RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 44, NUMBER 12 15 DECEMBER 1991

Selberg's ζ function and the quantization of chaos

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(Received 29 April 1991)

We study Artin's billiard, an extremely chaotic system defined on the fundamental domain of the modular group PSL(2,$\mathbb{Z}$), and show that its quantum energy levels are given exactly by the nontrivial zeros of a certain Selberg ζ function expressed as an Euler product over the classical periodic orbits. We demonstrate that at least the first 73 energy levels can be determined by using only a finite number of periodic orbits.

PACS number(s): 05.45.+b, 03.65.−w

A beautiful application of Gutzwiller's semiclassical trace formula for chaotic systems [1,2] consists in rewriting it in terms of a ζ function, defined by an Euler product over the classical periodic orbits, whose nontrivial zeros approximate the quantum energy levels. In a recent paper it has been shown [3], in the case of the hyperbola billiard [4], that it is possible to compute the dynamical ζ function on the critical line and thus to determine semiclassically the quantum energy levels of this strongly chaotic system.

In this paper we study Artin's billiard [5] for which Gutzwiller's trace formula is *exact*, since it is identical to the Selberg trace formula [6], and for which the associated ζ function leads to an exact rather than semiclassical quantization rule for chaos. The system is a non-Euclidean billiard realized by a point particle sliding freely on a hyperbolic triangle given by the fundamental domain of the modular group PSL(2,$\mathbb{Z}$). On the Poincaré upper half-plane $\mathcal{H}=\{z=x+iy,\, y>0\}$ endowed with the measure $dx\,dy/y^2$, the triangle is identical to the modular domain $D=\{|z|>1 \text{ for } -\frac{1}{2}<x<0 \wedge |z|\geq 1 \text{ for } 0\leq x \leq \frac{1}{2}\}$. With the correct identification of the boundaries, the noncompact triangle (with finite area $A=\pi/3$) becomes a Riemannian surface of constant negative curvature with the topology of a sphere containing an open end (cusp) at infinity. The model has a long history; in fact, it was the first system where ergodicity could be demonstrated and where symbolic dynamics has been introduced [5]. The corresponding quantum system is governed by the Hamiltonian $\hat{H}=-\Delta$, where $\Delta=y^2(\partial_x^2+\partial_y^2)$ is the non-Euclidean Laplacian ($\hbar=2m=1$). The eigenfunctions of $\hat{H}$ have to be invariant under the modular group, i.e., they have to satisfy periodic boundary conditions. In this paper we are interested only in the desymmetrized system, where only the "odd" eigenfunctions satisfying $\psi_n^-(-z^*)=-\psi_n^-(z)$ are considered, and which can be viewed as a quantum billiard defined on the halved domain $\tilde{D}=\{|z|\geq 1,\ 0\leq x\leq \frac{1}{2}\}$. For this system the eigenfunctions vanish on $\partial\tilde{D}$ (Dirichlet problem). While the full system has both a discrete and continuous spectrum [7], the desymmetrized billiard possesses only a discrete spectrum with $\frac{1}{4}<E_1\leq E_2\leq \cdots$, since the Eisenstein series that are related to the continuous spectrum are even [8]. The first 73 odd eigenvalues with $E_n<2500$ have been computed by Hejhal [9]. The ground-state energy is $E_1=91.14134$.

Our starting point is Selberg's trace formula [6] for the odd eigenvalues as derived by Venkov [8], which plays the role of Gutzwiller's periodic-orbit sum for our system:

$$\sum_{n=1}^{\infty} h(p_n)=\frac{1}{24}\int_{-\infty}^{+\infty} dp\, h(p)p\tanh(\pi p)+\frac{1}{4}\int_{-\infty}^{+\infty} dp\left[\frac{1}{4}+\frac{2}{3\sqrt{3}}\cosh\frac{\pi p}{3}\right]\frac{h(p)}{\cosh \pi p}$$
$$+\sum_{n=3}^{\infty}\sum_{k=1}^{\infty}\frac{g_n l_n}{4\sinh(kl_n/2)}g(kl_n)-\sum_{n=3}^{\infty}\sum_{k=0}^{\infty}\frac{j_n l_n}{4\cosh((k+1/2)l_n/2)}g((k+1/2)l_n))$$
$$-\frac{3}{4}g(0)\ln 2-\frac{1}{4\pi}\int_{-\infty}^{+\infty} dp\frac{\Gamma'(1/2+ip)}{\Gamma(1/2+ip)}h(p)\,. \tag{1}$$

Here all series and integrals converge absolutely under the following conditions on the smoothing function $h(p)$: (i) $h(-p)=h(p)$, (ii) $h(p)$ is holomorphic in the strip $|\mathrm{Im}p|\leq \frac{1}{2}+\epsilon$, $\epsilon>0$, and (iii) $h(p)=O(|p|^{-2-\epsilon})$ as $|p|\to\infty$. The function $g(x)$ is the Fourier transform of $h(p)$: $g(x)=(1/2\pi)\int_{-\infty}^{\infty}dp\, e^{ipx}h(p)$. On the left-hand side of (1) the sum runs over the odd eigenvalues parametrized by the momenta $p_n>0$, $E_n=\frac{1}{4}+p_n^2$. The two series on the right-hand side of (1) are the contributions from the periodic orbits whose primitive lengths on the full billiard are exactly given by $l_n=2\,\mathrm{arccosh}(n/2)$, $n=3,4,5,\ldots$. In the first series $g_n\in\mathbb{N}$ denotes the total multiplicity of the primitive periodic orbits on D having length l_n. The quantity $j_n\in\mathbb{N}_0$ gives the multiplicity of those primitive periodic orbits with length l_n which are symmetric with respect to the line $x=0$ ($j_n\equiv 0$ unless $n=m^2+2$, $m\in\mathbb{N}$).

