4.

Using the sandwich construction we are now able to construct all optical elements corresponding to the su(2,2) generators. The sandwich construction is only one special example of how sequences of optical elements can be designed yielding some desired overall effect. A few practical examples are listed below.

1. Rotation of the eigenpolarization

If we take the generator of optical activity (K_{02}) for σ_1 in Eq. (22a) or for κ_3 in Eq. (22b), we find that the sandwich construction rotates the axes of any element with linear eigenpolarization. For example, if we substitute for σ_2 the generator K_{03} for an antisymmetric linear retarder with axes x,y, the overall effect is that of K_{01}, which differs from K_{03} only by a rotation of the slow and fast axes over $\pi/4$. This is only one special example of how the linear eigenpolarization of an element is rotated by sandwiching between optical activity. A general rotation angle α would be obtained by simply replacing $\pi/4$ by α in Eq. (22a); see Fig. 4(b). In an experiment one would, of course, simply rotate the element itself, instead of inserting extra components for optical activity. The application of Eq. (22a) now works the other way around: it shows that an optical element with arbitrary linear eigenpolarization can be considered, for the sake of mathematical convenience, as the same element with eigenpolarizations x,y, but sandwiched between optical activity.

2. Adjustable optical activity

Optical activity normally occurs in, e.g., quartz crystals and the rotation angle for a given crystal is a fixed quantity. The sandwich construction shows how one can transform an adjustable *linear* retarder (such as an electro-optic modulator) into an adjustable *circular* retarder. To this end we sandwich an electro-optic modulator ($\sigma_2 \rightarrow K_{33}$, axes x,y) between mutually orthogonal quarter-wave plates ($\sigma_1 \rightarrow K_{31}$, axes $x \pm y$); see Fig. 4(c). The rotation angle of the resulting circular retarder is then adjustable simply by means of the voltage applied to the electro-optical modulator. Adjustable optical activity can also be obtained with a combination of two half-wave plates, the rotation angle being twice the angle between their fast axes [10].

B. Polarization-selective reflector

The sandwich construction provides a method by which one can achieve the situation that the amplitude reflection coefficients for two orthogonal polarizations have opposite signs. In an experiment the situation in which the reflection coefficients have different magnitude is often more interesting. The construction used in the experiments of Figs. 2(f) and 2(g) that achieves this situation is essentially a quarter-wave plate between two isotropic reflectors [Fig. 4(d)], viz., the two dielectric interfaces of the quarter-wave plate. This, however, is not a sandwich construction in the sense discussed above, since the two isotropic reflectors are described by noncompact generators. Moreover, since the relative phase of the reflectors is important, we have to account for the thickness of the quarter-wave plate so that the generator for free propagation K_{30} enters the analysis.

A qualitative description of this construction in terms of simple optical arguments has been given in Sec. II, in the discussion of Fig. 2(f). If the two isotropic reflectors interfere destructively for the y polarization, they automatically interfere constructively for the x polarization, as a consequence of the presence of the quarter-wave plate. Hence we have constructed an optical element that selectively reflects the x polarization. Note that, viewed in transmission, this composite element still imposes a phase difference between the x- and y-polarized components of the light.

C. Simulation of the Sagnac effect

The generator for the Sagnac effect, K_{00}, was found to play a special role in Sec. IV C: it commutes with every other generator and there are no generators that have K_{00} as their commutator. This implies, for example, that it is impossible to design a sandwich construction yielding the same effect. The Sagnac effect can only be achieved by actually rotating the ring cavity and cannot be completely simulated. It is possible, however, to obtain a simulation of the Sagnac effect in a U(1,1) context, i.e., the Sagnac effect as it occurs for scalar waves. We thus disregard the degree of freedom due to polarization and consider an isotropic cavity, described by the generators K_{k0}. The elements of the Lie algebra are the matrices

$$\begin{pmatrix} (a_0+a_3)I_2 & (ia_1+a_2)I_2 \\ (ia_1-a_2)I_2 & (a_0-a_3)I_2 \end{pmatrix}, \tag{23}$$

where $2a_0$ is the phase difference between the counterpropagating waves per round-trip through the cavity, due to the Sagnac effect. We can now use the extra degree of freedom, due to polarization, to simulate the Sagnac effect. The simulation is based on the observation that some generators cannot be distinguished from K_{00}, as long as only one polarization is considered. One such generator is K_{03} (T-antisymmetric linear retarder) which, like K_{00}, commutes with K_{10}, K_{20}, and K_{30}. If we replace the Sagnac generator K_{00} by K_{03}, the elements of the resulting Lie algebra are the matrices

$$\begin{pmatrix} a_0'+a_3 & 0 & ia_1+a_2 & 0 \\ 0 & -a_0'+a_3 & 0 & ia_1+a_2 \\ ia_1-a_2 & 0 & a_0'-a_3 & 0 \\ 0 & ia_1-a_2 & 0 & -a_0'-a_3 \end{pmatrix}. \tag{24}$$

We see that as long as we look at the x polarization only (first and third row and column), the system is formally identical to a rotating one with Sagnac phase a_0'. If we look at the y polarization, however, the Sagnac phase is just the opposite and equals $-a_0'$. We conclude that the generator K_{03} simulates a certain rotation rate for the x-polarized light, but at the same time the opposite rotation rate for the y-polarized light. A similar result is obtained if, instead of K_{03}, we use K_{01} (for the polarizations $x \pm y$) or K_{02} (for $\sigma^{\pm}$). Note that, although the Sagnac generator is T antisymmetric, it may be simulated, in the sense discussed above, by the T-symmetric generator K_{02} (optical activity).

