Effect of wiggler errors on free-electron-laser gain

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For a free-electron laser (FEL) operating in the exponential regime before saturation, we present an analytic description of the effect on the gain of longitudinal velocity variations arising from wiggler field errors. The average gain reduction and the width of the output power distribution are expressed in terms of the mean-square average of the ponderomotive phase shift per gain length. A scheme for correcting the electron trajectory using position monitors and dipole correctors is analyzed. Our work is directly applicable to the design of FEL amplifiers, and the results are encouraging for their feasibility.

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I. INTRODUCTION

Understanding the effects of wiggler errors [1–8] is of critical importance in the design of free-electron lasers. There has been recent work aimed at determining the tolerances which must be imposed on the allowable field errors. Analytic results have been obtained for the effect of wiggler errors on the electron trajectory [7], and for the gain reduction in the low-gain regime [5, 6]. Computer simulations have been carried out both in the low- [6, 8] and high-gain [3, 4] regimes. In this paper, we consider a free-electron laser (FEL) operating in the exponential regime before saturation and use one-dimensional FEL equations [9–11] to develop an analytic description of the effect of wiggler errors on the gain.

The effects of wiggler errors can be divided into two classes: (1) longitudinal velocity fluctuation and drift, which moves the electron beam away from resonance; and (2) transverse trajectory wander, which causes the centroid of the electron beam to move away from the radiation beam. In this paper we address the longitudinal effects and leave detailed consideration of the mode overlap problem to a planned future work based on a full three-dimensional computer simulation. The determination of the effect of longitudinal velocity variations is dominantly a one-dimensional problem for which we present a detailed analysis.

We divide longitudinal velocity effects into two types. The first type is magnetic-field amplitude errors, which are correlated over only a few wiggler periods and have no net field integral. The second type is cumulative steering errors, which produce a drift of the electron trajectory away from the wiggler axis.

For the case of amplitude errors, we find that the required tolerance on the magnetic-field fluctuation is relaxed, because it turns out that rather than having to satisfy $(\Delta B/B)_{\rm rms} \ll \rho$ one needs only meet $(\Delta B/B)^2_{\rm rms} \ll \rho$, where ρ is the Pierce parameter [9] typically of magnitude 10^{-3}.

Without correcting the electron trajectory periodically along the wiggler, achievable steering errors will cause a cumulative angular deviation away from the wiggler axis, moving the electron beam away from resonance and completely negating the gain. We have derived an expression for the gain reduction when there is trajectory correction and we have found that by installing a sufficient number of correction and monitoring stations along the wiggler, the required tolerance on magnetic-field steering errors can be relaxed to an achievable limit. The results of our analysis on both types of longitudinal errors are encouraging for the design of single-pass FEL amplifiers utilizing long wiggler magnets.

For both types of longitudinal error we have found that the criterion for small-gain reduction is determined by keeping the ponderomotive phase shift due to wiggler errors small in one gain length. The ponderomotive phase shift is

$$\delta = k \int dz \frac{1}{v_0}[v_{\|}(\gamma_0, z) - v_0] \tag{1.1}$$

for a monoenergetic electron beam of energy γ_0, where v_o is the average longitudinal velocity in the ideal wiggler and $v_{\|}(\gamma_0, z)$ is the longitudinal velocity in the presence of wiggler errors.

The importance of the ponderomotive phase shift in determining the gain reduction due to wiggler errors has been discussed in Refs. [5] and [6] for the low-gain regime. In this paper we consider the high-gain regime and explicitly express the gain reduction due to wiggler errors in terms of W, the average value of the square of the ponderomotive phase shift per gain length, defined by [see Eq. (6.16)]

$$W \equiv \frac{1}{T}\overline{\left(\int_0^T d\tau \frac{d\delta(\tau)}{d\tau}\right)^2}, \tag{1.2}$$

where $\tau = 2\rho k_w z$ is the scaled longitudinal coordinate [9] which changes by $2/\sqrt{3}$ in a gain length, and the average is over an ensemble of wiggler errors. The power growth of the radiation field is proportional to

$$e^{(\sqrt{3}-2W/9)\tau}. \tag{1.3}$$

For magnetic-field amplitude errors [see Eq. (6.31)]

$$W = \frac{\pi K_0^4}{(1+K_0^2/2)^2}\frac{(\Delta B/B)^2_{\rm rms}}{\rho}, \tag{1.4}$$

where K_0 is the magnetic-field strength parameter for the ideal wiggler. It follows that the criterion for small gain reduction is

$$\frac{2\pi}{9\sqrt{3}}\frac{K_0^4}{(1+K_0^2/2)^2}\frac{(\Delta B/B)^2_{\rm rms}}{\rho} \ll 1. \tag{1.5}$$

There is a simple physical interpretation of Eq. (1.4). The change δ_p in ponderomotive phase per wiggler period due to magnetic-field amplitude errors is

$$\delta_p = \frac{2\pi K_0^2}{1+K_0^2/2}\frac{\Delta B}{B}, \tag{1.6}$$

and the number of periods per gain length is approximately $N_G = 1/4\pi\rho$, hence

$$W = N_G(\delta_p)^2_{\rm rms}, \tag{1.7}$$

in agreement with Eq. (1.4).

We have derived an expression for the gain reduction due to steering errors assuming the trajectory is being corrected. We assume position monitors and trims are at the same locations spaced by N_S periods along the wiggler. The spacing is assumed to be shorter than a gain length. Trim strength of a given corrector is adjusted to center the beam at the following monitor. In this case we have found that the average value of the square of the ponderomotive phase shift per gain length is given by

$$W \approx \frac{24}{35}\frac{N_G}{N_S}(N_S\delta_p)^2 \tag{1.8}$$

with the rms phase shift per period δ_p in this case easily seen to be given by

$$\delta_p = \frac{2\pi\gamma^2 x'^2_{\rm rms}}{1+K_0^2/2}, \tag{1.9}$$

where the mean-square angular deviation $x'^2_{\rm rms}$ for the trajectory corrected beam is

$$x'^2_{\rm rms} = \frac{1}{6}N_C\overline{\theta^2}. \tag{1.10}$$

Here, N_C is the number of steering errors between correction stations and $\overline{\theta^2}$ is the mean-square angular deviation introduced by a single error.

The result of Eq. (1.8) also has a simple physical interpretation. One sees that $N_S\delta_p$ is the ponderomotive phase shift between two correction stations, because the phase drift is coherent between correctors. Since phase drifts for different sections are incoherent, the total phase drift per gain length is equal to the square root of the number of correction sections per gain length times $N_S\delta_p$.

As an example, consider a wiggler with $K_0 = \sqrt{2}$. We suppose the number of periods per gain length is $N_G = 100(\rho \simeq 0.8\times10^{-3})$, the number of periods between correction stations is $N_S = 50$, and the number of steering error kicks between correction stations is $N_C = 100$. We express the mean-square angular deflection per kick as

$$\gamma^2\overline{\theta^2} = 4K_0^2(\Delta B/B)^2_{\rm rms}. \tag{1.11}$$

Taking the achievable tolerance $(\Delta B/B)_{\rm rms} = 5\times10^{-3}$, we find $W = 0.37$, resulting in a modest 5% reduction in the growth rate.

We have also obtained an expression for the width of the output power distribution; see Eqs. (6.21)–(6.25). For a long wiggler containing many gain lengths, there is a contribution to the width independent of the total length, determined by the field errors at the beginning and end of the wiggler. The magnitude of this width is proportional to $\sqrt{W}$, and hence for small errors (and/or short wigglers) the spread in the output power is comparable to or larger than the average reduction in power.

Our paper is organized in the following manner. In Sec. II, we derive a third-order differential equation, Eq. (2.27), determining the effect of wiggler errors on the FEL gain. This equation is solved in Sec. III in a manner directly yielding the average growth rate of the radiation field. In Sec. III–V the contributions of errors at the beginning and end of the wiggler are ignored. They are considered in Sec. VI where they are shown to contribute to the width of the output power distribution. The case of an idealized permanent magnet wiggler with infinitely wide blocks is considered in Sec. IV. In this two-dimensional model, magnetization errors give rise to only local longitudinal velocity modulation, with no net steering errors. In Sec. V, the gain reduction in this idealized case is analyzed. Next, in Sec. VI, we present an alternate solution of the differential equation, Eq. (2.27), by reformulating it as an integral equation. For a stochastic ensemble of wiggler field errors, we express the average gain reduction and the width of the output power distribution in terms of the mean-square average of the ponderomotive phase shift per gain length. In Sec. VII we treat steering errors which have been ignored to this point. A particular trajectory correction scheme is analyzed in detail, and an analytic expression is derived for the resulting gain reduction. The results look encouraging for the development of FEL amplifiers based on long wiggler magnets.

II. FEL GAIN EQUATIONS

We assume a two-dimensional (2D) planar wiggler with magnetic field

$$B_y = \frac{\partial A_x}{\partial z}, \tag{2.1}$$

$$\frac{eA_x}{mc} = K(z)\cos[k_w z + \alpha(z)]. \tag{2.2}$$

Here $2\pi/k_w$ is the design wiggler period, and K_0 is the design value of the wiggler parameter K. We will leave until later the question of how to obtain the *two* functions $K(z)$ and $\alpha(z)$ unambiguously from the *single* field error function $\delta A_x(z)$ or $\delta B_y(z)$. Constancy of the x component of the canonical momentum leads to

$$x' = \frac{K(z)}{\gamma}\cos[k_w z + \alpha(z)] \ , \tag{2.3}$$

where we neglect the radiation field in calculating the electron trajectory and where the prime stands for the derivative with respect to z.

