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Signature of chaos in a quantum pseudointegrable billiard

Taksu Cheon
Department of Physics, Hosei University, Chiyoda-ku Fujimi, Tokyo 102, Japan

T. Mizusaki
Department of Physics, University of Tokyo, Bunkyo-ku Hongo, Tokyo 113, Japan

T. Shigehara and N. Yoshinaga*
Computer Centre, University of Tokyo, Bunkyo-ku Yayoi, Tokyo 113, Japan
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We analyze the quantum-level statistics of the zero-obstacle-size limit of the pseudointegrable billiard. While the nearest-neighbor-spacing distribution approaches to the Poissonian shape in the semiclassical limit, it is shown to become the Wigner distribution in a low-energy region. This fact is interpreted as a signature of the chaotic motion generated by the quantum uncertainty.

The behavior of nonintegrable systems under the quantum condition has been one of the outstanding problems in quantum physics from its early days.[1] A large number of numerical experiments have been accumulated in recent years on the dynamical systems with few degrees of freedom.[2–7] They have revealed various interesting phenomena which shed light on the question whether chaotic motion exists in quantum physics. One characteristic feature of the quantized system is so-called *quantum restriction of chaos.*[2,5,7,8] It was originally found in the one-dimensional kicked rotor, and is now believed to be a more-general phenomenon. It is seen typically in systems with compact classical phase space. Because of the finite dimensionality of their Hilbert space, one finds the recurrence of the quantum states in finite time periods. The ubiquity of this phenomenon, however, is now under question: Recent studies, one on a diffusive standard map[9] and another on a kicked rotor of higher dimension,[10] seem to indicate the existence of systems with chaos persisting in quantum limit.

The multifaceted nature of the quantum physics has been displayed in the discovery of the unexpected appearance of the signatures of chaotic motion in the quantum pseudointegrable billiard system whose classical counterpart is nonchaotic.[11–13] Different interpretations are possible on this peculiar billiard system which exhibits the Wigner-level spacing distribution.[12] According to a standard view, the chaotic system shows the Wigner-type–level spacing distribution, while the nonchaotic dynamics is reflected in the Poisson-type–level statistics.[4,14,15] Hence, the pseudointegrable billiard might be taken as a special counterexample to this widely believed association of the level statistics to the degree of chaos. A more attractive idea is to view it as a signature of the unique chaotic motion *generated by* the quantum condition. This latter possibility has been further strengthened by the observation by Seba,[13] who has noticed the appearance of "wave chaos" in addition to the chaotic-level spacing statistics. He has also shown that even an infinitely thin pole can cause the phenomena which are usually associated with chaotic motion. A very interesting and plausible scenario now emerges. When the motion of a point of mass is quantized, it acquires intrinsic fuzziness arising from the uncertainty principle, and the probability of hitting the pole of δ function becomes finite. Thus, the chaotic orbits, which occupy a set-of-measure-zero subspace of the classical phase space, could spread into the finite portion with the quantization. Such a picture, if it turns out to be valid, would establish a unique phenomenon in which the quantization induces the chaotic characteristics to a classically nonchaotic system.

In order to bring about more persuasive evidence to this yet speculative notion of quantum generation of chaotic motion, it is essential to prove that the Wigner statistics found so far is limited to the long-wavelength limit (quantum limit), and that it is to be gradually replaced by the Poisson statistics as the Planck's constant $\hbar$ becomes smaller. That is what we intend to show in this Rapid Communication through the numerical analysis of the pseudointegrable billiard. It must be noted as a warning that numerical examples can never substitute the proof, and the establishment of such general concept as "quantum generation of chaos" requires careful analysis of the quantization of the classical orbits in the presence of chaos.[16–18] This fundamental problem is, of course, beyond the scope of this work.