The simulation scheme using the generator K_{03} as a Sagnac generator was used in the experiments reported in Refs. [2–4]. The effect of the generator K_{03} was obtained by means of a sandwich construction of a Faraday rotator between two quarter-wave plates [Fig. 4(a)]. The x polarization was selected either by exciting the cavity modes with x-polarized injected light, so that the y-polarized modes were not excited or by filtering out the y-polarized modes with intracavity Brewster windows. As a last remark we note that in the field of ring-laser gyros "Faraday biasing" is a popular method to simulate rotation [7,8].

D. A universal SU(2,2) gadget?

So far we have shown in this section that the optical elements associated with the generators in Table I are either standard components or can be composed from standard components by means of a sandwich construction. This implies that the entire group SU(2,2) can be realized and one might even construct a universal SU(2,2) gadget in the spirit of the gadget developed for SU(2) [10]. Such a gadget would consist of a chain of at least 15 independently adjustable optical components, that can be set to any desired SU(2,2) element. The number 15 enters because SU(2,2) is a 15-parameter group. The design of the SU(2) gadget shows that the group can be realized with just two rotatable quarter-wave plates and one half-wave plate, i.e., with essentially one type of optical element: the T-symmetric linear retarder. In our language the SU(2) gadget covers the generators K_{31}, K_{02}, and K_{33}. If we want to realize the entire group SU(2,2), we have to add three extra types of optical elements. The addition of free propagation (K_{30}) and isotropic reflectors (K_{10}) brings all time-reversal symmetric SU(2,2) elements within reach. Addition of one T-antisymmetric type of element, such as a Faraday rotator, is then sufficient to realize the entire group SU(2,2).

VI. SUBALGEBRAS OF su(2,2)

We saw in the preceding section that the entire group SU(2,2) can be realized, provided that sufficient distinct optical elements are available. If the optical elements are restricted to a few types we may not be able to realize the entire group. For example, if no reflectors are present, it is clear that we cannot leave the group of the retarders. In general, such restrictions in the choice of optical elements will confine the system to a subgroup of SU(2,2). Associated with these subgroups are subalgebras of su(2,2), which can be found by looking for subsets of generators with closed commutation relations, such that if K and K' are in the subspace spanned by the set of generators, then $i[K,K']$ is also in that subspace [the factor i is an artifact of our definition, Eq. (10), where we chose not to absorb the factor i in the matrix K]. The confinement of the optical system to a subgroup has important consequences for the mode structure of a ring cavity as will be shown in Sec. VIII, where a connection will be made between subalgebras of su(2,2) and a common symmetry in the optical elements.

A. Abelian subalgebras

The simplest subalgebra is the one-dimensional algebra u(1), generated by an arbitrary generator. Since every element of the algebra is a real number times the generator, all elements commute and u(1) is Abelian. If the generator is compact, the corresponding group is isomorphic with $U(1) \cong SO(2)$, or the group of rotations about a fixed axis. If the generator is noncompact, the group is isomorphic with SO(1,1) or the Lorentz group in two-dimensional space-time (or the group of translations along a straight line).

If there is a set of commutating generators, they obviously have closed commutation relations and generate just the direct sum of their one-dimensional algebras. We can find up to three such commuting generators and all the elements of the resulting three-dimensional algebra will commute. The algebra thus constructed is the maximal Abelian subalgebra u(1)⊕u(1)⊕u(1).

B. Non-Abelian three-dimensional algebras

There are only two distinct non-Abelian subalgebras of dimension 3, namely, su(2) and su(1,1). The commutation relations of these algebras were given in Eq. (21). The algebra su(2) is compact so that its three generators must all be compact or nonreflecting. The algebra su(1,1) is noncompact, so that it cannot be generated by nonreflecting generators alone. This algebra has one compact and two noncompact generators.

An example of the algebra su(2) is provided by K_{31}, K_{02}, and K_{33}, corresponding to the group of nonreflecting elements with time-reversal symmetry. It represents all optical elements that can be produced by ordinary quarter- and half-wave plates and optical activity [10]. As is well known, the order of these elements is important for the overall effect. The corresponding group of M matrices is isomorphic to SU(2), or the group of transformations of the Poincaré sphere. The M matrices are block diagonal, each block being an SU(2) matrix and complex conjugate to the other as a consequence of time-reversal symmetry:

$$M=\begin{bmatrix} U & 0 \\ 0 & U^* \end{bmatrix} \text{ with } U=\begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix} \tag{25}$$

and $|\alpha|^2+|\beta|^2=1$.