We now consider a plane-polarized radiation field

$$E_x(z,t) = cB_y(z,t) = \frac{1}{2}E(z)e^{ik(z-ct)} + \frac{1}{2}E^*(z)e^{-ik(z-ct)} \tag{2.4}$$

corresponding to a long radiation pulse. Energy is exchanged between the jth electron and the radiation field such that

$$\gamma_j' = -\frac{eE(z)K(z)}{2mc^2\gamma_j}\cos[k_w z + \alpha(z)]e^{ik(z-ct_j)} + \text{c.c.} \tag{2.5}$$

The radiation wave number k is chosen to satisfy the resonant condition for the ideal wiggler,

$$k = \frac{2\gamma_0^2 k_w}{1 + K_0^2/2}, \tag{2.6}$$

with γ_0 being the design electron energy in units of mc^2. Let us now define a phase angle for each electron as

$$\psi_j = (k_w + k)z - kct_j , \tag{2.7}$$

where t_j is the arrival time of the jth electron at longitudinal position z. The rate of change of ψ_j with respect to z is

$$\psi_j' = k_w + k\left(1 - \frac{c}{\dot z_j}\right) \cong k_w\left[1 - \frac{2\gamma_0^2}{1 + K_0^2/2}\left(1 - \frac{\dot z_j}{c}\right)\right] . \tag{2.8}$$

But

$$\frac{\dot z_j}{c} \cong \frac{v_j}{c} - \frac{x'^2}{2} \cong 1 - \frac{1 + K^2(z)\cos^2[k_w z + \alpha(z)]}{2\gamma_j^2} . \tag{2.9}$$

Assuming $(\gamma_j - \gamma_0)/\gamma_0$ is small, we find

$$\psi_j' \cong \frac{2k_w(\gamma_j - \gamma_0)}{\gamma_0} - k_w\frac{\{K^2(z)\cos^2[k_w z + \alpha(z)] - K_0^2/2\}}{1 + K_0^2/2} . \tag{2.10}$$

The final equation governing the FEL gain in the exponential gain regime is the stimulation of radiation by the electron motion in the wiggler. This is governed by the wave equation

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \mu_0\frac{\partial J_x}{dt} = -ik\mu_0 ec^2\sum_j x_j'\delta(z - z_j(t)) . \tag{2.11}$$

We use the paraxial approximation, assuming $E(z)$ [Eq. (2.4)] is a slowly varying function whose second derivative can be neglected. We average the right-hand side of Eq. (2.11) over a small volume, and change to z as the independent variable. Defining n_0 to be the number of electrons per unit volume, we find [9–11]

$$E' = n_0\mu_0 ec^2K(z)e^{ik_w z}\cos[k_w z + \alpha(z)]\left\langle\frac{e^{-i\psi_j(z)}}{\gamma_j}\right\rangle, \tag{2.12}$$

where $< >$ denotes the average over the electron distribution.

The ponderomotive phase $\psi_j(z)$ was defined in Eq. (2.7), and the FEL is described by the differential equations (2.5), (2.10), and (2.12). The dominant effect of the wiggler errors comes from Eq. (2.10), which describes the deviation from resonance. We have verified that, for small errors in the wiggler field, it is a good approximation to neglect the error fields in Eq. (2.5) and (2.12), i.e., take $K(z) \cong K_0$ and $\alpha(z) \cong 0$. In this case

$$\gamma_j' = -\frac{eK_0E(z)}{2mc^2\gamma_0}e^{ik_w z}\cos\ k_w z\ e^{i\psi_j} + \text{c.c.}, \tag{2.13}$$

$$\psi_j' = \frac{2k_w(\gamma_j - \gamma_0)}{\gamma_0} - k_w\frac{\{K^2(z)\cos^2[k_w z + \alpha(z)] - K_0^2/2\}}{1 + K_0^2/2} , \tag{2.14}$$

$$E' = \frac{n_0\mu_0 ec^2K_0}{\gamma_0}e^{ik_w z}\cos\ k_w z < e^{-i\theta_j} > . \tag{2.15}$$

In order to extract the synchronous terms we express the ponderomotive phase ψ_j in the form

$$\psi_j(z) = \phi_j(z) + \delta(z) - \frac{K_0^2/4}{1 + K_0^2/2}\sin\ 2k_w z \ , \tag{2.16}$$

where

$$\delta'(z) = -k_w\frac{K^2(z)\cos^2[k_w z + \alpha(z)] - K_0^2\cos^2 k_w z}{1 + K_0^2/2} . \tag{2.17}$$

We then discard the nonsynchronous terms by using the approximation [10]

$$e^{ik_w z}\cos k_w z\ e^{ib\sin 2k_w z} \approx J_0(b) - J_1(b). \tag{2.18}$$

The difference of Bessel functions on the right-hand side corresponds to the dc component, and the oscillating components have been neglected. In this manner we derive

$$\gamma_j' = -\frac{k_w D_2}{\gamma_0}(Ee^{i\delta}e^{i\phi_j} + \text{c.c.}), \tag{2.19}$$

$$\phi_j' = \frac{2k_w(\gamma_j - \gamma_0)}{\gamma_0}, \tag{2.20}$$

$$E' = \frac{k_w D_1}{\gamma_0}e^{-i\delta} < e^{-i\phi_j} >, \tag{2.21}$$

where

$$D_1 = \frac{eK_0[JJ]}{2mc^2k_w} \quad , \quad D_2 = \frac{n_0\mu_0 ec^2K_0[JJ]}{k_w},$$

$$[JJ] = J_0\left(\frac{K_0^2/4}{1+K_0^2/2}\right) - J_1\left(\frac{K_0^2/4}{1+K_0^2/2}\right).$$

We now proceed to derive the gain equation in the approximation of terms linear in $(\gamma_j - \gamma_0)/\gamma_0$, ϕ_j', and E. It is useful to introduce the quantity

$$y(z) = E(z)e^{i\delta(z)}. \tag{2.22}$$

Expressing E in terms of y, we take two derivatives of Eq. (2.21) with respect to z, and discard the quadratic term in $(\phi_j')^2$ to obtain

$$y''' - i(\delta' y)'' = \frac{k_w D_1}{\gamma_0} < -i\phi_j''\, e^{-i\phi_j} > . \tag{2.23}$$

From Eqs. (2.19) and (2.20), it follows that

$$\phi_j'' = -\frac{2k_w^2 D_2}{\gamma_0^2}(ye^{i\phi_j} + y^* e^{-i\phi_j}). \tag{2.24}$$

Assuming the initial electron distribution is unbunched, $< e^{-2i\phi_j} >= 0$; hence

$$y''' - i(\delta' y)'' = i(2\rho k_w)^3 y, \tag{2.25}$$

where the Pierce parameter ρ is defined by

$$(2\rho)^3 = \frac{2D_1D_2}{\gamma_0^3}. \tag{2.26}$$

Changing from z to Bonifacio's [9] variable, $\tau = 2\rho k_w z$, we write the linear gain equation in the form

$$\frac{d^3y}{d\tau^3} - i\frac{d^2}{d\tau^2}[f(\tau)y] = iy, \tag{2.27}$$

where

$$f(\tau) = \frac{d\delta}{d\tau} \,. \tag{2.28}$$

III. SOLUTION OF FEL GAIN EQUATIONS WITH ERRORS

We now proceed to solve Eq. (2.27) in the approximation of small $f(\tau)$. Specifically we have

$$\frac{d^3y}{d\tau^3} - iy = \frac{id^2(fy)}{d\tau^2} \,, \tag{3.1}$$

which has the unperturbed solution ($f = 0$)

$$y = Ae^{-i\omega\tau} + Be^{-i\omega^*\tau} + Ce^{-i\tau}. \tag{3.2}$$

Here ω, ω^*, and 1 are the three cube roots of 1, with

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \tag{3.3}$$

The exponential growth of the term proportional to A corresponds to the growth rate

$$r_g = \frac{\sqrt{3}}{2} \quad \text{in } \tau, \quad (r_g)_z = \frac{\sqrt{3}}{2}2\rho k_w \quad \text{in } z. \tag{3.4}$$

We now consider A, B, and C to be slowly varying functions of τ. Because we have replaced one variable y by three, we are free to set two additional constraints, which we do by arranging that

$$\dot{y} = -i\omega Ae^{-i\omega\tau} - i\omega^* Be^{-i\omega^*\tau} - iCe^{-i\tau} \tag{3.5}$$

and

$$\ddot{y} = -\omega^2 Ae^{-i\omega\tau} - \omega^{*2} Be^{-i\omega^*\tau} - Ce^{-i\tau}. \tag{3.6}$$

The dot(s) indicate derivative(s) with respect to τ. The constraints leading to Eqs. (3.5) and (3.6) are

$$\dot{A}e^{-i\omega\tau} + \dot{B}e^{-i\omega^*\tau} + \dot{C}e^{-i\tau} = 0, \tag{3.7}$$

$$\omega\dot{A}e^{-i\omega\tau} + \omega^*\dot{B}e^{-i\omega^*\tau} + \dot{C}e^{-i\tau} = 0, \tag{3.8}$$

and Eq. (3.1) leads to

$$\omega^2\dot{A}e^{-i\omega\tau} + \omega^{*2}\dot{B}e^{-i\omega^*\tau} + \dot{C}e^{-i\tau} = -i\frac{d^2(fy)}{d\tau^2}. \tag{3.9}$$

