Patterns produced by a short-wave instability in the presence of a zero mode

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One-dimensional pattern-forming systems with a quadratic nonlinearity and a long-wave zero mode in the governing equations are considered. The pattern-generating instability, which may be both steady and oscillatory, is assumed to set in at a finite wave number. Physical examples are the instability of a front of the laser-sustained evaporation of a solid and the instability of seismic waves in a viscoelastic medium. A system of generalized Ginzburg-Landau equations for the complex envelope of a fundamental harmonic and for the real slowly relaxing zero mode are derived in a general form [for the case of the steady instability, essentially the same equations have been proposed earlier by Coullet and Fauve, Phys. Rev. Lett. **55**, 2857 (1985)]. Using these equations, stability of spatially periodic patterns is investigated. The main result is that the stability band is anomalously narrow in comparison with the classical Eckhaus band. For the case of the steady instability, the evolution of long-wave modulations of the patterns is investigated in the geometric-optics approximation. It is demonstrated that, unlike the usual Ginzburg-Landau equation, the equations derived in the present work allow for modulation profiles of a permanent shape. The profiles may be transient layers moving at a constant velocity, or quiescent periodic "superstructures," including a "soliton" as a limiting case. At least some of those profiles can be stable.

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I. INTRODUCTION

The present work is devoted to an extension of the classical Eckhaus analysis [1,2] of stability of one-dimensional patterns produced by the short-wave steady (nonoscillatory) instability to a special case when a corresponding physical system possesses a zero mode (i.e., the damping rate of infinitesimal perturbation vanishes at the zero wave number), and, additionally, an effective nonlinear governing equation contains quadratic terms. As is well known, in the standard situation, when there is no zero mode, the quadratic nonlinearity gives rise to the slaved second and zeroth harmonics, and interaction of those harmonics with the fundamental (first) instability-generating one leads to stabilization of a pattern at a finite (but small) amplitude. In the situation analyzed in the present work, the peculiarity is that the slowly relaxing zero harmonic cannot be slaved. This situation is generic for instabilities of plane fronts of phase transitions (provided the instability is short wave), where the existence of the zero mode is stipulated by the translational invariance. A particular example is furnished by the instability of the plane front of laser-sustained evaporation of a condensed matter [3]. It is important that an effective evolution equation for this instability contains a quadratic term, which reflects the physical difference between the phases before the front and behind it (a cubic term would imply the symmetry between the two phases) [4]. Note that the same features, i.e., the presence of the zero mode and the quadratic nonlinearity, are also inherent in the famous Kuramoto-Sivashinsky (KS) equation which governs the development of the thermodiffusion instability of the front of gas combustion, and has other important applications [5]. However, the instability in the KS equation is long-wave. Another particular example is the equation to govern longitudinal seismic waves in a viscoelastic medium [6], which can be written in the form

$$u_t+\frac{\partial^2}{\partial x^2}\left[-\left(\frac{\partial^2}{\partial x^2}+1\right)^2+\epsilon\right]u+uu_x=0\ , \qquad (1.1)$$

where the order parameter $u(x,t)$ has the physical meaning of the displacement velocity, and ϵ is a control parameter. Note that Eq. (1.1) conserves the net order parameter $\int_{-\infty}^{+\infty}u(x)dx$.

As is well known, in the absence of the zero mode, evolution of a slowly varying complex envelope of the fundamental harmonic is governed by the Ginzburg-Landau (GL) equation, while the enslaved amplitudes of the second and zeroth harmonics can be excluded. In Sec. II, a generalized GL equation for the complex envelope of the basic harmonic, coupled to an equation for the real zero mode, is derived. These equations coincide in form with those considered earlier in Ref. [7]. The equations of Ref. [7] were put forward as a general one-dimensional model for a short-wave pattern (in convection or in the Couette flow) interacting with a mean flow. However, in the presentation given in Ref. [7] the small overcriticality parameter ϵ was not singled out. In Sec. II of the present paper it is demonstrated that, taking account of that small parameter and making a rescaling, one can simplify the generalized GL equations, omitting some terms in them. In its final form, the generalized system, unlike the usual parameter-free GL equation, contains three real parameters, one of them being ϵ. The important peculiarity of this system is that ϵ cannot be scaled out (the purport of this fact is discussed below).

In Sec. III, results for stability of stationary spatially periodic solutions of the generalized GL equations are set forth [8]. As usual, the crucial role is played by the so-called sideband disturbances, which may be regarded as long-wavelength modulations of the underlying homogeneous pattern. The sideband stability has been investigated in Ref. [7] within the framework of the above-mentioned equations to which the present generalized GL system is equivalent. In particular, the analysis presented in a brief form in Sec. III recovers the important fact reported in Ref. [7]: While for the usual GL equation the band of the wave numbers k of the stable patterns is symmetric with respect to the point $k=0$, in the present case the stability band is confined to the negative values of k. Reformulation of the stability conditions in terms of the overcriticality parameter ϵ reveals another interesting result: unlike the usual stability band, which has a width $\Delta k \sim \sqrt{\epsilon}$ [1,2], in the present case it is anomalously narrow, $\Delta k \sim \epsilon$. The latter result is closely related to the above-mentioned fact that the small overcriticality parameter ϵ cannot be excluded from the generalized GL equations.

In Sec. IV the patterns modulated at a large spatial scale are considered within the framework of the so-called geometric optics approximation [9,10]. As is known, modulations governed by the usual GL equation suffer diffusive spreading out, so that no stationary modulation may exist [9,11]. The situation is different in the present case: Both moving and quiescent modulated profiles of a permanent shape are demonstrated to exist. The moving profiles are always transient layers [12,9], i.e., they match two asymptotically uniform patterns with different wave numbers. A structure of the transient layers may be both monotone and oscillating. On the contrary to this, all the quiescent profiles are periodic, and they include as a limiting case a solitonlike profile. It is easy to provide for the stability of moving transient layers, demanding that the corresponding asymptotic uniform patterns be stable. As for the quiescent profiles, I did not investigate their stability in detail, but some qualitative arguments allow one to hope that the "soliton" and, at least, those periodic profiles which may be regarded as rarefied chains of the "solitons" are stable.