For an example of the algebra su(1,1) we can take the isotropic generators K_{10}, K_{20} (both noncompact) and K_{30} (compact). In this case the M matrices are SU(1,1) matrices with each matrix element multiplied by the 2×2 unit matrix I_2:

$$M=\begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{bmatrix}\otimes I_2=\begin{bmatrix} \alpha I_2 & \beta I_2 \\ \beta^* I_2 & \alpha^* I_2 \end{bmatrix}, \tag{26}$$

with $|\alpha|^2-|\beta|^2=1$. We can work with M as if it were a 2×2 matrix, i.e., as if the polarization degree of freedom did not exist and the optical system were constructed from reflectors and sections of free propagation for scalar waves. The non-Abelian character is obvious, since a displacement of a reflector (an interchange with a section of free space) changes the way in which it interferes with other reflectors. The possibilities for realizing su(1,1) are numerous, but in practice there will always be sections of free space in the optical system and hence it makes sense to choose K_{30} as the compact generator. With this restriction four triples are left, consisting of K_{1s}, K_{2s}, and K_{30} ($s=0,\ldots,3$).

C. Algebras of dimensions 6 and 7

The three-dimensional algebras su(2) and su(1,1) can be used as building blocks for the direct sums su(2)⊕ su(2), su(1,1)⊕su(2), and su(1,1)⊕su(1,1), all with dimension 6. The generators that commute with K_{30} (free propagation) are precisely the compact generators and generate su(2)⊕su(2)⊕u(1), where the u(1) component is generated by K_{30} itself. (In order to see the direct product structure, consider the commutation relations of the six linear combinations $K_{0s}\pm K_{3s}$, $s=1,2,3$.) The corresponding group S[U(2)⊗U(2)] is the maximal compact subgroup of SU(2,2) [24,25]. It contains all block-diagonal (and therefore unitary) M matrices with unit determinant, representing all nonreflecting optical elements. These can obviously be displaced along the optic axis (they commute with free propagation) as long as their order is not changed.

The generators that commutate with K_{03} generate su(1,1)⊕su(1,1)⊕u(1), with the u(1) component generated by K_{03}. (The direct product structure becomes manifest if one makes the six linear combinations $K_{k0}\pm K_{k3}$, $k=1,2,3$.) The elements of the corresponding group S[U(1,1)⊗U(1,1)] consist of diagonal 2×2 submatrices. The optical system can be thought of as composed of one U(1,1) element for the x-polarized light waves and another one for the y-polarized waves. All the group elements preserve the x and the y polarizations.

An example of the algebra su(2)⊕su(1,1) arises if we generate the su(2) component by K_{31}, K_{02}, and K_{33} (T-symmetric retarders.) The su(1,1) component is then generated by K_{30}, K_{12}, and K_{22} (T-antisymmetric partial $\sigma^\pm$reflectors). It is shown in Sec. VIII that this subalgebra leads to Kramers's degeneracy in the mode spectrum. Another example of su(2)⊕su(1,1) is generated by K_{01}, K_{02}, K_{03}, K_{10}, K_{20}, and K_{30}. Also in this case we have Kramers's degeneracy in the spectrum.

As a last example of a six-dimensional algebra we consider all generators that commute with a particular noncompact generator. The six generators commuting with K_{10} (apart from K_{10} itself) are K_{01}, K_{11}, K_{02}, K_{12}, K_{03}, and K_{13}. They generate the algebra sl(2,c). We note however that this example is of limited practical interest, since the algebra does not contain K_{30} as a generator and does not commute with K_{30} either. In an experiment all sections of free propagation would have to be exactly an integer number of wavelengths, in order to ensure that effectively the generator K_{30} is absent.

D. Algebra of dimension 10

Finally we mention a class of subalgebras of dimension 10, an example of which is given by all T-symmetric generators. The condition that the M matrix commutes with T [Eq. (9)] is equivalent with the condition that M preserves a bilinear antisymmetric matrix: $\tilde{M}\,B\,M=B$, where the tilde denotes transposition and B is the antisymmetric matrix

$$B=\begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}. \tag{27}$$

Such matrix groups preserving a bilinear antisymmetric metric are called *symplectic* [25,27] and the group of T-symmetric M matrices is therefore isomorphic to the group USp(2,2). Since in the experimental setup (Fig. 1) we used only time-reversal symmetric optical components, all experimental examples in this paper fall in this category. Not in this category are configurations in which a Faraday rotator is used, such as a multioscillator ring-laser gyro [7,8] or the experiments reported in Refs. [2–4].

E. Application to the experiments

Now that we have identified the subalgebras of su(2,2), we can apply these results to the experiments described in Sec. II and find the subalgebras that are relevant for the different configurations. In the experiment of Fig. 2(a) we used only an electro-optic modulator with fast and slow axes along x,y, described by K_{33} and sections of free space (K_{30}). These two generators commute so that the relevant algebra is u(1)⊕u(1). The maximal Abelian subalgebra would have been obtained by adding a T-antisymmetric retarder for the x,y polarizations (K_{03}). The Abelian nature tells us that the band structure is not affected if the optical components are interchanged.

For the experiments of Figs. 2(b) and 2(c) an additional modulator with axes along $x \pm y$ was added (K_{31}). Now the generators K_{30}, K_{33}, and K_{31} do not have closed commutation relations, but do if we add the generator for optical activity, K_{02}. The algebra relevant for Figs. 2(b) and 2(c) is therefore su(2)⊕u(1), with the u(1) component generated by K_{30}.