The solutions to Eqs. (3.7)–(3.9) are

$$\begin{aligned} 3\dot{A} &= -i\omega e^{i\omega\tau}\frac{d^2(fy)}{d\tau^2} \\ &= -i\omega[Au(\tau) + Bv(\tau)e^{i\lambda_1\tau} + Cw(\tau)e^{i\lambda_2\tau}], \end{aligned} \tag{3.10}$$

$$\begin{aligned} 3\dot{B} &= -i\omega^* e^{i\omega^*\tau}\frac{d^2(fy)}{d\tau^2} \\ &= -i\omega^*[Au(\tau)e^{-i\lambda_1\tau} + Bv(\tau) + Cw(\tau)e^{i\lambda_3\tau}], \end{aligned} \tag{3.11}$$

$$\begin{aligned} 3\dot{C} &= -ie^{i\tau}\frac{d^2(fy)}{d\tau^2} \\ &= -i[Au(\tau)e^{-i\lambda_2\tau} + Bv(\tau)e^{-i\lambda_3\tau} + Cw(\tau)], \end{aligned} \tag{3.12}$$

where we have used Eqs. (3.2), (3.5), and (3.6), and where

$$\lambda_1 \equiv \omega - \omega^*, \quad \lambda_2 \equiv \omega - 1, \quad \lambda_3 \equiv \omega^* - 1, \tag{3.13}$$

$$u(\tau) \equiv \ddot{f} - 2i\omega\dot{f} - \omega^* f, \tag{3.14}$$

$$v(\tau) = \ddot{f} - 2i\omega^*\dot{f} - \omega f, \tag{3.15}$$

$$w(\tau) = \ddot{f} - 2i\dot{f} - f. \tag{3.16}$$

We intend to include terms up to and including second order in f.

In the calculation that follows, we make the simplifying assumption that the error function f and all its derivatives vanish in a region of order a few gain lengths at the beginning and end of the wiggler. The utility of this approximation is that it allows us to directly determine the reduction of the exponential growth rate due to wiggler errors. We suppose the error-free end regions to be fixed in length, and consider increasing the length of the wiggler. We determine the term in log y proportional to wiggler length. In this approximation A, B, and C are constant in the initial and final regions, in which the A term grows exponentially, while the B and C terms become negligible. In Sec. VI, we present a more complete calculation including the effect of errors at the ends of the wiggler. These errors are shown to give rise to a term in $\log y$ independent of the wiggler length; see Eq. (6.21).

The right-hand side of Eq. (3.10) contains terms in A, B, and C. The factors multiplying the terms in B and C decrease exponentially with τ. As a result, only those parts of B and C which increase exponentially will contribute to a modification of the growth rate. But the terms which increase exponentially in Eqs. (3.11) and (3.12) for B and C are those which arise from A on the right-hand side of these equations. Since the result for B and C will then be proportional to f, and since the B and C terms on the right-hand side of Eq. (3.10) will be multiplied by f, the contribution of these terms to $\dot{A}$ will be quadratic in f. Moreover, A can be considered constant in these terms since $\dot{A}$ itself is first order in f. Thus we have

$$3\dot{B} \cong -i\omega^* A e^{-i\lambda_1\tau} u(\tau), \quad 3\dot{C} = -iAe^{-i\lambda_2\tau} u(\tau). \tag{3.17}$$

The increment to the growth rate is related to the real part of $\dot{A}/A$ averaged over the full wiggler. Thus we calculate $\dot{A}/A$ from Eq. (3.10):

$$\frac{\dot{A}}{A} = -\frac{i\omega u}{3} - \frac{i\omega v}{3}\frac{B}{A}e^{i\lambda_1\tau} - \frac{i\omega w}{3}\frac{C}{A}e^{i\lambda_2\tau}. \tag{3.18}$$

According to Eq. (3.17), B/A and C/A can be written as

$$\begin{aligned} \frac{B}{A} &= -\frac{i\omega^*}{3}\int_0^\tau d\tau' e^{-i\lambda_1\tau'} u(\tau')\ , \\ \frac{C}{A} &= -\frac{i}{3}\int_0^\tau d\tau' e^{-i\lambda_2\tau'} u(\tau'). \end{aligned} \tag{3.19}$$

The average of $\dot{A}/A$ over the full wiggler is therefore

$$\left\langle\frac{\dot{A}}{A}\right\rangle = -\frac{i\omega}{3} < u > -\frac{I_{1vu}}{9} - \frac{\omega I_{2wu}}{9}, \tag{3.20}$$

where

$$I_{1vu} \equiv \frac{1}{T}\int_0^T d\tau\, e^{i\lambda_1\tau} v(\tau)\int_0^\tau d\tau' e^{-i\lambda_1\tau'} u(\tau'), \tag{3.21}$$

$$I_{2wu} \equiv \frac{1}{T}\int_0^T d\tau\, e^{i\lambda_2\tau} w(\tau)\int_0^\tau d\tau' e^{-i\lambda_2\tau'} u(\tau'). \tag{3.22}$$

Here $T = (2\rho)2\pi N_P$, where N_P is the number of wiggler periods, and where the symbol $\langle\,\rangle$ stands for the average over the interval $0 < \tau < T$.

All terms in Eqs. (3.21) and (3.22) are of the form

$$J_{mn} = \frac{1}{T}\int_0^T d\tau\, e^{i\lambda\tau}\frac{d^m f}{d\tau^m}\int_0^\tau d\tau' e^{-i\lambda\tau'}\frac{d^n f}{d\tau^n} \tag{3.23}$$

for $m = 0, 1, 2$ and $n = 0, 1, 2$. Integrating by parts and discarding terms at the end points $\tau = 0, T$, we find

$$J_{01} = -J_{10} = \frac{J_{02}}{i\lambda} = -\frac{J_{11}}{i\lambda} = \frac{J_{20}}{i\lambda} = < f^2 > +i\lambda J_{00} \tag{3.24}$$

and

$$J_{12} = -J_{21} = -\frac{J_{22}}{i\lambda} = < \dot{f}^2 > +\lambda^2 < f^2 > +i\lambda^3 J_{00}. \tag{3.25}$$

We then obtain

$$I_{1vu} = i(\omega - \omega^*) < \dot{f}^2 > +J_{00}(\lambda = \lambda_1), \tag{3.26}$$

$$\omega I_{2wu} = i(\omega^* - \omega) < \dot{f}^2 > +J_{00}(\lambda = \lambda_2), \tag{3.27}$$

leading finally to

$$\left\langle\frac{\dot{A}}{A}\right\rangle = \frac{i}{3}\langle f\rangle - \frac{1}{9T}\int_0^T d\tau\int_0^\tau d\tau' f(\tau)f(\tau')(e^{i(\omega-\omega^*)(\tau-\tau')} + e^{i(\omega-1)(\tau-\tau')}), \tag{3.28}$$

where $f(\tau)$, defined in Eq. (2.28), is

$$f(\tau) = \frac{d\delta}{d\tau}. \tag{3.29}$$

We now assume that $f(\tau)$ arises from individual block errors ϵ_j in a form

$$f(\tau) = \sum_j \epsilon_j g(\tau - \tau_j), \tag{3.30}$$

where the error is centered at $\tau = \tau_j$ and where the function $g(u)$ has an appreciable value only for $\mid u \mid$ less than one or two wiggler periods. Specifically, we assume $g(u)$ is negligibly small for $\mid u \mid$ of order 1 (a gain length).

The double integral in Eq. (3.28) is therefore made up of terms which take the form

$$\sum_j\sum_k \epsilon_j\epsilon_k\int d\tau\int d\tau' g(\tau - \tau_j)g(\tau' - \tau_k)e^{i\lambda(\tau-\tau')}. \tag{3.31}$$

If the errors ϵ_j are uncorrelated over distances much larger than a wiggler period, only values of $\mid \tau_j - \tau_k \mid$ of order one wiggler period or less will contribute in Eq. (3.31). Since the function $g(u)$ has an appreciable value only for $\mid u \mid$ less than one or two wiggler periods, the only significant contributions to the double integral over τ and τ' will occur for $|\tau - \tau'|$ less than a few wiggler periods. But for this range of $(\tau - \tau')$ the exponential factor in Eq. (3.31) can be replaced by unity, since $[\mathrm{Im}(\lambda)]^{-1}$ is of order a *gain* length, which is much larger than a wiggler period. Thus we finally obtain for the change in the average growth rate

$$\begin{aligned} \langle\Delta r_g\rangle &= \mathrm{Re}\left\langle\frac{\dot{A}}{A}\right\rangle \\ &= \mathrm{Re}\left[\frac{i}{3T}\int_0^T d\tau\, f(\tau) - \frac{1}{9T}\left(\int_0^T d\tau f(\tau)\right)^2\right]. \end{aligned} \tag{3.32}$$

This result is valid even if

$$G = \int_0^T d\tau f(\tau) \tag{3.33}$$

is large, as long as G/T is small. An alternate derivation, involving the assumption that G is small leads to exactly the same result. Finally, we note that the form in Eq. (3.32) is valid only if we neglect long-range error correlations. If we wish to include the effects of long-range error correlations, we must use Eq. (3.28).

We now assume that T corresponds to many gain lengths. We further assume that the error function vanishes for $\tau \leq 0$ and $\tau \geq T$, so that the limits of integration in Eq. (3.33) can be extended to $-\infty < \tau < \infty$, thereby obtaining

$$G = \int_{-\infty}^{\infty} dz \left(\frac{d\delta}{dz}\right). \tag{3.34}$$

The (real) detuning parameter $\delta(\infty)-\delta(-\infty)$ contributes to the change in gain and we have, from Eq. (2.17),

$$G = -\frac{k_w}{1+K_0^2/2}\int_{-\infty}^{\infty} dz\{\, K^2(z)\cos^2[k_w z+\alpha(z)] - K_0^2\cos^2 k_w z\}. \tag{3.35}$$

Thus the first-order term is imaginary, and only the second-order term contributes, giving

$$\langle \Delta r_g\rangle \cong -\frac{G^2}{9T} \tag{3.36}$$

for the growth rate, where $r_g = \sqrt{3}/2$ is the growth rate without magnet errors.