Section V is devoted to consideration of patterns generated by an oscillatory short-wave instability in a dispersive (wave) system with the zero mode. As a particular example, one may take the most general form of an equation for the seismic waves derived in Ref. [6]:

$$u_t+\frac{\partial^2}{\partial x^2}\left[-\left(\frac{\partial^2}{\partial x^2}+1\right)^2+\epsilon\right]u+uu_x +\beta_1 u_{xxx}+\beta_2 u_{xxxxx}=0\ , \qquad (1.2)$$

where the additional dispersive terms $\sim\beta_{1,2}$ account for the oscillatory character of the instability setting in at $\epsilon=0$ at the wave numbers $k=\pm1$ [note that Eq. (1.2), as well as Eq. (1.1), conserves the net order parameter]. Nonlinear waves produced by a short-wave oscillatory instability were investigated in a number of papers by means of different techniques [13–17]. A fundamental fact inherent in the short-wave oscillatory instability in the usual situation (studied in the papers quoted) is that the critical wave numbers $k=\pm1$ give rise to competing traveling waves with opposite signs of the group velocity. However, the group velocity determined by Eq. (1.2) is

$$V_{\rm gr}=-3\beta_1 k^2+5\beta_2 k^4\ . \qquad (1.3)$$

According to Eq. (1.3), in the present case both values $k=\pm1$ give rise to the same group velocity. This implies that the waves produced by the instability will be described by a single complex amplitude instead of two amplitudes in the usual situation [14–17]. The analysis developed in Sec. V is based upon a generalized GL equation (with complex coefficients) for that amplitude coupled to the equation for the real zero mode. These equations have a family of traveling-wave solutions. A central point of Sec. V is the analysis of the stability of those solutions. The resulting stability band is again anomalously narrow as compared to that for the usual situation.

At last, in Sec. VI perspectives for further study are discussed. In particular, a codimension-2 situation, when the onset of an instability in the system with a conserved real order parameter concurs with the change of the character of the instability from long wave to short wave, is considered, and a universal governing equation for this situation is proposed. A two-dimensional generalization is discussed too.

II. THE GENERALIZED GINZBURG-LANDAU EQUATIONS

To arrive at the GL equations governing the evolution of patterns produced by the steady short-wave instability in systems with the zero mode, let us start the analysis with the particular equation (1.1). A solution is looked for in the form

$$u(x,t)=U_0(x,t)+U_1(x,t)e^{ix}+U_1^*(x,t)e^{-ix} +U_2(x,t)e^{2ix}+U_2^*(x,t)e^{-2ix}+\cdots\ , \qquad (2.1)$$

where the real zero mode $U_0(x,t)$ and the complex envelopes $U_1(x,t)$ and $U_2(x,t)$ are assumed to be slowly varying functions of x in comparison with the fundamental harmonic $e^{\pm ix}$. As usual, the second harmonic is slaved by the fundamental one. Indeed, inserting Eq. (2.1) into Eq. (1.1) and equating coefficients in front of the second harmonic, one readily finds, in the lowest approximation with respect to derivatives of the slowly varying amplitude functions, that

$$U_2=-(i/9)U_1^2\ . \qquad (2.2)$$

At last, inserting Eq. (2.2) along with Eq. (2.1) into Eq. (1.1) and equating the coefficients in front of the fundamental and zero harmonics, one arrives at the following coupled equations:

$$(U_1)_t=\epsilon U_1+4(U_1)_{xx}+\tfrac{1}{9}|U_1|^2U_1-iU_0U_1 -U_0(U_1)_x\ , \qquad (2.3)$$

$$(U_0)_t=(U_0)_{xx}-(|U_1|^2)_x-\tfrac{1}{2}(U_0^2)_x \; . \tag{2.4}$$

As was mentioned above, the same equations (with general values of the coefficients) have been proposed earlier in Ref. [7] to describe the situation when, in convection or in the Couette flow, the short-wave pattern is coupled to a mean flow. However, the scaling properties of terms of these equations with respect to the small parameter ϵ were not analyzed in Ref. [7]. To do this, it is natural to rescale the variables as follows:

$$\begin{aligned} &U_1=\sqrt{\epsilon}\tilde{U}_1, \quad x=\tilde{x}/\sqrt{\epsilon} \; , \\ &t=\tilde{t}/\epsilon, \quad U_0=\epsilon\tilde{U}_0 \; . \end{aligned} \tag{2.5}$$

Inserting Eqs. (2.5) into Eqs. (2.3) and (2.4) one can see that the terms $-U_0(U_1)_x$ in Eq. (2.3) and $-\frac{1}{2}(U_0^2)_x$ in Eq. (2.4) acquire a small multiplier $\sim\sqrt{\epsilon}$, while the term $(|U_1|^2)_x$ gets a large factor $\sim\epsilon^{-1/2}$. In fact, the rescaled variables defined by Eq. (2.5) will not be directly employed in what follows; however, they suggest that the above-mentioned terms acquiring the small multipliers on rescaling may be omitted in the first approximation. At last, a straightforward consideration of the simplified system (with the two terms omitted) demonstrates that, to write it in the most general form, one should replace the coefficient $\frac{1}{9}$ in front of the cubic term in Eq. (2.3) by an arbitrary one κ ($\kappa>0$), and insert an arbitrary positive coefficient q^2 in front of the term $(U_0)_{xx}$ in Eq. (2.4):

$$(U_1)_t=\epsilon U_1+4(U_1)_{xx}-\kappa|U_1|^2U_1-iU_0U_1=0 \; , \tag{2.6}$$

$$(U_0)_t=q^2(U_0)_{xx}-(|U_1|^2)_x \; . \tag{2.7}$$

For instance, the coefficient κ in Eq. (2.6) will be different from $\frac{1}{9}$ if the cubic term u^2u_x with an independent coefficient in front of it is added to the underlying equation (1.1).

The system of Eqs. (2.6) and (2.7) gives the GL equations for the situation when the short-wave steady instability is coupled by a quadratic nonlinearity to the zero mode. In their general form, these GL equations contain the small overcriticality parameter ϵ [which cannot be scaled out because of the presence of the last term in Eq. (2.7), see above], and two additional positive parameters κ and q^2. It is natural to assume that κ and q^2 take values of order one.

The presence of the unremovable small parameter ϵ suggests that, to compensate it, another small parameter must show up in an expansion of solutions of Eqs. (2.6) and (2.7). In the next section it will be demonstrated that, as a matter of fact, this additional small parameter is the wave number of the continuous-wave (cw) solutions, which represent the short-wavelength patterns in an underlying physical problem.

To conclude this section, it seems worthy to mention Ref. [18], where a different but somewhat cognate situation was considered: an oscillatory instability set in simultaneously with the steady one. Two coupled *ordinary* differential equations governing evolution of the, respectively, complex and real amplitudes of the corresponding unstable modes have been derived in Ref. [18].

