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$1/f$ noise for driven interfaces

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Velocity-fluctuation spectra are proposed as a probe of the dynamic scaling of moving interfaces. Using the theory of Kardar, Parisi, and Zhang [Phys. Rev. Lett. **56**, 889 (1986)] it is shown that the spectrum of the spatially averaged displacement velocity diverges as $1/f^{\alpha}$ at low frequencies, where $\alpha=\frac{1}{3}$ ($\alpha\simeq0.7$) for a one- (two-) dimensional interface. This implies superdiffusive motion of the average interface position, which is verified numerically. The fluctuation spectrum of the stationary interface width diverges as $1/f^{2+\alpha}$. Simulations of interfaces driven by non-Gaussian noise are also presented.

Power-law divergencies in the low-frequency fluctuation spectra of macroscopic quantities, often referred to as $1/f$ noise, are a common characteristic of nonequilibrium steady states [1–4]. Despite the apparent universality of the phenomenon, however, its dynamical origins are quite diverse and a detailed analysis of the relevant mechanisms is required in each case [3,4]. When successful, such studies can turn $1/f$ noise into a useful probe of the physical system at hand [4].

Recently it has been suggested [5] that the ubiquity of $1/f$ noise could be traced back to the generic occurrence of power-law spatiotemporal correlations in noisy nonequilibrium systems [6]. However, in general, this plausible assertion does not stand closer scrutiny, since it is not guaranteed that the internal dynamics of the system actually shows up in the time evolution of spatially averaged quantities [7,8]. Hence further clarification of the conditions under which these systems show nontrivial fluctuation spectra is needed.

The present paper addresses these questions for the case of a steadily moving interface with local noisy dynamics [9,10]. Here the natural macroscopic observable is the displacement velocity v averaged over a region of linear size L which is large compared to microscopic length scales. I show that the fluctuation spectrum of v diverges as

$$\langle|\hat{v}(\omega)|^2\rangle\simeq L^{-d}\omega^{-\alpha} \tag{1}$$

for a d-dimensional interface, where [11]

$$\alpha=\frac{d+4}{z}-3 \tag{2}$$

in terms of the dynamic exponent [12] z of the interfacial fluctuations. The power-law (1) holds down to a cutoff frequency $\omega_0\sim1/L^z$ where the spectrum becomes constant. The numerical value of z is known [10,12] to depend on the dimensionality d and the nature of the microscopic noise driving the fluctuations. For Gaussian noise with short-range correlations, $z=\frac{3}{2}$ for $d=1$ and [13] $z\simeq1.62$ for $d=2$, leading to the noise exponents $\alpha=\frac{1}{3}$ and $\alpha\simeq0.7$, respectively [14].

Apart from the general questions alluded to above, this work was motivated by recent experiments on the motion of fluid interfaces in porous media, where the interfacial fluctuations were investigated by directly imaging the position of the interface during the displacement process [15]. Unfortunately this technique appears to lead to somewhat ambiguous results [16], and the comparison with theoretical predictions has remained controversial. I therefore propose to use the fluctuation spectrum of the average flow velocity, or equivalently the pressure in the fluid, as a probe for the interfacial dynamics. In the high-frequency range where fluctuations are dominated by the microscopic pore structure, an experiment was reported by Stokes, Kushnick, and Robbins [17].

The noisy interface motion is described by the Kardar-Parisi-Zhang (KPZ) equation for the interface position $h(\mathbf{x},t)$ relative to some d-dimensional reference plane [12],

$$\frac{\partial}{\partial t}h(\mathbf{x},t)=v_0+\frac{\lambda}{2}(\nabla h)^2+\sigma\nabla^2 h+\eta\,. \tag{3}$$

The first two terms on the right-hand side describe the inclination dependence of the local displacement velocity [18], the Laplacian reflects local smoothening due to an effective interfacial tension σ, and the noise term $\eta(\mathbf{x},t)$ is at this point only assumed to have short-range *temporal* correlations. We consider a finite system of linear extension L in the d transverse directions, and decompose $h(\mathbf{x},t)$ into discrete Fourier modes $\hat{h}_{\mathbf{q}}(t)$. The instantaneous, spatially averaged velocity is then $v(t)=\partial\hat{h}_{\mathbf{q}=0}/\partial t$. The nonlinear term in (3) couples $v(t)$ to the internal degrees of freedom of the interface,

$$v(t)=v_0+\frac{\lambda}{2}\sum_{\mathbf{q}}\mathbf{q}^2|\hat{h}_{\mathbf{q}}(t)|^2+\eta_{\mathbf{q}=0}(t)\,. \tag{4}$$