The configuration of Fig. 2(d) was obtained from 2(a) by adding an isotropic reflector (K_{10}). Again the commutation relations are not closed and we have to add the generators K_{20}, K_{13}, and K_{23}. The relevant algebra is therefore the example of su(1,1)⊕su(1,1) that was discussed above. The configuration of Fig. 2(f) was obtained from 2(a) by adding a reflector that reflected only the x polarization. Note that such an element is different from the group element $M_{13}(\phi)$, since the latter reflects the x and y polarizations with a π-phase difference. As discussed in Sec. V B, the element used for Fig. 2(f) was essentially a quarter-wave plate (K_{33}) between two isotropic reflectors. We thus need the generators for isotropic reflection K_{10} and K_{20}, so that we arrive at the same algebra as for Fig. 2(d).

Figure 2(e) was obtained from 2(b) by adding an isotropic reflector. Due to this addition of an extra generator (K_{10} or K_{20}) the commutation relations are no longer closed and the set must be extended to include all T-symmetric generators, so that we have the ten-dimensional algebra usp(2,2). The same situation occurs in Fig. 2(g). The quarter-wave plate reflecting the $x+y$ polarization adds the generators K_{10}, K_{20}, and K_{31} to those of Fig. 2(b) and the smallest set of generators with closed commutation relations contains all T-symmetric generators.

VII. MODE SPECTRA OF OPTICAL RING CAVITIES

In this section we discuss how the mode spectrum of an optical ring cavity is calculated using the transmission-matrix formalism. The method described here is slightly different from the one introduced by Lenstra and Geurten [9]. The eigenmode spectrum of a ring cavity is determined by the M matrix for one round-trip through the system $M_{\mathrm{RT}} = M_n \cdots M_2 M_1$. In Ref. [9] the mode spectrum of the ring was obtained from M_{RT} by looking for vectors that reproduce themselves after one round-trip: $M_{\mathrm{RT}}|\Psi\rangle = |\Psi\rangle$, i.e., a periodic boundary condition was imposed. Hence one has to solve the secular equation for eigenvalue unity: $\det(M_{\mathrm{RT}} - I_4) = 0$. This equation determines the eigenfrequencies, since M_{RT} is a function of the optical frequency, e.g., through the appearance of phase factors $\exp(ikL)$ in some matrix elements. In general we find four eigenfrequencies per free spectral range of the cavity. Some examples of such calculations were given in Ref. [9]. In an experiment this procedure corresponds to injecting light from an external laser into the ring cavity and looking for cavity resonances as the laser frequency is scanned [30].

An alternative and sometimes more practical method is to keep the frequency of the external laser fixed and to scan the length of the ring cavity, e.g., by moving one of the mirrors over a few wavelengths. We give here a theoretical procedure to derive the mode spectrum as measured with this second method. When the cavity length is changed, effectively a section of free space is inserted, say at point P in Fig. 3. We now define the cavity modes as those vectors satisfying the periodic boundary condition after the insertion of a suitable length of free space. Inspection of the M matrix for free propagation [Eq. (3)] then tells us that we must solve the equation

$$e^{i\phi J} M_{\mathrm{RT}}|\phi\rangle = \begin{bmatrix} e^{i\phi} I_2 & 0 \\ 0 & e^{-i\phi} I_2 \end{bmatrix} M_{\mathrm{RT}}|\phi\rangle = |\phi\rangle \ . \quad (28)$$

One thus solves the modified secular equation $\det[\exp(i\phi J) M_{\mathrm{RT}} - I_4] = 0$. The resulting values $\exp(i\phi) = \exp(ikL)$, which we call "pseudoeigenvalues," give the length of free propagation needed to satisfy the periodic boundary condition. In general, we will find four pseudoeigenvalues and the spectrum of the ring is the set of the four values $\{\phi_i\}_{i=1,\ldots,4}$. A plot of the spectrum as one parameter is varied then yields a mapping of the four-level system (band structure) like the experimental examples of Sec. II.

Rather than calculating the mode spectrum for many specific optical configurations, we want to use the formalism to investigate how symmetries in the optical system influence the mode structure. A problem we encounter in doing so is that symmetries are usually formulated in terms of unitary matrices (or Hermitian Hamiltonians), the eigenvalues of which give the spectrum of interest [27,28]. We therefore introduce a unitary scattering matrix S that describes the same optical element as the pseudounitary transmission matrix M. We recall that the M matrix was defined by relating the fields on one side of an optical element to those on the other side; see Fig. 3(b). We define the S matrix by relating the outgoing to the incoming fields:

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{B} \end{bmatrix} = S \begin{bmatrix} \mathbf{A} \\ \mathbf{D} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{D} \end{bmatrix} , \quad (29)$$

where the s_{ij} are 2×2 submatrices. It follows from the definitions, Eqs. (2) and (29), that M and S are related by a mapping f:

$$S=f(M)=\begin{bmatrix}(m_{11}^{\dagger})^{-1} & m_{12}m_{22}^{-1}\\ -m_{22}^{-1}m_{21} & m_{22}^{-1}\end{bmatrix}, \tag{30}$$

$$M=f(S)=\begin{bmatrix}(s_{11}^{\dagger})^{-1} & s_{12}s_{22}^{-1}\\ -s_{22}^{-1}s_{21} & s_{22}^{-1}\end{bmatrix}, \tag{31}$$

where we used the pseudounitarity of M [Eq. (5)]. Note that $f(f(M))=f(S)=M$. It is straightforward to verify that an S matrix obtained from this mapping is indeed unitary: $S^{\dagger}S=SS^{\dagger}=I_4$. For matrices close to I_4 the mapping is approximated to second order in ϕ by

$$S=f(M)=f(e^{i\phi K})\approx e^{i\phi JK}\ . \tag{32}$$

As JK is Hermitian, $\exp(i\phi JK)$ is unitary (for real ϕ).