III. STABILITY OF THE STATIONARY PATTERNS

Equations (2.6) and (2.7) possess the family of spatially periodic cw stationary solutions which have the same form as in the case of the usual GL equations;

$$U_0=0, \quad U_1=\sqrt{\kappa^{-1}(\epsilon-4k^2)}e^{ikx} \; , \tag{3.1}$$

where k is an arbitrary wave number belonging to the *existence band* $k^2<\epsilon/4$. In terms of the underlying real order parameter $u(x)$, the solution (3.1) corresponds to a stationary pattern with the full wave number $K\equiv 1+k$, see Eq. (2.1).

The central point of the subsequent analysis is to investigate the stability of the solutions (3.1) against infinitesimal disturbances, and to find a *stability band* inside the existence one. To this end, it is natural to represent the complex amplitude in the form

$$U_1(x,t)=a(x,t)\exp[i\phi(x,t)] \tag{3.2}$$

with real functions $a(x,t)$ and $\phi(x,t)$. In terms of this representation, Eqs. (2.6) and (2.7) take the form

$$a_t=\epsilon a+4a_{xx}-4a\phi_x^2-\kappa a^3 \; , \tag{3.3}$$

$$\phi_t=4\phi_{xx}-8a^{-1}a_x\phi_x-U_0 \; , \tag{3.4}$$

$$(U_0)_t=q^2(U_0)_{xx}-(a^2)_x \; . \tag{3.5}$$

A perturbed solution, to be inserted into Eqs. (3.3)–(3.5), must be taken in the form

$$\begin{aligned} &a=a^{(0)}+a^{(1)}e^{\gamma t+ipx} \; , \\ &\phi=\phi^{(0)}+\phi^{(1)}e^{\gamma t+ipx} \; , \end{aligned} \tag{3.6}$$

$$U_0=U_0^{(1)}e^{\gamma t+ipx} \; , \tag{3.7}$$

where $a^{(0)}\equiv\sqrt{\kappa^{-1}(\epsilon-4k^2)}$, $\phi^{(0)}\equiv kx$, $a^{(1)}$, $\phi^{(1)}$, and $U_0^{(1)}$ are infinitesimal amplitudes of the disturbance, and γ and p are, respectively, its instability growth rate and wave number.

After straightforward calculations, insertion of Eqs. (3.6) and (3.7) into Eqs. (3.3)–(3.5) yields the following equation relating γ to p:

$$\begin{aligned} &[(\gamma+4p^2)^2+2(\epsilon-4k^2)(\gamma+4p^2)-64k^2p^2] \\ &\qquad\times(\gamma+4p^2)-16\kappa^{-1}(\epsilon-4k^2)kp^2=0 \; . \end{aligned} \tag{3.8}$$

As is well known [1,2] for the stability problem a critical role is played by the so-called sideband perturbations, i.e., those with infinitely small p. At $p=0$, Eq. (3.8) has the stable root $\gamma=-2(\epsilon-4k^2)$, and two neutral ones $\gamma=0$. Expanding the two branches of the dependence $\gamma(p)$, that give $\gamma(0)=0$, in powers of p, one finds at order $\gamma=O(p)$,

$$\gamma^2=8\kappa^{-1}kp^2 \; . \tag{3.9}$$

According to Eq. (3.9), the stability condition $\mathrm{Re}\gamma(p)\le 0$ can only be satisfied at this order if k is negative, i.e., γ is purely imaginary. This result has been first obtained in Ref. [7]. Thus, in terms of the full wave number $K=1+k$ of the underlying spatially periodic pattern, the stability band must lie to the left of the wave number $K_0=1$, at which the stability sets in. This is a drastic difference from the usual Eckhaus stability band which, in the first approximation, is symmetric with respect to the point $K_0=1$ (an asymmetry of the Eckhaus stability band appears in the next approximation [19]).

The critical step of the sideband-stability analysis is to extend it to the second order:

$$\gamma(p)=\pm i\Gamma_1 p+\Gamma_2 p^2+O(p^3) , \tag{3.10}$$

where, according to Eq. (3.9), $\Gamma_1\equiv\sqrt{-8\kappa^{-1}k}$ (it is assumed $k\le 0$). Analysis of Eq. (3.8) demonstrates that the coefficient Γ_2 in the expansion (3.10) is purely real. Thus its sign completely determines the sideband stability in the range $k\le 0$. A straightforward calculation yields

$$\Gamma_2=-(2\epsilon)^{-1}[4\kappa^{-1}k+(4+q^2)\epsilon]+O(k^2/\epsilon) . \tag{3.11}$$

Finally, it follows from Eqs. (3.9) and (3.11) that the stability band is

$$k\le 0,\quad |k|\le k_0\equiv\tfrac{1}{4}\kappa(4+q^2)\epsilon+O(\epsilon^2) . \tag{3.12}$$

The $O(\epsilon^2)$ correction in Eq. (3.12) can also be found [for instance, it is $-\frac{13}{9}(\frac{5}{36})^2\epsilon^2$ in the case $q^2=1$, $\kappa=\frac{1}{9}$ corresponding to Eq. (1.1)], but is not of great interest.

The fundamental difference of the stability band (3.12) from the standard Eckhaus band [1,2],

$$k\gtrless 0,\quad |k|\le k_0\equiv\sqrt{\epsilon/12}+O(\epsilon) , \tag{3.13}$$

[the $O(\epsilon)$ corrections in Eq. (3.13) are different for $k>0$ and for $k<0$ [19]], is that the band (3.12) is anomalously narrow in comparison with (3.13). One can readily trace back a reason for this difference to the presence of the term linear in k in the square brackets of Eq. (3.11). In turn, that term stems from the last term in Eq. (2.7) [which becomes large after the rescaling (2.5)].

So far, only the sideband disturbances were analyzed. However, applying the Routh-Hurwitz criterion to Eq. (3.8), one can verify that the condition of the stability against infinitesimal disturbances with an arbitrary p gives rise to no additional restriction of the stability band.

To conclude this section, it is relevant to note that the anomalous narrowness of the stability band in the systems with the zero mode makes the famous wave-number-selection problem (see, e.g., Ref. [26]) less important for them.