Finally, we return to Eq. (2.2) and write

$$K(z)\cos[k_w z+\alpha(z)] = \frac{eA_x}{mc} = K_0\cos k_w z + \frac{e}{mc}\delta A_x(z). \tag{3.37}$$

Keeping only linear terms in $\delta A_x(z)$ and then averaging over the rapid wiggler oscillations, we obtain [12]

$$G \cong -\frac{2k_wK_0^2}{1+K_0^2/2}\frac{1}{A_x^{\max}}\int_{-\infty}^{\infty} dz\cos k_w z\;\delta A_x(z), \tag{3.38}$$

where $A_x^{\max} = mcK_0/e$ is the maximum wiggler vector potential magnitude without wiggler errors. An alternate form that shows the explicit dependence of G on individual magnet block errors ϵ_j is

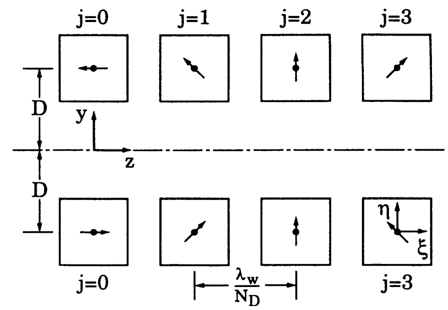

FIG. 1. Idealized wiggler (one half wiggler period for $N_D = 8$).

$$G = \sum_j \epsilon_j G_j\,,$$

$$G_j = -\frac{2k_wK_0^2}{1+K_0^2/2}\frac{1}{A_x^{\max}}\int_{-\infty}^{\infty} dz\cos k_w z\;\delta A_x^j(z), \tag{3.39}$$

where $\delta A_x^j(z)$ is error in the vector potential per unit error in the jth magnet block. Equations (3.38) and (3.39) are now free from any ambiguity with regard to separating wiggler errors into phase and amplitude components. They can be used directly with specific wiggler errors to determine their effect on the FEL gain.

IV. WIGGLER FIELD ERRORS

Errors in the wiggler field can occur because of errors in block magnetization magnitude and direction, as well as errors in block size, shape, position, and orientation. Since geometrical parameters can usually be held to much tighter tolerances than magnetization errors, we assume only magnetization errors (magnitude and direction) in the present calculation. We further assume that we have only a two-dimensional problem, that is, that the blocks are uniform and infinite in the x direction.

In this error calculation, we assume that the wiggler without errors is a periodic array of line dipoles, as shown in Fig. 1, located in two rows at

$$y_j = \pm D,\quad z_j = j\lambda_w/N_D, \tag{4.1}$$

where there are N_D dipoles per wiggler period in each row. The components of magnetization are

$$y_j = D, M_y^j = M_0\sin 2\pi j/N_D,\quad M_z^j = -M_0\cos 2\pi j/N_D, \tag{4.2}$$

$$y_j = -D, M_y^j = M_0\sin 2\pi j/N_D,\quad M_z^j = M_0\cos 2\pi j/N_D, \tag{4.3}$$

and the vector potential at $y=0$ is

$$A_x(z) = \frac{M_0}{\pi}\sum_{j=-\infty}^{\infty}\frac{D\,\cos(2\pi j/N_D)-(z-z_j)\sin(2\pi j/N_D)}{(z-z_j)^2+D^2}. \tag{4.4}$$

It is clear that $A_x(z)$ is a periodic function of z with period λ_w. Its Fourier coefficients are defined by

$$A_x(z) = \sum_{n=-\infty}^{\infty} A_n e^{-2in\pi z/\lambda_w}, \tag{4.5}$$

where

$$A_n = \frac{M_0}{\pi\lambda_w} \sum_j \int_0^{\lambda_w} dz\; e^{2in\pi z/\lambda_w} \frac{D\cos(2\pi j/N_D) - (z-z_j)\sin(2\pi j/N_D)}{(z-z_j)^2 + D^2}. \tag{4.6}$$

If we let

$$z = z_j + s\ , \tag{4.7}$$

we can write

$$A_n = \frac{M_0}{\pi\lambda_w} \sum_{j=-\infty}^{\infty} e^{2inj\pi/N_D} \int_{-j\lambda_w/N_D}^{\lambda_w - j\lambda_w/N_D} ds\, e^{2in\pi s/\lambda_w} \frac{D\cos(2\pi j/N_D) - s\sin(2\pi j/N_D)}{s^2 + D^2}. \tag{4.8}$$

We now replace the single sum over j by the double sum over p,q where

$$j = pN_D + q, \quad -\infty < p < \infty, \quad 1 \le q \le N_D. \tag{4.9}$$

This leads to

$$A_n = \frac{M_0}{2\pi\lambda_w} \sum_{q=1}^{N_D} e^{2inq\pi/N_D} \sum_{p=-\infty}^{\infty} \int_{-p\lambda_w - q\lambda_w/N_D}^{\lambda_w - p\lambda_w - q\lambda_w/N_D} ds e^{2in\pi s/\lambda_w} \left(\frac{e^{2\pi iq/N_D}}{D - is} + \frac{e^{-2\pi iq/N_D}}{D + is} \right). \tag{4.10}$$

The sum over p now permits us to extend the integral over s from $-\infty$ to ∞, and we find

$$A_n = \frac{M_0}{2\pi\lambda_w} \sum_{q=1}^{N_D} e^{2in\pi q/N_D} \int_{-\infty}^{\infty} ds e^{2in\pi s/\lambda_w} \left(\frac{e^{2\pi iq/N_D}}{D - is} + \frac{e^{-2\pi iq/N_D}}{D + is} \right). \tag{4.11}$$

It is clear that the only surviving harmonics are the ones where $n = \pm 1, \pm 1 \pm N_D, \pm 1 \pm 2N_D, \ldots$. We will only evaluate the $n = \pm 1$ coefficients, which are

$$A_{\pm 1} = \frac{M_0 N_D}{2\pi\lambda_w} \int_{-\infty}^{\infty} \frac{ds e^{\pm 2i\pi s/\lambda_w}}{D \pm is} = \frac{M_0 N_D}{\lambda_w} e^{-2\pi D/\lambda_w}. \tag{4.12}$$

The main wiggler potential (first harmonic) is therefore

$$A_x^{(1)}(z) = \frac{2M_0 N_D}{\lambda_w} e^{-2\pi D/\lambda_w} \cos(2\pi z/\lambda_w). \tag{4.13}$$

If we have a block (line dipole) with magnetization error components δM_y^j and δM_z^j, according to Eq. (4.4) it will contribute an error in the vector potential which is

$$\delta A_x^j(z) = -\frac{1}{2\pi} \frac{[\pm D\delta M_z^j + (z - z_j)\delta M_y^j]}{(z-z_j)^2 + D^2}, \tag{4.14}$$

where the $\pm$ depends on whether the block is located at $y = \pm D$. Thus the parameter G in Eq. (3.38) is

$$\begin{aligned} G &= \frac{k_w K_0^2}{\pi(1 + K_0^2/2)} \frac{\lambda_w}{2M_0 N_D} e^{2\pi D/\lambda_w} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} dz \cos k_w z \left(\frac{\pm D\delta M_z^j + (z - z_j)\delta M_y^j}{(z-z_j)^2 + D^2} \right) \\ &= \frac{K_0^2 e^{2\pi D/\lambda_w}}{M_0 N_D (1 + K_0^2/2)} \left(\mathrm{Re} \sum_{j=-\infty}^{\infty} e^{ik_w z_j} \int_{-\infty}^{\infty} ds e^{ik_w s} \frac{(\pm D\delta M_z^j + s\delta M_y^j)}{D^2 + s^2} \right) \\ &= \frac{\pi K_0^2}{N_D(1 + K_0^2/2)} \sum_{j=-\infty}^{\infty} \left(\pm \frac{\delta M_z^j}{M_0} \cos k_w z_j - \frac{\delta M_y^j}{M_0} \sin k_w z_j \right) = -\frac{\pi K_0^2}{N_D(1 + K_0^2/2)} \sum_{j=-\infty}^{\infty} \frac{\delta \mathbf{M}^j \cdot \mathbf{M}^j}{M_0^2}, \end{aligned} \tag{4.15}$$

where the components of $\mathbf{M}^j$ are given in Eqs. (4.2) and (4.3) and where the sum over j extends over both rows of blocks (dipoles).

It is clear from the last form for G in Eq. (4.15) that only the component of $\delta\mathbf{M}$ in the direction of magnetization contributes to the reduction in gain. This is not surprising since Eq. (3.38) suggests that only the in-phase component of δA_x enters. Errors in direction of magnetization therefore only change the out-of-phase component of δA_x, which does not change the gain in lowest order.

For a wiggler consisting of identically shaped blocks of finite size, the first-harmonic amplitude in Eq. (4.12) is replaced by

$$A_{\pm 1} = \frac{\tilde{M} N_D}{\lambda_w} \int\int d\xi \, d\eta \, e^{-2\pi(D+\eta)/\lambda_w}, \tag{4.16}$$

where $\tilde{M}$ is the magnetization per unit area. The integral extends over the entire cross section of a block, where the origin of the ξ, η coordinate system is at the centroid of the block.