Note that this implies a renormalization of the bare growth velocity v_0 which has been shown [19] to generate universal finite-size corrections to the average velocity $\langle v\rangle$. Here we wish to compute the stationary velocity correlation function $\langle v(t)v(s)\rangle-\langle v\rangle^2$ and its Fourier transform. Apart from the trivial short-range contribution of the noise this involves a sum over four point correlation functions of the $h_{\mathbf{q}}$. We make the conventional assumption that the four point functions are dominated by products of two point functions and use the scaling form [10,12] for the latter,

$$\langle\hat{h}_{\mathbf{q}}(t)\hat{h}_{-\mathbf{q}}(0)\rangle=L^{-d}q^{-(d+2\zeta)}g(q^z t) \tag{5}$$

for small $q=|\mathbf{q}|$ and long times. Here $\zeta=2-z$ is the static roughness exponent [10,12] of the interface and the scaling function g decays rapidly for large arguments. Within this approximation it follows from (4) that

$$\langle|\hat{v}(\omega)|^2\rangle\simeq\frac{\lambda^2}{L^d}\int_{\Lambda_0}^{\Lambda_1}dq\,q^{3z-d-5}\hat{g}(\omega/q^z)\,, \tag{6}$$

where $\hat{g}$ is the Fourier transform of g and the cutoffs $\Lambda_0=O(1/L),\Lambda_1=O(1)$. Clearly for $\omega\ll\Lambda_0^z\sim L^{-z}$ the spectrum becomes constant. To evaluate the integral in the opposite limit $1/L^z\ll\omega\ll 1$ we require some information about the scaling function $\hat{g}$. We expect that $\hat{g}(\omega/q^z)\sim(\omega/q^z)^{-\nu}$ for $q\ll\omega^{1/z}$, where the exponent ν will be determined later. If $\nu>(d+4)/z-3$ the integral converges at the lower cutoff, the limit $\Lambda_0\rightarrow 0$ can be taken and Eqs. (1) and (2) follow. We shall see below that the fluctuation spectra of other quantities can nevertheless be dominated by the lower cutoff.

To determine ν we note that the velocity fluctuations can also be obtained from the $q\rightarrow 0$ limit of the height fluctuations,

$$\langle|\hat{v}(\omega)|^2\rangle=\lim_{q\rightarrow 0}\omega^2\langle|\hat{h}(\mathbf{q},\omega)|^2\rangle\,. \tag{7}$$

Using the scaling form (5) for the height fluctuations and the postulated behavior of the scaling function for small q we see that the existence of the limit requires the relation $\nu=(d+4)/z-1=2+\alpha$. Hence $\nu>\alpha$ is always true and my main result is established.

The long-range temporal velocity correlations imply anomalous diffusion for the center of mass $\bar{h}(t)=\hat{h}_{\mathbf{q}=0}(t)$. For short times ($t\ll L^z$) the motion is superdiffusive, $\xi_c^2(t)=\langle[\bar{h}(s+t)-\bar{h}(s)-vt]^2\rangle\simeq L^{-d}t^{1+\alpha}$ for $\alpha<1$ and $\xi_c^2(t)\simeq L^{-4(z-1)}t^2$ for [20] $\alpha>1$. At long times the fluctuations are diffusive, however with an anomalous size dependence, $\xi_c^2(t)\simeq L^{-(3z-4)}t$. The two regimes are combined in the scaling form

$$\xi_c^2(t,L)=D(t/L^z)L^{-(3z-4)}t\,, \tag{8}$$

where the function D saturates for $t\gg L^z$ and $D(x)\sim x^\alpha$ for small $x[D(x)\sim x$ if $\alpha>1]$. Numerical support for (8) from simulations of one- and two-dimensional interfaces is shown in Fig. 1. Note that at the crossover time $t\sim L^z$ the center-of-mass fluctuations are comparable to the interface width [10,12] $\xi^2=\langle[h(\mathbf{x},t)-\bar{h}(t)]^2\rangle$, $\xi_c^2\sim L^{4-2z}=L^{2\zeta}\sim\xi^2$, and $\xi_c\gg\xi$ at later times [21].

I have directly simulated velocity-fluctuation spectra for a one-dimensional model recently introduced by Zhang [22] where the noise in (3) is taken from a power-law distribution $P(\eta)\sim\eta^{-(\mu+1)}$, $\mu>2$. This kind of noise, which appears to be present [22] in the fluid displacement experiments referred to above, can be shown [23] to produce scaling exponents which vary continuously with μ. A lower bound [23] on α, which is expected to become exact in the limit $\mu\rightarrow 2$, is given by $\alpha=(8-\mu)/$

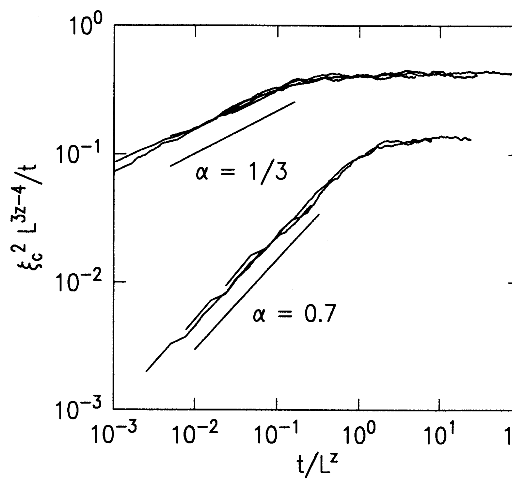

FIG. 1. Center-of-mass fluctuations scaled according to Eq. (8). The upper set of curves was obtained from simulations of the one-dimensional single-step model [28] on lattices of size $L=26$–400. The lower set of curves shows data for the two-dimensional restricted solid-on-solid model [29] for lattice sizes $L=10$–40. Averages were taken over typically 1000 independent runs.