IV. MODULATED PATTERNS

A. General analysis

The next natural step in the analysis of dynamics of the patterns produced by the short-wave instability in the systems with the zero mode is to consider patterns with the local wave number and amplitude subject to a modulation at a spatial scale L much greater than the local spatial period of the pattern. Under this assumption, the underlying equations (3.3)–(3.5) may be simplified in the so-called geometric optics (GO) approximation [9,10]. In the GO approximation, Eq. (3.3) amounts to the quasistationary relation between the local amplitude $a(x)$ and the local wave number $k(x)\equiv\phi_x$, where $a(x)$ and $k(x)$ are regarded as slowly varying functions of x and t:

$$a^2=\kappa^{-1}(\epsilon-4k^2) . \tag{4.1}$$

Differentiating Eq. (4.1) in x yields the additional relation

$$a_x/a=-4kk_x/(\epsilon-4k^2) . \tag{4.2}$$

Next, one should insert Eqs. (4.1) and (4.2) into Eqs. (3.4) and (3.5), retaining in the latter equations terms of the lowest order with respect to small derivatives of the slowly varying functions. A simple analysis of the dimensionality demonstrates that the consistent GO approximation is possible if the quantities $k(x)$, $U_0(x)$, $\phi_t(x)$, and the characteristic scales L and T of variation of these quantities in x and t are ordered as follows:

$$k\sim L^{-2},\quad U_0\sim\phi_t\sim L^{-3},\quad T\sim L^2 . \tag{4.3}$$

To bring the resultant system of equations into an eventual form, it is necessary to express the quantity ϕ_t on the left-hand side of Eq. (3.4) in terms of k. To do this, one should additionally differentiate this equation in x and make use of the identity $\phi_{tx}\equiv k_t$. The resulting form of the simplified equations is

$$k_t=4k_{xx}-(U_0)_x , \tag{4.4}$$

$$(U_0)_t=q^2(U_0)_{xx}+4\kappa^{-1}(k^2)_x . \tag{4.5}$$

Strictly speaking, the term $4k_{xx}$ in Eq. (4.4) is obtained in the form

$$4\left[\frac{\epsilon-12k^2}{\epsilon-4k^2}k_x\right]_x . \tag{4.6}$$

However, the modulated patterns may be meaningful only in the case when the function $k(x,t)$ takes values in the stability band (3.12), i.e., from the very beginning one may assume $k^2\ll\epsilon$ and take the term (4.6) as written in Eq. (4.4).

One can exclude the variable U_0 and rewrite Eqs. (4.4) and (4.5) in the form of the single equation for the variable $k(x,t)$:

$$k_{tt}-(4+q^2)k_{txx}+4q^2k_{xxxx}+4\kappa^{-1}(k^2)_{xx}=0 . \tag{4.7}$$

To compare, let us recall that for the usual GL equation the GO approximation leads to the nonlinear diffusion equation [9,11]

$$k_t=4\left[\frac{\epsilon-12k^2}{\epsilon-4k^2}k_x\right]_x \tag{4.8}$$

[cf. Eq. (4.6)]. The fact that the evolution of the long-wave modulations of a pattern coupled to the zeroth mode is governed by a second-order (in time) equation, unlike the first-order one (4.8), has been mentioned earlier in Ref. [7], but in that work no solutions were presented. Equation (4.8) has no nontrivial stationary solution, and any modulation profile suffers diffusional spreading [27]. The situation is altogether different for Eq. (4.7). Its solutions describing modulation profiles of a permanent shape moving at a constant velocity should be looked for in the form

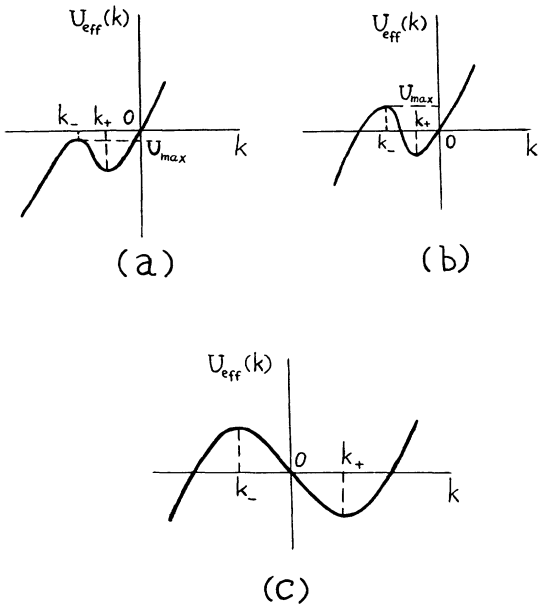

FIG. 1. The effective potential (4.11): (a) $V\neq 0$, $C>0$, $U_{\max}<0$ [see Eq. (4.16)]; (b) $V\neq 0$, $C>0$, $U_{\max}>0$ [see Eq. (4.17)]; (c) $V=0$, $C<0$.

$$k=k(z),\ z\equiv x-Vt\ . \tag{4.9}$$

Inserting Eq. (4.9) into Eq. (4.7) and integrating twice, one arrives at the following equation:

$$4q^2k''+(4+q^2)Vk'+\frac{4}{\kappa}k^2+V^2k+2C=0\ , \tag{4.10}$$

C being an arbitrary integration constant. One may regard Eq. (4.10) as an equation of motion of a mechanical particle with the mass $m=4q^2$ and the coordinate $k(z)$ in the effective potential (Fig. 1)

$$U_{\text{eff}}=\frac{4}{3\kappa}k^3+\frac{V^2}{2}k^2+2Ck \tag{4.11}$$

in the presence of the friction force with the friction coefficient $\alpha=(4+q^2)V$. Evidently, the potential (4.11) has two equilibrium positions,

$$k_{\pm}=\frac{\kappa}{8}(-V^2\pm\sqrt{V^4-32C/\kappa})\ , \tag{4.12}$$

provided $V^4\geq 32C/\kappa$.

B. Moving transient layers ($V\neq 0$)

In the case $V>0$, meaningful modulation profiles with a permanent shape are generated by trajectories of the mechanical system (4.10) that connect the equilibrium position k_- (reached asymptotically at $z\to-\infty$) with the one k_+ (reached at $z\to+\infty$). In the case $V<0$, the equilibria k_- and k_+ are attained, respectively, at $z\to+\infty$ and $z\to-\infty$. These profiles may be regarded as transient layers (TL's) between two asymptotically uniform states with $k=k_{\mp}$. Evidently, the TL may not be stable unless both asymptotic values $k_{\pm}$ belong to the stability band (3.12). As it follows from Eq. (4.12), this means that the following inequalities must hold:

$$0\leq C\leq \kappa V^4/32\ , \tag{4.13}$$

$$C\geq(\kappa/8)(4+q^2)\epsilon[V^2-(4+q^2)\epsilon]\ , \tag{4.14}$$

$$V^2\leq 2(4+q^2)\epsilon \tag{4.15}$$

[it is easy to see that the inequalities (4.13) and (4.14) are compatible]. Note that when V^2 attains the maximum value, equal to $2(4+q^2)\epsilon$ according to Eq. (4.15), Eqs. (4.13) and (4.14) imply $C=(\kappa/8)(4+q^2)^2\epsilon^2$, and $k_+=k_-$ according to Eq. (4.12). Thus the TL disappears when its velocity attains the maximum value. Note also that the stability condition (3.12), together with the relation (4.3), gives rise to the restriction on the modulation scale L: $L\gtrsim\epsilon^{-1/2}$.