A similar averaging is needed in the calculation of the integral over $\delta A_x(z)$ in Eq. (3.38), since the factor $\cos k_w z$ in the integrand extracts only the first harmonic of the error. Thus both the numerator $\int_{-\infty}^{\infty} dz \cos\, k_w z \delta A_x(z)$ and the denominator $A_x^{\max}$ contain the form factor

$$\frac{1}{\mathcal{A}} \int\int_{\mathcal{B}} d\xi d\eta \, e^{-2\pi\eta/\lambda_w} \tag{4.17}$$

which cancels in calculating G and the growth rate. As a consequence, the final result for the change in the growth rate due to errors in the strength and direction of the magnetization is *independent* of the shape of the (two-dimensional) block. This exact result is only an approximate result for errors in block position since the form factor for such errors corresponds to a somewhat different weighting in Eq. (4.17).

V. REDUCTION IN GAIN

The reduction in gain in Eq. (3.36) can now be evaluated for a collection of magnetization strength errors

$$\epsilon_j = \frac{\delta\mathbf{M}^j \cdot \mathbf{M}^j}{M_0^2}. \tag{5.1}$$

Note that only the component of ϵ_j parallel to the magnetization axis enters.

We return to Eqs. (3.30), (3.33), (3.39), and (4.15) to write

$$f(\tau) = \sum_j \epsilon_j g(\tau - \tau_j) \quad , \quad G_j = \int_0^T g(\tau - \tau_j) d\tau, \tag{5.2}$$

$$G_j = -\frac{\pi K_0^2}{N_D(1 + K_0^2/2)} \equiv H, \tag{5.3}$$

and

$$g(\tau - \tau_j) = -\frac{2K_0^2}{(1 + K_0^2/2)} \frac{1}{2\rho A_x^{\max}} \cos(k_w z) \delta A_x^j(z). \tag{5.4}$$

Because δA_x^j is appreciable only near $\tau = \tau_j$, with $\tau = 2\rho k_w z$, we can approximate $g(\tau - \tau_j)$ by a δ function in evaluating any integrals like those in Eq. (3.28) in which the factors in the parentheses vary appreciably only over a gain length. Thus we write

$$f(\tau) = \sum_j \epsilon_j g(\tau - \tau_j) \cong \sum_j \epsilon_j H \delta(\tau - \tau_j). \tag{5.5}$$

We therefore obtain from Eq. (3.28)

$$\langle \Delta r_g \rangle = -\frac{H^2}{9T} \left(\sum_j \epsilon_j^2 + \sum_{j>k}\sum \epsilon_j \epsilon_k \mathrm{Re}(e^{i(\omega-\omega*)(\tau_j-\tau_k)} + e^{i(\omega-1)(\tau_j-\tau_k)}) \right), \tag{5.6}$$

where we have separated out the diagonal term and taken only half of it in order to properly take into account the requirement that $\tau \geq \tau_j$ in the region of integration.

For an uncorrelated set of errors whose rms value is

$$\overline{\epsilon_j \epsilon_k} = \overline{\epsilon_\parallel^2} \delta_{jk} \ , \tag{5.7}$$

we find for the change in gain for a statistical ensemble

$$\overline{\langle \Delta r_g \rangle} = -\frac{H^2 \overline{\epsilon_\parallel^2} 2N_D N_P}{9T} = -\frac{\pi^2 K_0^4 2N_D N_P \overline{\epsilon_\parallel^2}}{9TN_D^2(1 + K_0^2/2)^2}, \tag{5.8}$$

where N_P is the number of wiggler periods in the FEL, with $2N_D$ magnets per wiggler period. Since $T = 2\rho 2\pi N_P$, we finally have

$$\overline{\langle \Delta r_g \rangle} = -\frac{\pi}{9N_D} \frac{K_0^4}{(1 + K_0^2/2)^2} \frac{\overline{\epsilon_\parallel^2}}{2\rho}. \tag{5.9}$$

If the blocks within a distance small compared to a gain length but comparable with a wiggler wave length are correlated to reduce wiggler error, this correlation should be used in evaluating the double sum in Eq. (5.6). Clearly this has the potential to lower any reduction in gain.

It is also possible to calculate the width of the distribution in gain reduction. Specifically we calculate

$$\overline{[\langle \Delta r_g \rangle]^2} = \frac{H^4}{81T^2} \left[\overline{\left(\sum_j \epsilon_j^2\right)\left(\sum_\ell \epsilon_\ell^2\right)} + \overline{\sum_{j>k}\sum \epsilon_j \epsilon_k \sum_{\ell>m}\sum \epsilon_\ell \epsilon_m Q_{jk} Q_{\ell m}} \right], \tag{5.10}$$

where

$$Q_{jk} = \mathrm{Re}(e^{i(\omega-\omega*)(\tau_j-\tau_k)} + e^{i(\omega-1)(\tau_j-\tau_k)}). \tag{5.11}$$

For uncorrelated errors we have, with $2N_D N_P \gg 1$,

$$\overline{\sum_j \epsilon_j^2 \sum_\ell \epsilon_\ell^2} \cong 4N_D^2 N_P^2 (\overline{\epsilon_\parallel^2})^2 \tag{5.12}$$

and

$$\overline{\sum\sum_{j>k} \epsilon_j\epsilon_k \sum\sum_{\ell>m} \epsilon_\ell\epsilon_m Q_{jk}Q_{\ell m}} = (\epsilon_\parallel^2)^2 \sum\sum_{j>k} Q_{jk}^2. \tag{5.13}$$

Converting the sum over j and k in Eq. (5.13) to an integral, we eventually obtain

$$\frac{\overline{(\langle\Delta r_g\rangle - \overline{\langle\Delta r_g\rangle})^2}}{(\overline{\langle\Delta r_g\rangle})^2} = \frac{17\sqrt{3}}{24T} - \frac{49}{144T^2}. \tag{5.14}$$

Thus the relative width of the distribution in growth-rate reduction is of the order of 1 over the square root of the number of gain lengths in the FEL.

At this point we wish to emphasize that we have calculated that portion of the gain reduction and distribution width which is proportional to T. This is the case because the integrals over the error function $f(\tau)$ have been extended from $-\infty$ to ∞ where appropriate, consistent with our assumption following Eq. (3.16). This model will be reexamined in Sec. VI.

We have checked the results for $\overline{\langle\Delta r_g\rangle}$ given in Eq. (5.9) by direct numerical integration of Eq. (3.1) using $f(\tau)$ as obtained from Eqs. (5.2), (5.4), and (4.14) with 4000 different sets of random errors and obtain close agreement. Specifically, with 4000 different sets of random errors, we find the following: (i) $\overline{\langle\Delta r_g\rangle}$ is approximately the same for 25 magnet block errors and for 250 magnet block errors. Except as noted, all simulations have been made with approximately 25 magnet block errors to save computer time. (ii) $\overline{\langle\Delta r_g\rangle}$ agrees with Eq. (5.9) to 7%. (iii) $\overline{\langle\Delta r_g\rangle}$ is proportional to $\overline{\epsilon_\parallel^2}$ to better than 2%. (iv) $\overline{\langle\Delta r_g\rangle}$ is independent of $\overline{\epsilon_\perp^2}$ (for 250 magnet block errors). (v) $\overline{\langle\Delta r_g\rangle}$ is independent of $k_w D$ to better than 10%. (vi) $\overline{\langle\Delta r_g\rangle}$ is inversely proportional to $2\rho N_D$. (vii) $\overline{\langle\Delta r_g\rangle}$ is proportional to

$$\left(\frac{K_0^2}{(1+K_0^2/2)}\right)^2$$

to better than 1%. (viii) A sample distribution in the reduction in growth rate is shown in Fig. 2. The ratio of width to average value agrees with Eq. (5.14) to about 10%.

VI. SOLUTION BASED ON INTEGRAL EQUATION

Some additional insight can be gained by reformulating Eq. (3.1) as an integral equation. We shall consider the iterative solution of this integral equation and compare the results to a computer simulation. Recall that the effect of wiggler field errors on the FEL gain is described

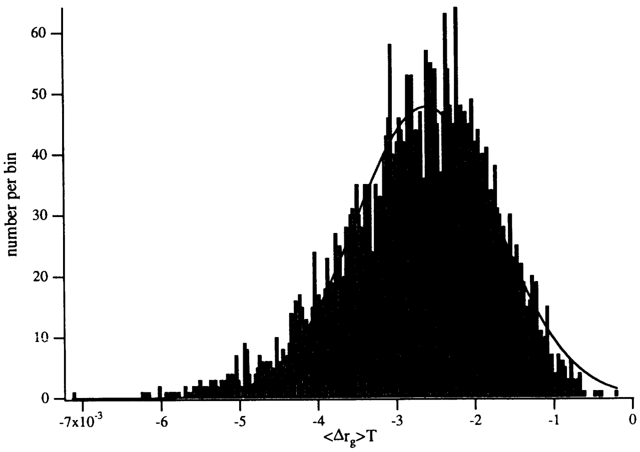

FIG. 2. Distribution in the reduction in gain for a typical simulation with 4000 random error seeds in $\epsilon_\parallel^j$ for 25 magnet blocks. The curve is a fit to a Gaussian. The average value of $\overline{\langle\Delta r_g\rangle T}$ is -2.72×10^{-3} and the width $= T(\overline{\langle\Delta r_g\rangle^2} - \overline{\langle\Delta r_g\rangle}^2)^{1/2}$ is 9.30×10^{-4}.

by the differential equation

$$\frac{d^3y}{d\tau^3} - iy = i\frac{d^2(fy)}{d\tau^2}. \tag{6.1}$$

We consider $f(\tau)$ to have the form given in Eq. (3.30),

$$f(\tau) = \sum_{j=1}^{N_T} \epsilon_j g(\tau-\tau_j), \tag{6.2}$$

and we assume that $g(u)$ has appreciable value only for $|u|$ less than a few wiggler periods, a distance short compared to a gain length. This local nature of $g(u)$ corresponds to neglecting the effect of steering errors, which are treated in Sec. VII.