Seba's zero-size-obstacle limit of the two-dimensional pseudointegrable billiard has been instrumental in identifying the essence of the seemingly contradictory phenomena of the appearance of chaotic-level statistics in a classically nonchaotic system. It is also helpful in reducing the computational burden because of its minimal number of parameters and of its simple analytical expression for the matrix elements. We consider the motion of a point of mass in a bounded plane with a singular pole. The classical system is described by the Hamiltonian

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+V\delta(x-x_0)\delta(y-y_0)+W\,, \qquad (1)$$

where V specifies the strength of the singular point (x_0,y_0), and W represents the surrounding rectangular wall which is formally given by

$$W=\begin{cases}0, & 0<x<l \text{ and } 0<y<1/l\,,\\ \infty, & \text{otherwise}\,.\end{cases} \tag{2}$$

In order to avoid redundant scale parameters, we set both the mass of the particle and the area of the billiard to one. The length of the sides l (or $1/l$) is left to our disposal for the later purpose of avoiding unwanted symmetry of the system. Almost all orbits (apart from those of set of measure zero which accidentally hit the singular point) are quasiperiodic, thus are nonchaotic. For those exceptional set-of-measure-zero orbits, Eq. (1) is incomplete in the sense that it does not specify the precise way of the reflection. However, since we are primarily interested in the quantized system, we proceed with a note that more rigorous definition of the classical dynamics is possible.[13] The quantized system is described by the Schrödinger equation

$$\varepsilon\Psi(x,y)=-\tfrac{1}{2}(\nabla_x^2+\nabla_y^2)\Psi(x,y)+\upsilon\delta(x-x_0)\delta(y-y_0)\Psi(x,y) \tag{3}$$

with the boundary condition

$$\Psi(0,y)=\Psi(l,y)=0,\ \ \Psi(x,0)=\Psi(x,1/l)=0\,. \tag{4}$$

In obtaining Eq. (3), we have divided Eq. (2) by $\hbar^2$ to get the two parameters that characterize the system

$$\varepsilon=\frac{E}{\hbar^2},\ \ \upsilon=\frac{V}{\hbar^2}\,. \tag{5}$$

Equation (5) shows that the semiclassical limit is achieved by letting *both* ε *and* υ be large. It can also be seen that there are two distinct regions of quantum limit; that is, $\varepsilon\rightarrow 0$ and $\upsilon\rightarrow 0$.

Two statistical quantities which characterize the eigenvalue sequence, namely the nearest-neighbor spacing $P(s)$ and the rigidity $\Delta(L)$ have been extensively studied as possible quantum indicators of chaotic dynamics.[19,20] Although the rigorous theoretical foundation is still lacking, a large number of numerical studies indicate that these statistical quantities of a quantum system do reflect the degree of order or chaos in a classical counterpart. The pseudointegrable billiard[11–13] seems to provide a counterexample to the general association of Wigner distribution to the chaotic motion. It was argued that this correspondence only holds at a semiclassical limit.[12] However, this does not explain why the Wigner distribution appears at a long-wavelength limit. This puzzle can be resolved if the Wigner distribution is related to the quantum chaotic motion which occurs only at the region away from the semiclassical limit. If one replaces the motion of a point of mass in the pseudointegrable billiard equation (1) by that of a quantum wave packet, it can be easily seen that almost all orbits eventually come across the pole (x_0,y_0) within finite time, and the direction of the motion of wave packet is unpredictable because of the very probabilistic nature of the quantum physics. In this sense, the quantum motion of the point of mass can be regarded as chaotic. The degree of chaos should increase in larger wavelengths, thus the gradual appearance of Wigner distribution at a lower-momentum region is expected, while at higher momentum the Poisson distribution should be recovered. This gradual change of the nature of the level statistics in a different momentum region is what we are to show here by numerically solving the eigenvalue problem Eqs. (3) and (4). In this paper, we restrict ourselves to the eigenvalue statistics. A fuller account of our study which includes other aspects of the problem is now under preparation. We only note that the preliminary results on wave-function-related properties also seem to confirm our assertion.

In the study of quantum-level statistics, it is necessary to avoid the degeneracy not related to the dynamics in question. If we chose mutually incommensurate values for l, $1/l$, x_0, and y_0, all the spatial symmetry of the billiard can be removed. We diagonalize Eq. (3) with the Fourier basis

$$\Phi_{n,m}(x,y)=2\sin\left(n\frac{\pi}{l}x\right)\sin(m\pi l y)\,, \tag{6}$$

where

$$n,m=1,2,3,\ldots,n_{\max}\,.$$

For the truncation of the basis states, we take $n_{\max}$ to be up to about 40. The problem now becomes the numerical diagonalization of matrices of dimension $n_{\max}\times n_{\max}$ which, in our case, is about 1600. About 600 lowest eigenvalues are estimated to be sufficiently accurate for the