We make a few more remarks concerning this relation between M and S matrices. Since the mapping involves inversion of the submatrices m_{ii} or s_{ii}, these must be nonsingular. From the condition of pseudounitarity, Eq. (5), written in submatrices, it follows that $m_{11}^{\dagger}m_{11}=I_2+m_{21}^{\dagger}m_{21}$. Since $m_{21}^{\dagger}m_{21}$ is non-negatively definite, m_{11} cannot have an eigenvalue zero and is therefore indeed nonsingular. The same reasoning applies to m_{22}. Nonsingularity of s_{ii} means that reflection coefficients are not allowed to equal unity. A reflection coefficient unity would imply the divergence of some matrix elements of M; optically speaking the ring resonator would then be a standing-wave resonator in disguise. We restrict ourselves here to situations where reflection coefficients are smaller than unity, so that M exists and the submatrices m_{ii} and s_{ii} are nonsingular [31].

The mapping f can in principle be applied to calculate the S matrix for every optical element in the system. However, these S matrices have the disadvantage that they cannot simply be multiplied to find the S matrix for one round-trip through the system. The round-trip S matrix is found by first multiplying the individual M matrices and applying the mapping f to the round-trip M matrix $M_{\rm RT}$. The resulting S matrix $S_{\rm RT}$ has the property that its eigenvalues are identical with the pseudoeigenvalues of $M_{\rm RT}$, as follows directly from the definitions [Eqs. (2) and (29)]. Thus one solves the eigenvalue equation:

$$S_{\rm RT}|\psi\rangle=e^{i\phi}|\psi\rangle\ . \tag{33}$$

The definitions also show that the eigenvector $|\psi\rangle$ of $S_{\rm RT}$ and the pseudoeigenvector $|\phi\rangle$ of $M_{\rm RT}$ are related by

$$|\phi\rangle=\begin{bmatrix}e^{i\phi/2}I_2 & 0\\ 0 & e^{-i\phi/2}I_2\end{bmatrix}|\psi\rangle=e^{i\phi J/2}|\psi\rangle\ . \tag{34}$$

Now that the mode spectrum of a ring cavity has been reexpressed as the eigenvalue spectrum of a unitary matrix $S_{\rm RT}$, it is possible to apply the usual symmetry arguments.

VIII. SYMMETRY IN OPTICAL RING CAVITIES

In this section we investigate the influence of symmetries on the cavity-mode structure. Symmetries are usually expressed as a property of a unitary operator (or Hermitian Hamiltonian), the eigenvalues of which give the spectrum of interest. We show that the symmetries of interest here can equally well be expressed directly in terms of the M matrix. The mapping f is used to make the connection with the familiar formulation in terms of the unitary S matrix.

The symmetries of interest are those which, if shared by all optical components in the system, are transferred to the entire chain of elements. Hence, if the M matrices of two optical elements, M_1 and M_2, share a certain symmetry property, then the product M_1M_2 must have that property as well. In principle, one could also consider situations where the optical system has a global symmetry that is not present in the individual components. Such "accidental" symmetries can only be found by actually calculating the matrix $S_{\rm RT}$ or $M_{\rm RT}$ for that particular configuration. We do not consider such situations here.

Let us suppose that a certain symmetry property is respected by the group product. Clearly then, all U(2,2) elements (M matrices) sharing this property form a subgroup of U(2,2) and the corresponding algebra is a subalgebra of u(2,2). The generators of the subalgebra are found by considering infinitesimal elements of the subgroup, i.e., elements close to the unit element I_4. In order to answer the question of how the symmetry property is expressed in terms of S matrices, we can apply the mapping f to the infinitesimal elements, using Eq. (32). We illustrate this procedure with the example of time-reversal symmetry.

Time-reversal symmetry was introduced in Sec. III as the condition $[T,M]=0$. If we now substitute a matrix $M=\exp(i\phi K)$ with infinitesimal ϕ, we find for K the condition $[T,iK]=0$ or, since T is antiunitary, $\{T,K\}=0$ [cf. Eq. (17)]. The generators K_{ks} that anticommute with T are precisely the ten T-symmetric generators, which indeed generate an algebra (see Sec. VI). In order to reexpress time-reversal symmetry in terms of the S matrix, we note that $\{T,J\}=0$, from which it follows that $\{T,iJK\}=0$. Since the infinitesimal S matrix is given by $S=\exp(i\phi JK)$ [see Eq. (32)], we have

$$TS=S^{\dagger}T\ . \tag{35}$$

In Eq. (35) we recognize the familiar formulation of time-reversal symmetry [26–28].

The above procedure to reexpress a symmetry of M in terms of S has been carried out for symmetry operations, either unitary (U) or antiunitary (A), that commute with M. An additional distinction was made between symmetry operations commuting and anticommuting with J, i.e., between symmetries of the block-diagonal and off-diagonal type. We denote these two cases as type I and type II, respectively. For example, type U-II will stand for a unitary symmetry operation with off-diagonal blocks only. The results are shown in Table III.