Denoting the unperturbed solution (that for $f=0$) by $y_0(\tau)$, we recast Eq. (6.1) as the integral equation,

$$y(\tau) = y_0(\tau) + \frac{i}{3}\int_0^\tau d\tau' \tilde{\Phi}(\tau-\tau') f(\tau') y(\tau'), \tag{6.3}$$

where the Green function $\tilde{\Phi}(\tau)$ is

$$\tilde{\Phi}(\tau) = e^{-i\omega\tau}\Phi(\tau)\ , \ \Phi(\tau) = 1 + e^{i(\omega-\omega^*)\tau} + e^{i(\omega-1)\tau}. \tag{6.4}$$

We solve Eq. (6.3) to second order in f by iterating the kernel (in scattering theory this is called the Born approximation). We wish to study the reduction of the growth rate of the exponentially growing mode. Assuming no initial energy or spatial modulation of the electron beam and normalization $y_0(0) = 1$, we can take $y_0(\tau) = \exp(-i\omega\tau)\Phi(\tau)/3$, and write

$$y(\tau) = \frac{e^{-i\omega\tau}}{3}\left(\Phi(\tau) + \frac{i}{3}F_1(\tau) - \frac{1}{9}F_2(\tau)\right), \tag{6.5}$$

where we have defined

$$F_1(\tau) = \int_0^\tau d\tau' \Phi(\tau - \tau') f(\tau') \Phi(\tau'), \tag{6.6}$$

$$F_2(\tau) = \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \Phi(\tau - \tau') \Phi(\tau' - \tau'') \times f(\tau') f(\tau'') \Phi(\tau''). \tag{6.7}$$

At the end of the wiggler ($\tau = T = 4\pi\rho N_P$, N_P is number of wiggler periods), we shall calculate

$$\overline{|y|^2} = e^{\sqrt{3}T}\left(1 + \frac{i}{3}(\overline{F_1 - F_1^*}) - \frac{1}{9}(\overline{F_2 + F_2^*}) + \frac{1}{9}(\overline{F_1 F_1^*})\right), \tag{6.8}$$

$$\overline{|y|^4} = e^{2\sqrt{3}T}\left(1 + \frac{2i}{3}(\overline{F_1 - F_1^*}) - \frac{2}{9}(\overline{F_2 + F_2^*}) + \frac{4}{9}(\overline{F_1 F_1^*}) - \frac{1}{9}(\overline{F_1^2 + F_1^{*2}})\right), \tag{6.9}$$

where the average is over a stochastic ensemble of wiggler errors. We assume the correlation functions

$$\overline{f(\tau)} = 0, \tag{6.10}$$

$$\overline{f(\tau_1) f(\tau_2)} = w(\tau_1 - \tau_2), \tag{6.11}$$

where $w(v) = w(-v)$ is nonvanishing only over a distance short compared to a gain length. We then obtain

$$\overline{F_1} = 0, \tag{6.12}$$

$$\overline{F_2} = \tfrac{3}{2}(T + 2i\omega^*)W, \tag{6.13}$$

$$\overline{F_1^2} = (T + \tfrac{19}{3} i\omega^*)W, \tag{6.14}$$

$$\overline{F_1 F_1^*} = (T + \tfrac{11}{3}\sqrt{3})W, \tag{6.15}$$

with W defined by

$$W = \int_{-T}^{T} dv\, w(v) = \frac{1}{2T}\int_{-T}^{T} dv \int_0^{2T} du\, \overline{f\left(\frac{u+v}{2}\right) f\left(\frac{u-v}{2}\right)} = \frac{1}{T}\overline{\left(\int_0^T d\tau\, f(\tau)\right)^2}. \tag{6.16}$$

Recalling from Eq. (2.28) that $f = d\delta/d\tau$, we see that W is the mean-square average of the ponderomotive phase shift per gain length as given in Eq. (1.2) of the Introduction. Employing Eqs. (6.7), (6.8), and (6.10), we determine

$$\overline{|y|^2} = \frac{e^{\sqrt{3}T}}{9}\left(1 - \frac{2W}{9}T + \frac{2\sqrt{3}}{27}W\right), \tag{6.17}$$

$$\overline{|y|^4} = \frac{e^{2\sqrt{3}T}}{81}\left(1 - \frac{4W}{9}T + \frac{7\sqrt{3}}{27}W\right), \tag{6.18}$$

$$(\overline{\delta|y|^2}) = [\overline{|y|^4} - \overline{|y|^2}^2]^{1/2} = \frac{1}{3}\left(\overline{|F_1 - F_1^*|^2}\right)^{1/2} = \frac{e^{\sqrt{3}T}}{9}\left(\frac{\sqrt{3}}{9}W\right)^{1/2}. \tag{6.19}$$

In this manner we arrive at the estimate

$$\overline{|y|^2} = \frac{e^{\sqrt{3}T}}{9}\left[1 - \frac{2}{9}WT + \frac{2\sqrt{3}}{27}W \pm \left(\frac{\sqrt{3}}{9}W\right)^{1/2}\right]. \tag{6.20}$$

Let us suppose that $\overline{|y|^2}$ can be expressed in the form

$$\overline{|y|^2} = \frac{1}{9}\exp\{[\sqrt{3} + 2\overline{\Delta r_g} \pm 2\delta(\Delta r_g)]T + c \pm \delta c\}. \tag{6.21}$$

By taking the logarithm of Eq. (6.20) and keeping terms to second order in f, we find

$$c = \frac{2W\sqrt{3}}{27}, \tag{6.22}$$

$$\delta c = \left(\frac{W\sqrt{3}}{9}\right)^{1/2}, \tag{6.23}$$

$$\overline{\Delta r_g} = -\frac{W}{9}. \tag{6.24}$$

The effect of wiggler errors on the coefficient of the exponential growth factor is given by $\exp(c \pm \delta c)$. Note that for small W, the width δc is large compared to the average value c. The development of the theory given in Secs. III–V purposefully ignored the τ-independent term $c \pm \delta c$, which arises from integrals dominated by the two ends of the wiggler which contribute equally. On the other hand, the treatment given here based on iterating the integral equation does not yield the spread $\delta(\Delta r_g)$ in the growth rate. To determine the width it would be necessary to keep terms up to fourth order in f when iterating the integral equation. From Eq. (5.14) we know that

$$\delta(\Delta r_g) = \overline{\Delta r_g}\left(\frac{17\sqrt{3}}{24T} - \frac{49}{144T^2}\right)^{1/2}. \tag{6.25}$$

We consider $f(\tau)$ to have the form given in Eq. (6.2), and employ $\overline{\epsilon_j \epsilon_k} = \overline{\epsilon^2}\delta_{jk}$ and $H = \int_0^T d\tau\, g(\tau - \tau_j)$ in Eq. (6.16) to derive

$$W = \frac{H^2}{T}\overline{\epsilon^2} N_T. \tag{6.26}$$

For the permanent magnet wiggler considered in Secs. IV and V, the total number of blocks $N_T = 2N_D N_P$, where $2N_D$ is the number of blocks per period and N_P the number of periods. In this case, $H = \pi K^2/[N_D(1 + K_0^2/2)]$ and

$$W = \frac{\pi}{N_D} \frac{K_0^4}{(1 + K_0^2/2)^2} \frac{\overline{\epsilon_\parallel^2}}{2\rho}. \tag{6.27}$$

The average reduction of gain $\overline{\Delta r_g}$ given in Eq. (6.24) is seen to agree with Eq. (5.9).

We also consider a model consisting of magnetic amplitude errors correlated within each period to yield zero net deflection, with errors in different periods being independent of each other. Although this model is not strictly physically realizable, it is convenient for numerical simulation. The model is specified by the error in the vector potential

$$\delta A_x(z) = \sum_{j=1}^{N_P} \frac{\delta B_j(z)}{k_w}(1 - \cos k_w z), \tag{6.28}$$

$$\delta B_j(z) = \begin{cases} \Delta B_j, & j\lambda_w \le z \le (j+1)\lambda_w \\ 0 & \text{otherwise}\ . \end{cases} \tag{6.29}$$

Here ΔB_j are random variables with

$$\overline{\Delta B_j \Delta B_k} = \delta_{jk}(\Delta B)^2_{\text{rms}}. \tag{6.30}$$

In this case, $N_T = N_P$ and $H = 2\pi K_0^2/(1 + K_0^2/2)$, and we find

$$W = \frac{\pi K_0^4}{(1 + K_0^2/2)^2} \frac{(\Delta B/B_{\text{max}})^2_{\text{rms}}}{\rho}, \tag{6.31}$$

where B_{max} is the peak value of the ideal wiggler field.