Depending on the value of the control parameter V^4/C, one may get both monotone and oscillating trajectories connecting the equilibrium positions k_- and k_+. Accordingly, one may expect that the shape of the TL may be both monotone and oscillatory [numerical integration of Eq. (4.10) is needed to discern between the two possibilities]. The stability of the monotone profile seems to be ensured if all the local values of the wave number $k(x)$ lie inside the stability band (3.12). In the case of the oscillatory profile, this is guaranteed only if the maximum value $U_{\max}$ of the potential (4.11) is negative, i.e., the local values of $k(x)$ lying in the unstable range $k>0$ are not accessible [see Fig. 1(a)]. As it follows from Eqs. (4.11) and (4.12), the condition $U_{\max}\leq 0$ is equivalent to the inequality

$$3\kappa V^4/128\leq C\leq\kappa V^4/32 \tag{4.16}$$

[cf. Eq. (4.13)].

In the case $U_{\max}>0$, i.e.,

$$0\leq C\leq 3\kappa V^4/128\ , \tag{4.17}$$

the TL profile may have segments lying in the range $k>0$ [Fig. 1(b)]. Analysis of Eq. (3.8) reveals that the instability growth rate takes a maximum value at $p\sim\sqrt{k}$, provided $0<k\lesssim\epsilon$. According to Eq. (4.3), the corresponding wavelength $2\pi/p$ of the disturbance proves to be of the same order as the modulation length L. Thus, it is not clear *a priori* whether the segments with positive $k(x)$ will indeed be unstable, and this requires further investigation.

At last, it is worthy to mention that, irrespective of the sign of V, the TL always moves so that one has $k=k_+$ in front of it, and $k=k_-$ behind it (recall $|k_+|<|k_-|$).

C. Quiescent modulation profiles ($V=0$)

In the case $V=0$, Eq. (4.10) can be integrated explicitly. The stationary points (4.12) exist provided $C<0$. The simplest nontrivial solution is the "soliton"

$$k(x)=k_+[3\,\text{sech}^2(q^{-1}\sqrt{k_+/2\kappa}x)-1]\ , \tag{4.18}$$

where $k_+=(-\frac{1}{2}C\kappa)^{1/2}$ according to Eq. (4.12). Physically, the "soliton" may be interpreted as a phase defect in the underlying short-scale pattern: The phase misfit across the "soliton" is

$$\Delta\phi \equiv \int_{-\infty}^{+\infty} [k_+ + k(x)]dx = 6q\sqrt{2\kappa k_+} \ . \tag{4.19}$$

Strictly speaking, expression (4.19) is small due to the definition of the stability band [see Eq. (3.12)]. However, at moderate values of the parameters q^2 and κ, and at sufficiently small ϵ, the phase misfit (4.19) may attain the threshold value 2π. At $\Delta\phi=2\pi$, the defect may be interpreted as insertion of an extra cell into the underlying short-scale pattern. For instance, at $q^2=4$ and $\kappa=1$, the equality $\Delta\phi=2\pi$ means $\sqrt{\epsilon}\geq \pi/24$.

I will not develop an accurate investigation of the stability of the "soliton." Instead, the following qualitative arguments will be given in favor of its stability. Equation (4.7) without the second term on its left-hand side is nothing but the classical Boussinesq equation (BE) with the "good" sign in front of the dispersive term k_{xxxx} (the "bad" BE has the opposite sign at this place, see, e.g., Ref. [20]). It is known that the soliton of the "good" BE is stable, unlike the unstable solution of the "bad" BE [21]. The additional second term on the left-hand side of Eq. (4.7) may be regarded as a dissipative one (of the diffusion type). It is very similar that the soliton, being a stable stationary solution of the "good" BE, remains stable if the dissipative term has been added.

A general periodic solution to Eq. (4.10) with $V=0$ and $C<0$ can be written in terms of the Jacobi elliptic function dn:

$$k(x)=2\kappa q^2 B[3\,\mathrm{dn}^2(\sqrt{B}x,s)-(2-s^2)] \ , \tag{4.20}$$

where the modulus s is an arbitrary parameter taking values $0\leq s\leq 1$, and

$$B\equiv \tfrac{1}{2}(1+s^4-s^2)^{-1/2}q^{-2}\sqrt{-C/2\kappa} \ . \tag{4.21}$$

In the limiting case $s=1$, the solution given by Eqs. (4.20) and (4.21) goes over into the solution (4.18). Assuming that the solution is stable indeed, one may expect that the periodic solution (4.20) with $1-s\ll 1$, which is a rare chain of the silitons, is stable also. In the opposite limiting case $s=0$, the periodic solution degenerates into the trivial unstable solution $k\equiv(-\frac{1}{2}C\kappa)^{1/2}$. Thus the periodic solution with $s\ll 1$, close to the unstable one, may not be stable. This means that, at a given value of C, there must exist a boundary value of the parameter s which separates stable and unstable periodic modulation profiles.

Physically, the periodic profile may be regarded as a large-scale stationary "superstructure" imposed upon the underlying short-scale pattern. Of course, the period $2K(s)/\sqrt{B}$ of the "superstructure" [$K(s)$ is the complete elliptic integral of the first kind] need not be commensurable with the period of the underlying pattern, close to 2π in the notation adopted.