The reason for distinguishing between the diagonal (I) and off-diagonal (II) types of symmetry becomes clear if one starts from the formulation in terms of the S matrix. For example, for an antiunitary symmetry $AS=S^{\dagger}A$ we

TABLE III. The formulation of the four different types of symmetry; U_1 and U_2 are 2×2 unitary matrices and $\mathcal{C}$ is the complex conjugation operator.

Type	Form	M matrix	Generators	S matrix
Unitary (I)	$U=\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$	$[U,M]=0$	$[U,K_{ks}]=0$	$[U,S]=0$
Unitary (II)	$U=\begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix}$	$[U,M]=0$	$[U,K_{ks}]=0$	$US=S^\dagger U$
Antiunitary (I)	$A=\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}\mathcal{C}$	$[A,M]=0$	$\{A,K_{ks}\}=0$	$[A,S]=0$
Antiunitary (II)	$A=\begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix}\mathcal{C}$	$[A,M]=0$	$\{A,K_{ks}\}=0$	$AS=S^\dagger A$

substitute an infinitesimal $S=\exp(i\phi JK)$ and find the condition $\{A,iJK\}=0$. This determines a set of generators, which we now require to have closed commutation relations (the symmetry property must be preserved by the group product). We then find that this requirement imposes on A the condition that it must be of the second type: $\{A,J\}=0$.

So far our arguments were based on infinitesimal M and S matrices and, looking at Eqs. (30) and (31), it is not immediately obvious that the results of Table III are still correct for finite group elements. It is only slightly more elaborate to show that the results remain valid also for finite group elements. For example, a U-I symmetry,

$$U=\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}, \qquad (36)$$

imposes conditions on the submatrices of M, such as $m_{11}U_1=U_1m_{11}$. With the mapping f we then obtain $(s_{11}^\dagger)^{-1}U_1=U_1(s_{11}^\dagger)^{-1}$ and from this $s_{11}U_1=U_1s_{11}$. After repeating this for all submatrices, the final result can be written as $[U,S]=0$.

We make only a few remarks about the U-II and A-I types of symmetry. We feel that they have only limited applicability, since propagation through free space (K_{30}) does not obey these symmetries. The inevitable presence of sections of free space therefore breaks such symmetry. It can be shown that in the presence of a U-II or A-I symmetry the eigenvalues of $S_{\rm RT}$ occur in complex conjugate pairs $\exp(\pm i\phi)$. In band structures like in Fig. 2 this would be visible as a fictitious horizontal line about which the band structure is reflection symmetric. In most of our experimental band structures we can actually find such a horizontal line. It could very well be that those configurations can be considered as consisting effectively of one optical element with U-II or A-I symmetry plus one section of free propagation. In such a situation, with effectively only one section of free propagation, one can show that the horizontal line is shifted in such a way that the eigenvalues of $S_{\rm RT}$ occur in pairs $\exp[i(kL\pm\phi)]$, with L the length of free propagation.

A. Type-I unitary symmetry

In the presence of a U-I symmetry U it is convenient to look for simultaneous eigenvectors of U and S. The eigenvalue problem of S can now be solved separately for every eigenspace of U. We give three examples of unitary symmetry. For the first example, we note that in the experiments of Figs. 2(a)–2(c) all relevant generators commute with J, so that $U=J$ is a unitary symmetry. The eigenvectors of S can therefore be found among the eigenvectors of J, which are of the form $(\mathbf{A},0)$ and $(0,\mathbf{B})$. These vectors represent traveling waves in the clockwise or counterclockwise direction. Therefore the eigenmodes in Figs. 2(a)–2(c) represent traveling waves. This result is not surprising because all optical elements are nonreflecting and there is no coupling between counterpropagating waves.

For our second example we note that all generators relevant for Figs. 2(d) and 2(f) commute with $U=K_{03}$. The eigenvectors of K_{03} represent light with either x or y polarization, so that every mode in Figs. 2(d) and 2(f) is either a linear combination of $|x,\mathrm{CW}\rangle$ and $|x,\mathrm{CCW}\rangle$ or a linear combination of $|y,\mathrm{CW}\rangle$ and $|y,\mathrm{CCW}\rangle$. This has been observed in our experiments by choosing the polarization of the injected laser light. With x-polarized injection the y-polarized modes could not be excited and half of the modes seemed to disappear [19].

As a last example of unitary symmetry we consider an isotropic cavity, described by the generators K_{k0}. It is intuitively clear that there must be polarization degeneracy in an isotropic system. The isotropic S matrix has the general form

$$S=e^{i\chi}\begin{bmatrix} \alpha I_2 & \beta I_2 \\ -\beta^* I_2 & \alpha^* I_2 \end{bmatrix}, \qquad (37)$$

with $|\alpha|^2+|\beta|^2=1$. It is now easily verified that every unitary matrix U composed of two identical 2×2 diagonal blocks is a symmetry of the system, so that the symmetry group of the system is SU(2) [disregarding a trivial prefactor $\exp(i\phi)$ in U]. Every eigenvalue is doubly degenerate and vectors in the same two-dimensional eigenspace differ only in polarization.