We have carried out a computer simulation of the model of wiggler errors given in Eqs. (6.28)–(6.30) using the computer program TDA [4, 13]. The transverse electron beam size is taken to be large enough so that the radiation field on axis evolves as in the one-dimensional limit. In particular, we considered the parameters $\rho = 1.29 \times 10^{-3}, K_0 = 1.95$, and $(\Delta B/B_{\text{max}})_{\text{rms}} = 2\%$. From Eq. (6.31) it follows that $W = 1.67$ and hence the results of our analytic calculation [Eqs. (6.22)–(6.25)] are

$$2\overline{\Delta r_g} = -0.37, \quad 2(\delta \Delta r_g) = 0.41/\sqrt{T}\ ,$$

$$c = 0.11, \quad \delta c = 0.40.$$

The computer simulation results are presented in Fig. 3. The ratio of the output to input power $|y|^2/|y_0|^2$ is shown as a function of distance for 16 sets of random wiggler errors, and good agreement is found with the analytic estimate:

$$\frac{\overline{|y|^2}}{|y_0|^2} = \frac{1}{9} e^{c \pm \delta c} e^{[\sqrt{3} + 2\overline{\Delta r_g} \pm 2\delta(\Delta r_g)]\tau}. \tag{6.32}$$

In particular the analytic estimate of the average reduction in gain $2\overline{\Delta r_g}$ agrees with the numerical results to a few percent. As mentioned earlier, the spread in the exponent consists of two contributions. One is the spread in the constant $\pm\delta c$ and the other is the spread in the slope $\pm 2\delta(\Delta r_g)$. The simulation is consistent with the width being dominated by the spread in the constant $\pm\delta c$ as predicted by the analytical theory taking into account the statistics of a limited number (16) of samples.

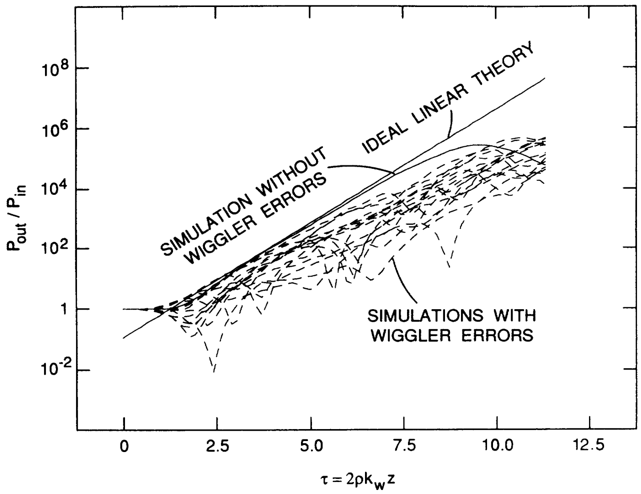

FIG. 3. The ratio $P_{\text{out}}/P_{\text{in}}$ of the output to input power is shown as a function of the scaled axial coordinate $\tau = 2\rho k_w z$, for 16 sets of random wiggler errors.

VII. EFFECT OF STEERING ERRORS

In Sec. IV we used a 2D wiggler model to calculate the error in the vector potential. According to Eq. (4.14), each block magnetization error makes a contribution to $\delta A_x(z)$ which is localized around the axial location of the block. Since conservation of the x component of the canonical momentum leads to

$$x'(z) = \frac{eA_x(z)}{\gamma mc}, \tag{7.1}$$

our 2D model leads to no net change in x' as the beam passes by the block with the error. As a consequence, we are led to the result in Eq. (3.32) for the growth-rate reduction because of the absence of long-range correlations in the terms which make up $f(\tau)$ and $f(\tau')$ in Eq. (3.28).

The situation in three dimensions is fundamentally different. Each magnet block is capable of deflecting the beam through a nonvanishing angle. As a result, the angle of the beam undergoes a random-walk process and there is now long-range correlation between $f(\tau)$ and $f(\tau')$ which cannot be neglected.

We shall first estimate the reduction in growth rate that occurs due to these uncorrelated angular deflections by assuming that $x'(z)$ changes abruptly by θ_j as the beam passes the jth magnet error. We then explore the result of introducing a correction scheme to control the buildup of the transverse angle and displacement.

Using Eqs. (3.29), (2.17), and (3.37), we can write

$$2\rho f(\tau) = -\frac{1}{1+K_0^2/2}\{[\gamma x'(z)]^2 + 2\gamma x'(z)K_0\cos k_w z\}, \tag{7.2}$$

where $\tau = 2\rho k_w z$. The rapidly oscillating factor makes the long-range correlation in the second term in the curly brackets unimportant. Assuming steps of θ_j in $x'(z)$ as we pass each magnet error, we have

$$f(\tau_n) = \alpha[x'(z)]^2 = \alpha\left(\sum_1^n \theta_j\right)^2 \tag{7.3}$$

where

$$\tau_n = 2\rho k_w z_n = nT/N_T,\ N_T = 2N_D N_P \tag{7.4}$$

and

$$\alpha = -\frac{\gamma^2}{2\rho(1+K_0^2/2)}. \tag{7.5}$$

We now evaluate the double integral in Eq. (3.28). Specifically

$$J_\lambda \equiv \frac{1}{T}\int_0^T d\tau\, f(\tau)\int_0^\tau d\tau' f(\tau')e^{i\lambda(\tau-\tau')} = \frac{T\alpha^2}{N_T^2}\int_0^{N_T} dn \sum_{j=1}^n \theta_j \sum_{k=1}^n \theta_k \int_0^n dn' \sum_{\ell=1}^{n'}\theta_\ell \sum_{m=1}^{n'}\theta_m e^{i\overline{\lambda}(n-n')}\ , \tag{7.6}$$

where $\overline{\lambda} = \lambda T/N_T$, and where we treat n and n' as continuous variables where appropriate. The most important contributions to the ensemble average are obtained by neglecting the $j=k=\ell=m$ terms and writing

$$\overline{\theta_j\theta_k\theta_\ell\theta_m} \cong \overline{\theta^2}^2(\delta_{jk}\delta_{\ell m} + \delta_{j\ell}\delta_{km} + \delta_{jm}\delta_{k\ell}), \tag{7.7}$$

leading to

$$\overline{J_\lambda} \cong \frac{T\alpha^2}{N_T^2}\overline{\theta^2}^2\int_0^{N_T} dn\int_0^n dn' e^{i\overline{\lambda}(n-n')}(nn'+2n'^2). \tag{7.8}$$

For $-\mathrm{Re}(i\overline{\lambda}N_T) = -\mathrm{Re}(i\lambda T) \gg 1$, the main contribution to the double integral comes from $n' \cong n$, and we find

$$\overline{J_\lambda} = \frac{N_T T\alpha^2\overline{\theta^2}^2}{-i\overline{\lambda}} = \frac{N_T^2\alpha^2\overline{\theta^2}^2}{-i\lambda}. \tag{7.9}$$

The reduction in average growth rate corresponding to Eq. (7.9) with $\lambda = \omega - \omega*$, and $\omega - 1$ in Eq. (3.28) for this quadratic term in $\delta A_x(z)$ is

$$<\overline{\Delta r_g^Q}> \cong -\frac{\sqrt{3}}{18}\left(\frac{\gamma^2 N_T\overline{\theta^2}}{2\rho(1+K_0^2/2)}\right)^2. \tag{7.10}$$

Notice that since $<\overline{\Delta r_g^Q}>$ is now quadratic in T, the integration of Eq (3.28) with respect to T gives an average power growth which is proportional to $\exp(\sqrt{3}+2\Delta r_g/3)\tau$ instead of $\exp(\sqrt{3}+2\Delta r_g)\tau$, as given by Eq (6.32).

If we now assume a correction scheme which returns $x(z)$ to zero in a distance short compared to the gain length, we can show that $x'(z)$ in each correction region is uncorrelated with $x'(z)$ in a different correction region. Assuming no focusing, the transverse displacement and angle at the end of an as yet uncorrected region having N_C magnets are given by

$$x(L) = x_0'L + \sum_{j=1}^{N_C}(L-z_j)\theta_j, \tag{7.11}$$

$$x'(L) = x_0' + \sum_{j=1}^{N_C}\theta_j. \tag{7.12}$$

We now arrange for a transverse impulse correction $\Delta x'$ at $z=0$ chosen to make $x(L)=0$. In this case the corrected values are

$$\tilde{x}(L) = (x_0'+\Delta x')L + \sum_{j=1}^{N_C}(L-z_j)\theta_j = 0\ , \tag{7.13}$$

$$x_0' + \Delta x' = -\sum_{j=1}^{N_C}\frac{L-z_j}{L}\theta_j\ , \tag{7.14}$$

and

$$\tilde{x}'(z_n) = x_0' + \Delta x' + \sum_{j=1}^n\theta_j = \sum_{j=1}^n\theta_j - \sum_{j=1}^{N_C}\frac{L-z_j}{L}\theta_j \equiv \sum_{j=1}^{N_C}\theta_j p_j(n)\ , \tag{7.15}$$

where

$$p_j(n) = \begin{cases} \frac{j}{N_C} & \text{for } 1 \le j \le n \\ -\frac{N_C-j}{N_C} & \text{for } n < j \le N_C. \end{cases} \tag{7.16}$$

Thus the angle in each correction region depends only on the errors in that region, which is assumed to be short compared with the gain length.