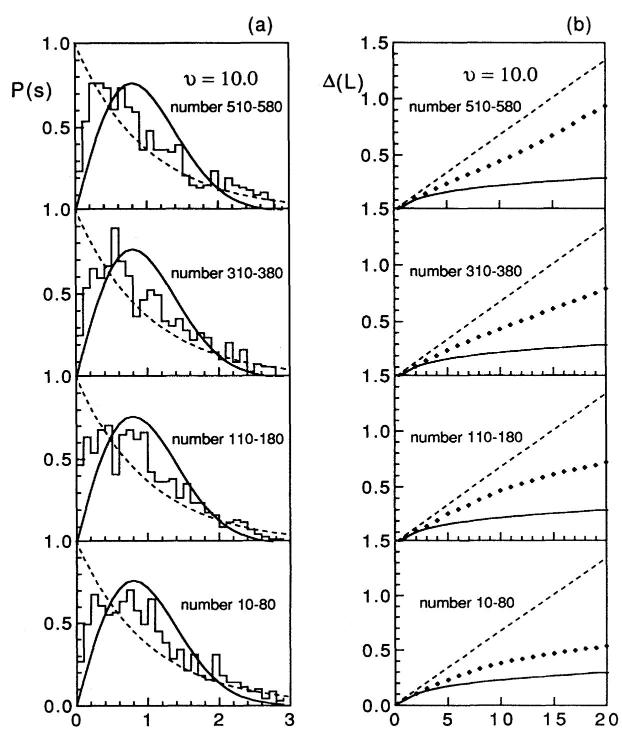

FIG. 1. (a) The nearest-neighbor-spacing distribution and (b) the rigidity of the pseudointegrable billiard, Eqs. (3) and (4). Statistics are taken within the subgroup of the eigenvalues as indicated in the figures. The coupling strength is chosen to be $\upsilon=10$.

coupling strength of $\upsilon \simeq 0.1-10$. After the unfolding[19] of the global level density, the accepted eigenstates are divided into smaller groups and the statistics are taken within each group. To obtain better statistics, we add levels from ten different billiard systems with different values of l. The actual values taken are $l = 0.350\pi$, 0.375π, $0.400\pi, \ldots, 0.575\pi$. The relative pole posistion to the edge of the wall is fixed as $x_0 = \sqrt{2}l/2$ and $y_0 = \sqrt{5}/5l$.

The results are shown in the figures. Figure 1 shows the level statistics for the coupling strength $\upsilon = 10$. Figure 2 is for $\upsilon = 0.1$. In both Figs. 1 and 2, graphs (a) are the nearest-neighbor spacing $P(s)$ and graphs (b) are the rigidity $\Delta(L)$. Four rows in each graph represent the samples from different energy regions, in descending order from top to bottom. In each graph, the dashed line represents the predictions of Poisson statistics and the solid lines are of Wigner statistics. One can see that the level statistics do change their nature in different energy regions as expected from our preceding arguments: Poisson-like distributions at higher energy and Wigner-like in low energy. The change is clear and unambiguous in $\Delta(L)$ statistics, but in $P(s)$, it is subtle and may require experienced interpretation. One can see the clear change in $P(s)$ only through such measures as its moments.[21] This is in accordance with more general arguments on the level statistics.[22,23] We conclude that we have obtained a signature that indicates the appearance of chaotic dynamics in the quantum pseudointegrable billiard.

An interesting point to note is the difference between Figs. 1 and 2, namely between strong-coupling and weak-coupling cases. The statistics of Fig. 2, the weaker-coupling system of the two, are generally more towards the Poisson limit. At first sight, this is somewhat puzzling because $\upsilon \rightarrow 0$ is the opposite of the semiclassical limit, and one might expect more Wigner-like behavior. This apparent difficulty is resolved when one looks into the structure of Hamiltonian matrix [Fig. 3(a)]. The parameter υ, as seen in Eq. (5), is the ratio between $\hbar^2$, typical level spacing in unperturbed system, and V, the strength of the perturbation by the pole. If the unperturbed level spacing is substantially larger than the strength of the perturbation, the system "resists" the weak perturbation that could have caused the chaotic motion in the absence of discrete levels. What we see in Fig. 2 is a very interesting combination of two opposite effects caused by quantization condition, one generating and the other restricting the chaotic motion. In the case of the stronger-coupling system shown in Fig. 1, the chaotic statistics are more fully developed in the low-energy region. At higher energy $\upsilon = 10$ it also approaches the Poisson limit. The full approach is presumably seen in the higher-energy region which lies beyond our current calculation.