V. THE SHORT-WAVE OSCILLATORY SYSTEM WITH THE ZERO MODE

A. The GL equations

In this section we will study traveling-wave patterns produced by the oscillatory short-wave instability in the dispersive system with the zero mode. To begin with, let us take as the simplest example Eq. (1.2) with $\beta_1\equiv\beta$, $\beta_2=0$:

$$u_t+\frac{\partial^2}{\partial x^2}\left[-\left(\frac{\partial^2}{\partial x^2}+1\right)^2+\epsilon\right]u+uu_x+\beta u_{xxx}=0 \ . \tag{5.1}$$

To single out the slowly varying amplitude functions, a general solution is looked for in the form [cf. Eq. (2.1)]

$$u=U_0(x,t)+U_1(x,t)e^{i(x+\beta t)}+U_1^*(x,t)e^{-i(x+\beta t)} + U_2(x,t)e^{2i(x+\beta t)}+U_2^*(x,t)e^{-2i(x+\beta t)}+\cdots \ . \tag{5.2}$$

The second harmonic remains enslaved and its amplitude can be expressed in terms of $U_1(x,t)$ [cf. Eq. (2.2)]:

$$U_2=-(i/3)(3+2i\beta)(9+4\beta^2)^{-1}[U_1(x,t)]^2 \ . \tag{5.3}$$

Insertion of Eqs. (5.2) and (5.3) into Eq. (5.1) leads to the following system of the generalized GL equations, cf. Eqs. (2.3) and (2.4):

$$(U_1)_t=\epsilon U_1+4(U_1)_{zz}-\tfrac{1}{3}(3+2i\beta)(9+4\beta^2)^{-1}|U_1|^2U_1 - 3i\beta(U_1)_{zz}-iU_0U_1-U_0(V_1)_z \ , \tag{5.4}$$

$$(U_0)_t=(U_0)_{zz}-(|U_1|^2)_z-\tfrac{1}{2}(U_0^2)_z-3\beta(U_0)_z \ , \tag{5.5}$$

where $z\equiv x+3\beta t$ is the coordinate traveling with the group velocity $V_{\rm gr}=-3\beta$ corresponding to $k=\pm 1$ [see Eq. (1.3)]. As well as in the case of the system of Eqs. (2.3) and (2.4), the straightforward dimension analysis [see Eq. (2.5)] demonstrates that the terms $-U_0(U_1)_z$ in Eq. (5.4) and $-\frac{1}{2}(U_0^2)_z$ in Eq. (5.5) may be omitted. If the dispersion coefficient β is not a small parameter ($|\beta|\sim 1$), the dominating role is played by the two terms: the familiar one $-(|U_1|^2)_z$ on the right-hand side of Eq. (5.5), and the new term $-3\beta(U_0)_z$.

In terms of the real amplitude and phase defined by Eq. (3.2), Eqs. (5.4) and (5.5) take the form [cf. Eqs. (3.3)–(3.5)]

$$a_t=\epsilon a+4a_{zz}-4a\phi_z^2-(9+4\beta^2)^{-1}a^3 + 3\beta a\phi_{zz}+6\beta a_z\phi_z \ , \tag{5.6}$$

$$\phi_t=4\phi_{zz}+8a^{-1}a_z\phi_2-U_0-\tfrac{2}{3}\beta(9+4\beta^2)^{-1}a^2 - 3\beta a^{-1}a_{zz}+3\beta\phi_z^2 \ , \tag{5.7}$$

$$(U_0)_t=(U_0)_{zz}-(a^2)_z-3\beta(U_0)_z \ . \tag{5.8}$$

B. Stability of the traveling waves

A traveling-wave solution of Eqs. (5.6)–(5.8) has the form

$$U_0=0,\quad [a^{(0)}]^2=(9+4\beta^2)(\epsilon-4k^2)\ ,$$
$$\phi^{(0)}=kz-\omega(k)t\ ,\quad \omega(k)\equiv\tfrac{2}{3}\beta(\epsilon-4k^2)-3\beta k^2 \tag{5.9}$$

[cf. Eq. (3.1)]. To investigate the stability of these solutions, a disturbed solution is again looked for in the form of Eqs. (3.6) and (3.7). The dispersion equation relating γ to p, analogous to Eq. (3.8), takes a rather cumbersome form displayed in the Appendix. Like in Sec. III, the critical step of the stability analysis is to consider the sideband disturbances. Inserting $\tilde{\gamma}(p)=\tilde{\gamma}_1+\tilde{\gamma}_2+O(p^3)$, where $\tilde{\gamma}\equiv\gamma+6i\beta kp$, $\tilde{\gamma}_1=O(p)$, and $\tilde{\gamma}_2=O(p^2)$ [cf. Eq. (3.10)], into Eq. (A1) and expanding it in powers of p, one arrives at the following equation for $\tilde{\gamma}_1$ [cf. Eq. (3.9)]:

$$\tilde{\gamma}_1^2-3i\beta p\tilde{\gamma}_1-8kp^2(9+2\beta^2)=0\ . \tag{5.10}$$

At the order $\tilde{\gamma}(p)=O(p)$, the stability condition $\mathrm{Re}\tilde{\gamma}\le 0$ can only be satisfied if $\tilde{\gamma}_1$ is purely imaginary. According to Eq. (5.10), this means that k lies in the range

$$k\le k_0\approx\beta^2/32\ , \tag{5.11}$$

provided $\beta^2/32$ is a small parameter ($\le\sqrt{\epsilon}$). In the opposite case, Eq. (5.10) imposes no real restriction on the stability band.

At the next order, expanding Eq. (A1) demonstrates that, like in the nondispersive case $\beta=0$, the expansion of $\tilde{\gamma}(p)$ goes as follows:

$$\tilde{\gamma}(p)=\tilde{\gamma}_1(p)+\tilde{\Gamma}_2p^2+O(p^3) \tag{5.12}$$

with a real $\tilde{\Gamma}_2$ [cf. Eq. (3.10)]. After some algebra, the stability condition $\tilde{\Gamma}_2\le 0$ takes the form

$$\tilde{\gamma}_1^2+8(\epsilon-4k^2)p^2-4\beta^2(\epsilon-4k^2)p^2-64k^2p^2$$
$$+(\tilde{\gamma}_1-3i\beta p-6i\beta kp)^{-1}[2(\epsilon-4k^2)p^2\tilde{\gamma}_1$$
$$-6i\beta(9+4\beta^2)(\epsilon-4k^2)p^3]\ge 0\ . \tag{5.13}$$

To investigate a stability domain on the plane (k,β) determined by the inequality (5.13), let us first consider the generic case, when β is not a small parameter ($\beta^2\gg\epsilon$, see below). In this case the preceding order $[\tilde{\gamma}=O(p)]$ gives no real restriction on p [see Eq. (5.11)], and Eq. (5.10) has the following two roots:

$$\tilde{\gamma}_1^{(+)}=3i\beta p-\tfrac{8}{3}i(9+2\beta^2)\beta^{-1}kp\ , \tag{5.14}$$

$$\tilde{\gamma}_1^{(-)}=\tfrac{8}{3}i(9+2\beta^2)\beta^{-1}kp\ . \tag{5.15}$$

Insertion of Eq. (5.14) into Eq. (5.13) yields the following stability condition in the eventual form:

$$0\le k\le 4(2+\beta^2)(36+17\beta^2)^{-1}\approx\tfrac{2}{9}\epsilon\ . \tag{5.16}$$

Insertion of the second root [Eq. (5.15)] into Eq. (5.13) yields no new restriction on the stability region.