In the calculation of $\overline{J_\lambda}$ in Eq. (7.6) we must now obtain the value of

$$\overline{\tilde{f}(\tau_n)\tilde{f}(\tau_n')} = \alpha^2\overline{\tilde{x}_n'^2\tilde{x}_{n'}'^2} = \alpha^2\sum_j\sum_k\sum_\ell\sum_m\overline{\theta_j\theta_k\theta_\ell\theta_m}p_j(n)p_k(n)p_\ell(n')p_m(n') \tag{7.17}$$

using Eq. (7.15). The $\delta_{jk}\delta_{\ell m}$ term in Eq. (7.7) corresponds to multiplying $\overline{x_n'^2}$ by $\overline{x_{n'}'^2}$. The additional terms from

$\delta_{j\ell}\delta_{km}$ and $\delta_{jm}\delta_{k\ell}$ (which are equal) exist only when there are correlations between the errors for n and n', that is, only in the same correction region where we can set $\exp[i\overline{\lambda}(n-n')]=1$. Thus we have

$$\overline{\tilde{f}(\tau_n)\tilde{f}(\tau_n')} = \alpha^2\overline{\tilde{x}_n'^2}\,\overline{\tilde{x}_{n'}'^2} + 2\alpha^2\overline{\theta^2}^2\left(\sum_j p_j(n)p_j(n')\right)^2, \tag{7.18}$$

where $\overline{\tilde{x}_n'^2} \equiv \overline{[\tilde{x}(n)]^2}$ is obtained from Eq. (7.15) as

$$\begin{aligned}\overline{[\tilde{x}'(n)]^2} &= \overline{\theta^2}\left(\frac{n^3+(N_C-n)^3}{3N_C^2}\right)\\ &= \overline{\theta^2}\left(\frac{N_C^2-3N_Cn+3n^2}{3N_C}\right) \equiv N_C\overline{\theta^2}q(n).\end{aligned} \tag{7.19}$$

The function $q(n)$ then has the shape shown in Fig. 4 and $\overline{\tilde{J}_\lambda}$ becomes

$$\overline{\tilde{J}_\lambda} = \frac{TN_C^2}{N_T^2}\alpha^2\overline{\theta^2}^2\left[\int_0^{N_T}dn\int_0^n dn'e^{i\overline{\lambda}(n-n')}q(n)q(n') + \frac{2N_T}{N_C^3}\int_0^{N_C}dn\int_0^n dn'\left(\sum_j p_j(n)p_j(n')\right)^2\right], \tag{7.20}$$

where the region of integration for the second term is restricted to a single correction region.

In order to evaluate the first term $\overline{\tilde{J}_\lambda}^{(1)}$ in $\overline{\tilde{J}_\lambda}$, we expand $q(n)$ in a Fourier series

$$q(n) = \sum_{\ell=-\infty}^{\infty} a_\ell e^{-2\pi i\ell n/N_C} \tag{7.21}$$

with

$$a_0 = \frac{1}{6},\quad a_\ell = \frac{1}{2\pi^2\ell^2}\quad (\ell\neq 0). \tag{7.22}$$

Setting $n-n'=n''$, we find

$$\begin{aligned}\overline{\tilde{J}_\lambda}^{(1)} &= \frac{N_C^2T}{N_T^2}\alpha^2\overline{\theta^2}^2\int_0^{N_T}dn\,q(n)\sum_{\ell=-\infty}^{\infty}a_\ell\int_0^n dn''e^{i\overline{\lambda}n''-2\pi i\ell n/N_C+2\pi i\ell n''/N_C}\\ &\cong \frac{N_C^2T}{N_T^2}\alpha^2\overline{\theta^2}^2\sum_{\ell=-\infty}^{\infty}\frac{a_\ell}{-(i\overline{\lambda}+2\pi i\ell/N_C)}\int_0^{N_T}dn\,q(n)e^{-2\pi i\ell n/N_C} \cong \frac{N_C^2T}{N_T}\alpha^2\overline{\theta^2}^2\sum_{\ell=-\infty}^{\infty}\frac{a_\ell^2}{-(i\overline{\lambda}+2\pi i\ell/N_C)}.\end{aligned} \tag{7.23}$$

But

$$\overline{\lambda}N_C = \lambda\tau_c,\quad T/N_T = \tau_c/N_C, \tag{7.24}$$

where $\tau_c \ll 1$ is the interval in τ corresponding to a correction region. Thus

$$\overline{\tilde{J}_\lambda}^{(1)} \cong N_C^2\alpha^2\overline{\theta^2}^2\left(\frac{a_0^2}{-i\lambda}+\tau_c\sum_{\ell\neq 0}\frac{a_\ell^2}{-(i\lambda\tau_c+2\pi i\ell)}\right). \tag{7.25}$$

Combining the terms for $-\ell$ with those for $+\ell$, and recognizing that $|\lambda\tau_c| \ll 1$, we find for the real part

$$Re\overline{\tilde{J}_\lambda}^{(1)} = N_C^2\alpha^2\overline{\theta^2}^2\left[a_0^2\mathrm{Re}\left(\frac{i}{\lambda}\right) - \frac{\tau_c^2}{2\pi^2}\sum_{\ell=1}^{\infty}\frac{a_\ell^2}{\ell^2}\mathrm{Re}(i\lambda)\right]. \tag{7.26}$$

We now use $\lambda=\omega-\omega*$, $\omega-1$, and $\sum_1^\infty \ell^{-6}=\pi^6/945$ and obtain

$$\mathrm{Re}\overline{\tilde{J}_\lambda}^{(1)} = \frac{\sqrt{3}}{108}\left(\frac{N_C\gamma^2\overline{\theta^2}}{2\rho(1+K_0^2/2)}\right)^2\left(1+\frac{\tau_c^2}{70}\right). \tag{7.27}$$

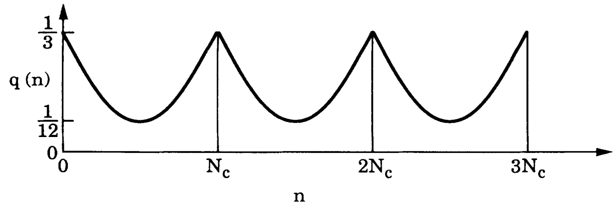

FIG. 4. The function $q(n)$.

The second term in $\overline{\tilde{J}_\lambda}$ can be evaluated directly using the form for $p_j(n)$ in Eq. (7.16). After considerable algebra, the result is

$$\mathrm{Re}\overline{\tilde{J}_\lambda}^{(2)} = \frac{2\tau_c}{105}\left(\frac{N_C\gamma^2\overline{\theta^2}}{2\rho(1+K_0^2/2)}\right)^2. \tag{7.28}$$

Combining Eqs. (7.26) and (7.28), we find for the change in growth rate

$$\overline{<\Delta\tilde{r}_g^Q>} = -\left(\frac{N_C\gamma^2\overline{\theta^2}}{2\rho(1+K_0^2/2)}\right)^2\left[\frac{\sqrt{3}}{972}\left(1+\frac{\tau_c^2}{70}\right)+\frac{2\tau_c}{945}\right]. \quad (7.29)$$

A comparison of Eqs. (7.29) and (7.10) indicates that our correction procedure has reduced the change in growth rate by a factor

$$\frac{\overline{<\Delta\tilde{r}_g^Q>}}{\overline{<\Delta r_g^Q>}} \cong \frac{1}{54}\left(\frac{N_C}{N_T}\right)^2 = \frac{1}{54}\left(\frac{\tau_c}{T}\right)^2. \quad (7.30)$$

It is now a simple matter to explore the consequences of detuning the FEL so as to remove the average detuning corresponding to Fig. 4. In this case the function $q(n)$ is replaced by $q(n)-\overline{q}$, corresponding to dropping the term in a_0. As shown in Eq. (7.29) this reduces the change in growth rate by an additional factor $72\tau_c/35\sqrt{3}$.

After correcting the average detuning, the third term in Eq. (7.29) proportional to $2\tau_c/945$ dominates. This dominant term comes from the second term of Eq. (7.18) and corresponds to correlation within one correction region. If the distance between correctors is smaller than the gain length, then the correlation length is shorter than the gain length, and the analysis of Sec. VI can be applied. In this case, the gain reduction is expressed in terms of the mean-square ponderomotive phase shift per gain length W as $\Delta\tilde{r}_g = -W/9$, with

$$W = \left(\frac{N_C\gamma^2\overline{\theta^2}}{2\rho(1+K_0^2/2)}\right)^2\frac{2\tau_c}{105}. \quad (7.31)$$

Noting that $\tau_c = 4\pi\rho N_S$, where N_S is the number of wiggler periods between correction stations, and defining $N_G = 1/4\pi\rho$, the number of periods in a gain length, we find

$$W \approx \frac{2}{105}N_G N_S N_C^2\left[\frac{2\pi\gamma^2\overline{\theta^2}}{1+K_0^2/2}\right]^2, \quad (7.32)$$

in agreement with Eq. (1.8) presented in the Introduction.

As a final point, we estimate the angular deflection caused by a magnet block of finite length L_x in the x direction symmetrically placed around the beam. This is compared with $\overline{\epsilon_\parallel^2}$ and $\overline{\epsilon_\perp^2}$ for the 2D blocks, where $\epsilon_\parallel^j$ and $\epsilon_\perp^j$ are, respectively, the fractional errors in the magnetization of the jth block parallel and perpendicular to the magnetization axis.

The result for $\overline{\theta^2}$ is given by

$$\gamma^2\overline{\theta^2} = \frac{8K_0^2D^2}{L_x^2+D^2}\frac{e^{2k_wD}}{k_w^2D^2}(\overline{\epsilon_\parallel^2}+\overline{\epsilon_\perp^2}). \quad (7.33)$$

For completeness we include the relation between rms field error and $\overline{\epsilon^2}$. Specifically it is

$$\overline{\left(\frac{\delta B}{B}\right)^2} = \frac{1}{16N_D}\frac{e^{2k_wD}}{k_w^2D^2}(\overline{\epsilon_\parallel^2}+\overline{\epsilon_\perp^2}). \quad (7.34)$$

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