B. Type-II antiunitary symmetry

Before giving examples of A-II symmetry, we note that this symmetry can be expressed in an alternative way. If we write $A=U\mathcal{C}$, with unitary U and $\mathcal{C}$ the complex conjugation operator, the condition $[M,A]=0$ can be expressed equivalently as

$$\tilde{M}(U^{\dagger}J)M=(U^{\dagger}J)\ . \tag{38}$$

This states that M preserves the bilinear metric $(U^{\dagger}J)$. If we now realize that $J\,U=-U\,J$ (see Table III), we see easily that $(U^{\dagger}J)$ is an antisymmetric matrix if $A^2=I_4$ and a symmetric matrix if $A^2=-I_4$. In the former case the M matrices form a *symplectic* group and in the latter case an *orthogonal* group [25,27]. An example of an A-II symmetry with $A^2=I_4$ is time-reversal symmetry, $A=T$. The case with $A^2=-I_4$ is well known to produce Kramers's degeneracy [26–28]. We give examples of both cases.

1. Time-reversal symmetry

The subalgebra corresponding to time-reversal symmetry is of course the algebra of the ten T-symmetric generators, for which $\{T,K_{ks}\}=0$. The M matrices preserve the antisymmetric bilinear metric

$$U^{\dagger}J=\begin{bmatrix}0 & -I_2\\ I_2 & 0\end{bmatrix}, \tag{39}$$

and form the *symplectic* group USp(2,2), as mentioned in Sec. VI; cf. Eq. (27). As in the case of unitary symmetry, we can now look for the eigenvectors of S among the eigenvectors of T. The eigenvectors of T being of the form $(\mathbf{A},\pm\mathbf{A}^*)$, we conclude that the eigenvectors of a time-symmetric S represent standing waves. Note that if $\mathbf{A}$ represents σ^+ polarization, then $\mathbf{A}^*$ represents σ^- polarization, so that $(\mathbf{A},\pm\mathbf{A}^*)$ represent so-called twisted modes.

If an eigenvector $|1\rangle$ of S does not represent a standing wave, it must be degenerate with its own time reverse $T|1\rangle$, and hence also with the linear superpositions $|1\rangle\pm T|1\rangle$, which *do* represent standing waves. The arguments apply to all experimental examples of Fig. 2, since only optical components associated with T-symmetric generators were used.

For the experiments of Figs. 2(a)–2(c) we concluded in Sec. VIII A that the eigenmodes were traveling waves, because there was a unitary symmetry $U=J$. Here, however, we conclude that the eigenmodes are standing waves, because there is time-reversal symmetry. These two conclusions are consistent only if there is degeneracy, such that two degenerate traveling-wave modes make up two standing-wave modes. In fact, since the M matrix commutes with J as well as with T, it also commutes with $J\,T$. Now $J\,T$ is an A-II symmetry and $(J\,T)^2=-I_4$, so that we have Kramers's degeneracy. For the experiments of Figs. 2(d) and 2(f) we concluded in Sec. VIII A that the eigenmodes were either x or y polarized. Since we conclude here that the eigenmodes are standing waves, we have fully determined the eigenmodes by symmetry arguments.

2. Kramers's degeneracy

A type-II antiunitary symmetry A with $A^2=-I_4$ always leads to degeneracy in the eigenvalue spectrum of S. For every eigenvector $|1\rangle$ there is an orthogonal eigenvector $A|1\rangle$ with the same eigenvalue. The orthogonality, $\langle 1|A|1\rangle=0$, is proven easily using the antiunitarity of A and $A^2=-I_4$. The resulting degeneracy is called Kramers's degeneracy [26–28]. Kramers's degeneracy is usually associated with time-reversal invariance in systems containing an odd number of fermions [26]. In such systems the time-reversal operator squares to -1 and all eigenstates of the Hamiltonian of the system are doubly degenerate. The best known example occurs in atoms with an odd total number of electrons and nucleons, as long as time-reversal symmetry is not broken, e.g., by a magnetic field.

In our case time-reversal invariance clearly cannot produce Kramers's degeneracy, because $T^2=I_4$ and we have to look for other antiunitary symmetries. One example was already mentioned above for the experiments of Figs. 2(a)–2(c). The symmetry operation in that case is

$$A_1=J\,T=\begin{bmatrix}0 & I_2\\ -I_2 & 0\end{bmatrix}\mathcal{C}\ . \tag{40}$$

The generators anticommuting with $J\,T$ are K_{31}, K_{02}, K_{33}, K_{30}, K_{12}, and K_{22}, which were shown to generate su(2)⊕su(1,1) in Sec. VI.

As another example we consider the operator

$$A_2=\begin{bmatrix}0 & i\sigma_2\\ i\sigma_2 & 0\end{bmatrix}\mathcal{C}\ . \tag{41}$$

In this case we find the generators K_{01}, K_{02}, K_{03}, K_{10}, K_{20}, and K_{30}, again generating an su(2)⊕su(1,1) algebra. This example of Kramers's degeneracy was present in the experiments described in Refs. [2–4]. The ring cavity contained sections of free space (K_{30}), an isotropic reflector (K_{10},K_{20}), and a T-antisymmetric retarder (K_{03}). The antisymmetric retarder was in fact a Faraday rotator sandwiched between two quarter-wave plates, according to Fig. 4(a). Note that in this example K_{01} and K_{03} are T-antisymmetric generators and that this type of Kramers's degeneracy is not broken by the magnetic field inside the Faraday rotator, in contrast to Kramers's degeneracy in atoms.