$(2\mu-1)$. The results presented in Fig. 2 are clearly consistent with this prediction.

I now turn to the fluctuations of the interface width ξ^2 in the stationary regime $t \gg L^z$. We have

$$\xi^2(t) = \sum_{\mathbf{q}} |h_{\mathbf{q}}(t)|^2 . \tag{9}$$

Up to a factor $\mathbf{q}^2$ this is identical to the expression (4) for the velocity, hence the temporal correlations of ξ^2 can be computed as described above. However the different $\mathbf{q}$-dependence implies that the integral corresponding to (6) is now dominated by the lower cutoff Λ_0, and consequently the fluctuation spectrum reflects the behavior of the scaling function $\hat{g}$ at large arguments. The general result is then

$$\langle |\hat{\xi}^2(\omega)|^2 \rangle \simeq L^{4\zeta-(\nu-1)z} \omega^{-\nu} . \tag{10}$$

I have shown above that $\nu = 2+\alpha$ and hence I predict that $\langle |\hat{\xi}^2(\omega)|^2 \rangle \sim \omega^{-7/3}$ ($\omega^{-2.7}$) for interfaces driven by Gaussian noise in one (two) dimensions. This agrees with recent numerical work of Sander and Yan [24], who find the spectral exponents to be 2.25 and 2.6 in the two cases.

It is instructive to consider the linear model

$$\frac{\partial}{\partial t} \hat{h}_{\mathbf{q}}(t) = -\sigma |\mathbf{q}|^z \hat{h}_{\mathbf{q}} + \hat{\eta}_{\mathbf{q}} , \tag{11}$$

where the width fluctuations can be easily calculated. For $z=2$ this is the Edwards-Wilkinson model [25] [the linearized version of (3)] while for $z=1$ it describes interface fluctuations in diffusion-limited erosion [26]. The roughness exponent for (11) is $\zeta=(z-d)/2$, so the interface is rough in dimensions $d \le z$. Here the fluctuation spectrum turns out to be dominated by the lower cutoff whenever the interface is rough. Since the scaling function $\hat{g}$ is Lorentzian in this case, $\nu=2$ and $\langle |\hat{\xi}^2(\omega)|^2 \rangle \simeq L^{z-2d}\omega^{-2}$ for $d \le z$ with logarithmic corrections in $d=z$. Nontrivial spectra appear in the range $z<d<3z$, where $\langle |\hat{\xi}^2(\omega)|^2 \rangle \simeq L^{-d}\omega^{d/z-3}$. In particular, a $1/\omega$ spectrum is expected for two-dimensional diffusion-limited erosion. For $d>3z$ the spectrum becomes flat.

To conclude, it is worth noting that the velocity fluctuations are always trivial, $\langle |\hat{v}(\omega)|^2 \rangle =$const, for a class of models recently introduced in the contexts of molecular-beam epitaxy [27] and evolving sandpiles [6]. These models are characterized by a relaxation dynamics conserving the height, and a nonconserving noise modeling the random flux onto the surface. The linear model (11) is a particular example of this class. The conservation law decouples the center-of-mass motion from the internal dynamics of the interface and hence $\langle |\hat{v}(\omega)|^2 \rangle = \langle |\hat{\eta}_{\mathbf{q}=0}(\omega)|^2 \rangle$. This can be exploited by reexpressing the velocity fluctuations through (7) to yield the exact scaling relations $\nu=2$ and [27] $z=d+2\zeta$. The width fluctuations behave as described above for the linear model (11).

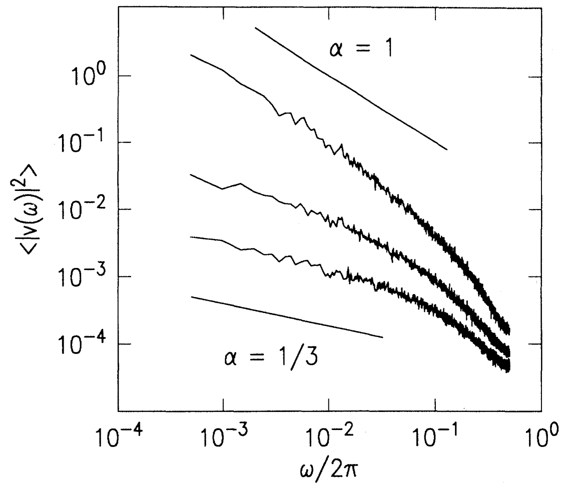

FIG. 2. Velocity-fluctuation spectra for a one-dimensional growth model with a power-law noise distribution. From top to bottom the spectra were obtained with power-law exponents $\mu=3$, 4, and 5. The lattice size was $L=1000$ and 100 time series of length 2048 were used for each spectrum. The theoretical prediction for the spectral exponent is $\alpha \simeq 1$ for $\mu=3$ and $\alpha \simeq \frac{1}{3}$ for $\mu=5$.

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