To understand the transition from Poisson to Wigner statistics in our system, it is again useful to look at the structure of Hamiltonian matrix [Fig. 3(b)]. The diagonal elements of the Hamiltonian Eq. (3) are an increasing function of the quantum numbers n and m, while nondiagonal elements are bounded by 4υ. The local structure of the matrix resembles that of the uniform random matrix in the top-left region, and to the diagonal random matrix in bottom-right region. Hence, the Wigner distribution for lower eigenvalues and the Poisson distribution for higher eigenvalues are naturally expected. It might be of some interest to note that the type of matrix shown in Fig. 3(b) has been already encountered in the theory of random matrix as the *additive random matrix*.[24,25] It is known that the additive random matrix does not have the scaling property.[21,25] This fact implies its nonuniformity, namely the existence of the different level statistics in different regions of eigenvalues. It now seems plausible that the pseudointegrable billiard belongs to a wider class of systems which show the transition in the level statistics

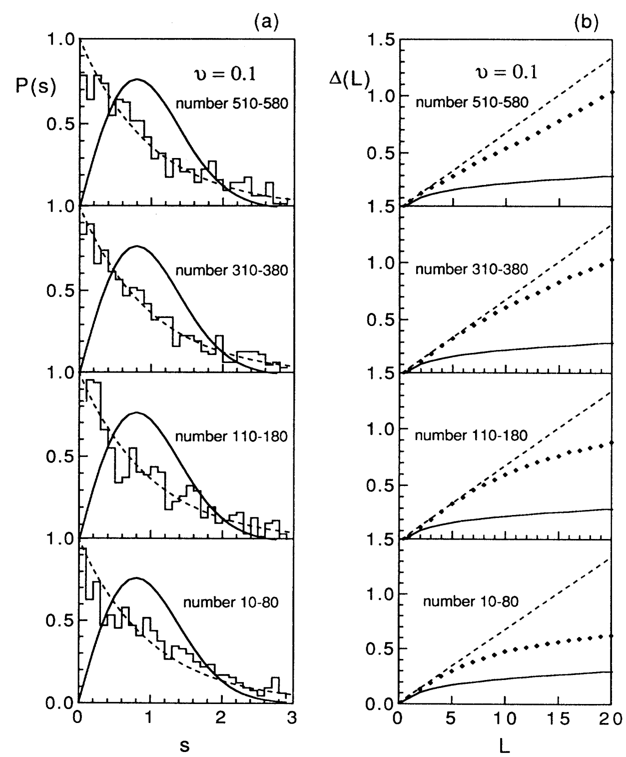

FIG. 2. Same as Fig. 1 except for the coupling strength, which in this case is $\upsilon = 0.1$.

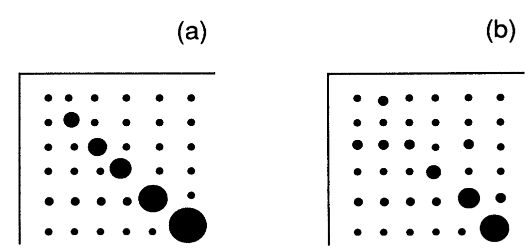

FIG. 3. Schematic representation of the structure of matrix elements of the Hamiltonian equation (3) in the Fourier basis equation (6). The size of the solid circle indicates the relative size of the matrix elements. (a) The case of weak coupling $\upsilon < 1$ where mixing effects of nondiagonal elements are small. (b) The strong-coupling case $\upsilon > 1$. The matrix is locally similar to the orthogonal random matrix at the top-left region, and to the diagonal random matrix at the bottom-right region.

from Wigner type in a low-energy region to the Poisson type in a high-energy region.[26] The relation between the quantum pseudointegrable billiard and the random matrix theory will be one of the subjects in our forthcoming publications.

In summary, we have observed the transition of the level statistics from Wigner-type to Poisson-type distributions with increasing energy in the zero-obstacle-size limit of the pseudointegrable billiards. We have argued that this can be naturally interpreted as the signature of the chaotic motion caused by the wavelike nature of a particle in quantum physics. In addition, with the different choice of the parameter controlling the coupling strength, we have observed a phenomenon indicating the restriction of chaotic spectra, which was generated originally as a pure quantum effect. Thus, our model seems to display an interesting interplay of chaos and order.

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*Permanent address: Department of Physics, College of Liberal Arts, Saitama University, Saitama 338, Japan.

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