Finally, we mention that an isotropic cavity, described by the generators K_{k0}, in general does not display Kramers's degeneracy, but does so when it is not rotating (no Sagnac effect; K_{00} is absent). In that case the symmetry operator is A_2 as in the above example.

Both examples given here, A_1 and A_2, define a six-dimensional algebra, which is consistent with the results of Scharf *et al.* [28]. They used a perturbation approach to show that, in the four-dimensional state space of two Kramers's degenerate levels, the Hamiltonian contains no more than six independent parameters. The symmetry that produces Kramers's degeneracy is the same symmetry that leads to quartic level repulsion in the context of random matrix theory of spectra [27].

IX. CONCLUSIONS

We have demonstrated experimentally a large variety of four-level systems in an optical ring cavity. Intracavity birefringent and reflecting elements were used to couple light waves with different polarizations and different propagation directions, respectively. The measurement of the cavity-mode frequencies as a function of a control parameter resulted in general in a diagram ("band structure") with four crossing and anticrossing levels. The four-level systems were adjustable by turning knobs in the experiment. In view of earlier work on optical realizations of driven two-level systems ("optical atoms") [3,4], this opens numerous possibilities for dynamical experiments in tailored four-level schemes. For example, one could realize a Λ- or V-type three-level scheme—leaving the fourth level unused—and drive it with a bichromatic field.

As a basis for the description of optical ring cavities we adopted the matrix formalism introduced by Lenstra and Geurten [9] which, under the restriction to loss-free conditions, leads naturally to the study of the Lie group U(2,2) and the associated Lie algebra u(2,2). The merit of the group-theoretical approach comes from the association of a specific type of optical element with each of the 16 generators of the algebra. The commutation relations provide at one glance information about what kind of effect may be produced with a sequence of optical elements. This notion was quantified in a recipe for a "sandwich" construction of nonstandard optical components out of standard (i.e., commercially available) ones. The commutation relations of the Lie algebra thus served as a starting point for "optical engineering." It was made clear that the entire group U(2,2) can be realized with optical components. The commutation relations also show how and to what extent the Sagnac effect in rotating ring cavities can be simulated as reported in previous experiments.

If the optical components are restricted to a few types, in such a way that the corresponding generators have closed commutation relations, the optical system is restricted to a subgroup of U(2,2). The number of independent parameters—16 for a general U(2,2) element—is then reduced to the number of generators of the subgroup. For example, in nonrotating ring cavities (no Sagnac effect) we deal with SU(2,2) instead of U(2,2). If the polarization degree of freedom remains unused, we deal with the group U(1,1), or SU(1,1) if the Sagnac effect is absent. If no reflectors are present, the relevant subgroup is $U(2)\otimes U(2)$.

The restriction to a subgroup was shown to be connected with a common symmetry in the optical components. We have used such symmetry properties to make general statements regarding the cavity-mode structure in our experimental configurations. In particular, in all the experiments discussed here time-reversal invariance was shown to apply, leading to the symplectic ten-parameter subgroup USp(2,2) and to standing-wave character of the cavity eigenmodes. In the experiment of Fig. 2(d) an additional unitary symmetry was identified, allowing the determination of the cavity eigenmodes completely by symmetry arguments. We identified antiunitary symmetries producing Kramers's degeneracy in the experiments of Figs. 2(a)–2(c) as well as in previously reported experiments lacking time-reversal symmetry. These are all examples of Kramers's degeneracy in bosonic systems (photons), contrary to its more familiar occurrence in fermionic systems with time-reversal invariance.

Although much has been said about symmetries, we have only marginally discussed the symmetrical appearance of some of the band structures of Fig. 2, most of which show reflection symmetry in a horizontal or vertical line. The reflection symmetries in a vertical line are determined by the functional dependence of the band structure on an (arbitrarily chosen) control parameter. The methods described in this paper do not seem well suited to analyze this problem. The horizontal reflection symmetry could point to one of the symmetries that were not discussed in detail, the type-II unitary and type-I antiunitary symmetries.

The application of the group U(2,2) presented here is to our knowledge its first application in optics. In some of the practical examples in this paper the group-theoretical approach may seem to be overkill because simple optical arguments work just as well. In other examples, however, it is our feeling that the group-theoretical approach is superior to optical intuition and in fact sharpens the latter. Obviously, this superiority will increase with the number of intracavity elements. Our experience is that the group U(2,2) is of direct practical use in optical experiments.

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[29] In fact, the group SU(2,2) is simply connected, implying that all its elements can be written in the form $M=\exp(i\phi K)$; see also Refs. [24] and [25]. The elements of U(2,2) can also be written in this form, as they consist of a phase factor $\exp(i\alpha)$ times an SU(2,2) element; see Sec. IV C.

[30] Strictly speaking, the injection of light implies that the cavity must have a leak to the outside world so that the cavity is no longer loss-free and its resonances have finite width. We assume here that the losses affect all modes in the same way, i.e., produce no coupling between the modes (cf. Ref. [5]).

[31] Note that f maps the noncompact set of pseudounitary matrices U(2,2) into the compact set of unitary matrices U(4). However, the image of U(2,2) is a noncompact subset of U(4) due to the exclusion of unitary matrices with singular s_{ii}.