Thus one may conclude that, while at $\beta=0$ the stability domain was confined to negative k [see Eqs. (3.12)], at β sufficiently large it transits as a whole to the range of positive k.

To follow the transition between the limiting cases $\beta=0$ and $\beta^2\gg\epsilon$, it is necessary to consider in more detail the case $\beta^2\lesssim\epsilon$. In this case Eq. (5.10) has the roots

$$\tilde{\gamma}_1^{(\pm)}=\frac{i}{2}p(3\beta\pm\sqrt{9\beta^2-288k}\ )\ . \tag{5.17}$$

Inserting Eq. (5.17) into Eq. (5.13), it is possible to see that, like in the case $\beta^2\gg\epsilon$, the root $\tilde{\gamma}_1^{(+)}$ plays a crucial role.

The subsequent analysis should be developed separately for $k>0$ and $k<0$. For k positive, the stability condition (5.13) with $\tilde{\gamma}_1=\tilde{\gamma}_1^{(+)}$ can be brought into the form

$$Z^3+3|\beta|Z^2-(9\beta^2+40\epsilon)Z+9|\beta|(56\epsilon-3\beta^2)\ge 0\ , \tag{5.18}$$

where

$$Z\equiv\sqrt{9\beta^2-288k}\ . \tag{5.19}$$

Investigation of the inequality (5.18) with regard to Eq. (5.19) demonstrates that in the limit $\beta^2\gg\epsilon$ it yields $k\le\frac{2}{9}\epsilon$, thus recovering the above result [Eq. (5.16)].

It is possible to demonstrate that the inequality (5.18) is satisfied in the whole region of definition of Z, $0\le Z^2\le 9\beta^2$ [see Eq. (5.19)], if the ratio ϵ/β^2 exceeds the critical value $(\epsilon/\beta^2)_{\rm cr}^{(+)}$ defined as the smaller root of the equation

$$34(\epsilon/\beta^2)_{\rm cr}-1=[1+\tfrac{10}{3}(\epsilon/\beta^2)_{\rm cr}]^{3/2}\ . \tag{5.20}$$

With a good accuracy, $(\epsilon/\beta^2)_{\rm cr}^{(+)}\approx\frac{1}{15}$. Thus, at $\beta^2<15\epsilon$ the sideband stability interval for positive k amounts to $0\le k\le\beta^2/32$ [see Eq. (5.13)].

At $\epsilon/\beta^2=(\epsilon/\beta^2)_{\rm cr}^{(+)}$, the inequality (5.18) is broken at the point

$$Z=Z_{\rm cr}\equiv\beta\{2[1+\tfrac{10}{3}(\epsilon/\beta^2)]^{1/2}-1\}\ , \tag{5.21}$$

at which $(k/\epsilon)=(k/\epsilon)_{\rm cr}\approx\frac{7}{18}$ according to Eq. (5.19). At $\epsilon/\beta^2<(\epsilon/\beta^2)_{\rm cr}^{(+)}$, there appears a lacuna around the point $k/\epsilon=(k/\epsilon)_{\rm cr}$, where the inequality (5.18) does not hold. In the same time, the inequality (5.18) remains to be satisfied at the point $Z=0$, i.e., at $k=\beta^2/32$, provided $\epsilon/\beta^2\ge\frac{3}{56}$. Thus, in the range $\frac{3}{56}\le\epsilon/\beta^2<\frac{1}{15}$ the stability band at $k>0$ splits into two subbands, and at $\epsilon/\beta^2=\frac{1}{15}$ the left subband disappears. The maximum value of k/ϵ attained at $\epsilon/\beta^2=\frac{3}{56}$ is $(k/\epsilon)_{\max}=\beta^2/32\epsilon=\frac{7}{12}$. The whole stability band on the plane $(k/\epsilon,\beta^2/\epsilon)$ is shown in Fig. 2.

Let us proceed to negative k. In this case the underlying stability condition (5.13) reduces to the same inequality (5.18), but with the opposite sign. At $\beta=0$ this inequality recovers the stability band (3.12). With the growth of β^2, the stability band at $k<0$ shrinks, and it disappears when the parameter ϵ/β^2 drops to the value $(\epsilon/\beta^2)_{\rm cr}^{(-)}$ equal to a larger root of Eq. (5.20) (Fig. 2). It is easy to find that $(\epsilon/\beta^2)_{\rm cr}^{(-)}\approx 31$. Thus the stability band at $k<0$ disappears very quickly with the growth of β^2. At $\epsilon/\beta^2>(\epsilon/\beta^2)_{\rm cr}^{-}$, this band is centered at the point $Z=Z_{\rm cr}$ [see Eq. (5.21)], i.e., at

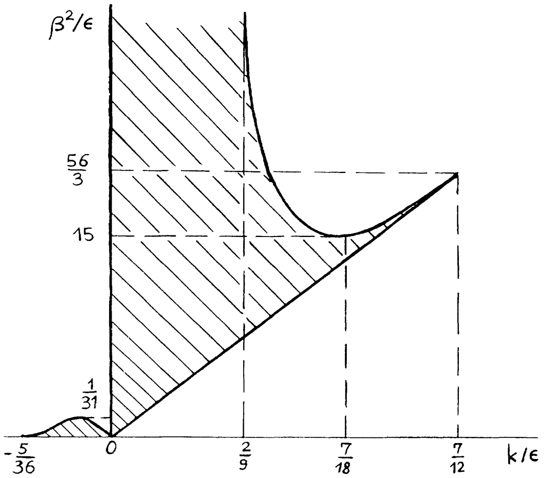

FIG. 2. The stability domain (shaded) on the plane $(k/\epsilon, \beta^2/\epsilon)$ for the traveling-wave solutions (5.9). The lower boundary of the stability domain at $k>0$ is the straight line defined by Eq. (5.11).

$$k/\epsilon=(k/\epsilon)_{\rm cr} \equiv \tfrac{1}{72}\{\beta^2/\epsilon+[(\beta^2/\epsilon)^2+\tfrac{10}{3}(\beta^2/\epsilon)-\tfrac{10}{3}]\} \ . \tag{5.22}$$

In particular, $(k/\epsilon)_{\rm cr}\approx\frac{1}{24}$ at $\epsilon/\beta^2=(\epsilon/\beta^2)_{\rm cr}^{(-)}$.

At last, analysis of the inequality (5.18) with the opposite sign demonstrates that, as is shown in Fig. 2, the boundary of the stability domain intersects the axis $\beta=0$ at a finite angle at the point $k=0$, and is tangent to the axis at the point $k=-\frac{5}{36}\epsilon$.

Note that, as well as in the case $\beta=0$, the stability region is confined to the values $|k|\le\epsilon$. This is a drastic difference from the usual short-wave oscillatory systems, where the stability band of the traveling waves has a width $\sim\sqrt{\epsilon}$ [14].

Thus far, only the sideband stability of the traveling further waves has been studied. A further analysis is necessary to guarantee the stability against the infinitesimal disturbances with arbitrary p. That analysis is rather laborious, and it will be deferred to another work. It is relevant to note that the full stability analysis for the traveling-wave solutions of the usual GL equation with complex coefficient has been performed in Ref. [22].

C. Generalizations

In the preceding parts of this section the consideration hinged around the particular system based on Eq. (5.1). Proceeding from the coupled GL equations (5.4) and (5.5) for this system, one can readily write the GL equations for the dispersive (oscillatory) systems in a general form [cf. Eqs. (2.6) and (2.7)]:

$$(U_1)_t=\epsilon U_1+4(U_1)_{zz}-(\kappa+i\alpha)|U_1|^2U_1 -i\beta(U_1)_{zz}-iU_0U_1 \ , \tag{5.23}$$

$$(U_0)_t=q^2(U_1)_{zz}-(|U_1|^2)_z-V(U_0)_z \ , \tag{5.24}$$

where all the parameters are real and arbitrary [according to the above, the terms $-U_0(U_1)_z$ and $-\frac{1}{2}(U_0^2)_z$ in Eqs. (5.4) and (5.5) may be omitted]. The stability analysis developed above can be extended to the system based on Eqs. (5.23) and (5.24). In particular, the sideband stability at the order $\gamma=O(p)$ amounts to the inequality [cf. Eq. (5.11)] $k\le k_0\equiv\kappa V^2/32$. At order p^2, the stability conditions in the general form are rather cumbersome.

At last, the long-scale modulations of the traveling waves can be studied in the framework of the GO approximation as well as for the dispersionless case $\beta=0$ (see Sec. IV). Results of this investigation will be set forth elsewhere. In this connection, it is relevant to mention that, for the usual GL equation with complex coefficients, the GO approximation makes it possible to find transient layers of a permanent shape moving with a constant velocity [9].

VI. CONCLUSION

Let us briefly discuss possible extensions of the present work. First of all, it is relevant to note some similarity between Eq. (1.1) and the KS equation[5]

$$u_t+u_{xx}+u_{xxxx}+uu_x=0 \ . \tag{6.1}$$

As was mentioned in the Introduction, Eq. (6.1) describes the onset of the long-wave steady instability in a system with the conserved order parameter, while Eq. (1.2) governs the onset of the short-wave instability. A next step is to analyze the so-called codimension-2 situation (see, e.g., Ref. [26]), when the onset of an instability concurs with the transition between the short-wave and long-wave instabilities. It is possible to deduce a universal equation governing the codimension-two bifurcation. In a rescaled form, the equation takes the form

$$u_t+\epsilon(u_{xx}+u_{xxxx})-u_{xxxxxx}-uu_x=0 \ . \tag{6.2}$$

The parameter ϵ in Eq. (6.2), unlike that in Eq. (1.1), is not assumed small. At $\epsilon>0$, Eq. (6.2) gives rise to the long-wave instability, and at $\epsilon<-4$ to the short-wave one. As a matter of fact, Eq. (6.2) goes over into the KS equation in the limit $\epsilon\to-\infty$, and at $0<-(\epsilon+4)\ll 4$ it can be reduced to Eq. (1.1). The most interesting case is $|\epsilon|\sim 1$. However, in this range Eq. (6.2) can only be attacked by numerical methods. Like in the case of the KS equation, one may expect both the presence of stable stationary periodic patterns [28] and chaotic (turbulent) behavior [5,29]. It would also be interesting to add the term βu_{xxx} (with $|\beta|\sim 1$) to the right-hand side of Eq. (6.2) [cf. Eq. (5.1)], in order to study the dynamics of the dispersive system near the codimension-two point. Let us note that a general evolution equation for the longitudinal seismic waves derived in Ref. [6] comprises Eq. (6.2) with $|\epsilon|\sim 1$ and its dispersive generalizations with the odd derivatives in the linear part.

Another feasible extension is related to two-dimensional equations. The simplest two-dimensional analog of Eq. (1.1) is

$$v_t+\triangle[-(\triangle+1)^2+\epsilon]v+\tfrac{1}{2}(\nabla V)^2-\tfrac{1}{2}\langle(\nabla v)^2\rangle=0\ , \tag{6.3}$$

where $\triangle$ and ∇ stand for the Laplacian and gradient, $\langle\ \rangle$ means spatial averaging, and in the one-dimensional case $u\equiv v_x$. The two-dimensional equation (6.3) has stationary solutions of the form [22]

$$v=\sum_{n=1}^{N}(a_n e^{i\mathbf{k}_n\mathbf{r}}+\text{c.c.})\ , \tag{6.4}$$

where $\mathbf{k}_n$ are the wave vectors lying in the instability region $(k_n^2-1)^2<\epsilon$, and a_n are some amplitudes. However, one can demonstrate that all the solutions (6.4) are unstable. Therefore, one may expect that Eq. (6.3) gives rise to a "two-dimensional turbulence," but this requires numerical simulations (numerical simulations of the two-dimensional analogs of the KS equation have been performed in Ref. [25]). It would be still more interesting to develop a numerical investigation of the two-dimensional version of Eq. (6.2) and its dispersive generalization.

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APPENDIX

The equation relating the instability growth rate γ to the disturbance wave number p for the traveling-wave solution (5.9) has the form

$$\begin{vmatrix} \tilde{\gamma}+4p^2+2(\epsilon-4k^2) & -8ikp+3\beta p^2 & 0 \\ 8ikp+4\beta(\epsilon-4k^2)-3\beta p^2 & \tilde{\gamma}+4p^2 & 1 \\ -2ip(a^{(0)})^2 & 0 & \tilde{\gamma}-3i\beta p+p^2-6i\beta kp \end{vmatrix}=0\ , \tag{A1}$$

where $\tilde{\gamma}=\gamma+6i\beta kp$.

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