To calculate the regularized trace of the Green's function of $\hat{H}$, we insert $h(p)=[p^2+(s-\frac{1}{2})^2]^{-1}-[p^2+(\sigma-\frac{1}{2})^2]^{-1}$ in (1) yielding (Res, Re$\sigma>1$)

$$\sum_{n=1}^{\infty}\left[\frac{1}{E_n+s(s-1)}-\frac{1}{E_n+\sigma(\sigma-1)}\right]=F(s)-F(\sigma)\,, \tag{2}$$

$$F(s)=\frac{1}{2}\frac{1}{2s-1}\left[\frac{Z'(s)}{Z(s)}-\frac{Y'(s)}{Y(s)}+G(s)\right]. \tag{3}$$

Here $Z(s)$ denotes the usual Selberg ζ function [6,7]

$$Z(s)=\prod_{n=3}^{\infty}\prod_{k=0}^{\infty}(1-e^{-(s+k)l_n})^{g_n},\quad \mathrm{Re}s>1\,, \tag{4}$$

while $Y(s)$ is a Selberg ζ function which was introduced in [10]

$$Y(s)=\prod_{n=3}^{\infty}\prod_{k=0}^{\infty}\left[\frac{(1-e^{-l_n[k+(1/2)s]})(1+e^{-l_n[k+(1/2)(s+1)]})}{(1-e^{-l_n[k+(1/2)(s+1)]})(1+e^{-l_n[k+(1/2)s]})}\right]^{j_n},\quad \mathrm{Re}s>1\,. \tag{5}$$

$G(s)$ is a meromorphic function in s whose only poles are in the left half-plane Re$s\le 0$. Since the left-hand side of (2) is a meromorphic function in s, Eq. (2) defines a meromorphic continuation of $F(s)$ and thus of the ratio $Z(s)/Y(s)$ to all $s\in\mathbb{C}$. Introducing yet another Selberg ζ function

$$Z^-(s)\equiv\left[\frac{Z(s)}{Y(s)}\right]^{1/2} \tag{6}$$

and letting the regulator σ in (2) go to $1+$, we arrive at the following representation for the logarithmic derivative of $Z^-(s)$ (Re$s>1$)

$$\frac{1}{2s-1}\frac{d}{ds}\ln Z^-(s)=\frac{H(s)}{2s-1}+\sum_{n=1}^{\infty}\left[\frac{1}{E_n+s(s-1)}-\frac{1}{E_n}\right]. \tag{7}$$

Here $H(s)$ is meromorphic, but has no poles for Re$s>0$. From (6), (4), and (5) one can derive the following Euler product representation

$$Z^-(s)=\prod_{\gamma}\prod_{k=0}^{\infty}(1-\chi_\gamma^{k+1}e^{-(s+k)l_\gamma})^{m_\gamma}, \tag{8}$$

where $\chi_\gamma=-1$, $l_\gamma=l_n/2$, and $m_\gamma=j_n$ for the symmetric periodic orbits, and $\chi_\gamma=+1$, $l_\gamma=l_n$, and $m_\gamma=(g_n-j_n)/2$ for the remaining ones. While (8) converges absolutely only for Re$s>1$, we infer from (7) that $Z^-(s)$ has an analytic continuation to all $s\in\mathbb{C}$. In fact, $sZ^-(s)$ is an entire function of s whose only zeros in the half-plane Re$s>0$ are located at $s=\frac{1}{2}\pm ip_n$, i.e., they lie on the "critical line" Re$s=\frac{1}{2}$ and the Riemann hypothesis is valid for $Z^-(s)$. Notice that the zero at $s=1$ which is present in both $Z(s)$ and $Y(s)$, and which is due to the ground-state energy $E_0=0$ of the full billiard, cancels out in $Z^-(s)$ [10]. This has led to the conjecture [11] that the "entropy barrier" at Re$s=1$, which we are forced to cross if we want to calculate the zeros on the critical line, and which is a serious obstruction in the Euler product (4), is "transparent" in the Euler product (8), allowing us in the most favorable case to compute $Z^-(s)$ on the critical line.

With the help of the functional equation for $Z^-(s)$ we deduce that the combination $Z^-(\frac{1}{2}-ip)e^{-i\pi\bar{N}^-(E)}$ is real on the critical line, where $\bar{N}^-(E)$ is the mean value of the spectral staircase $N^-(E)$ which counts the number of energy levels E_n smaller than or equal to E. From the small-t asymptotics of the trace of the heat kernel derived in [8], we obtain by using Theorem 6 of [12] the asymptotic expansion ($E\rightarrow\infty$)

$$\bar{N}^-(E)=\frac{1}{24}E-\frac{1}{4\pi}\sqrt{E}\ln E-\frac{3\ln 2-2}{4\pi}\sqrt{E}+\tfrac{23}{144}\,. \tag{9}$$

We are then led to define the real function [$p=(E-\frac{1}{4})^{1/2}>0$]

$$\xi^-(p)\equiv\mathrm{Re}\{Z^-(1/2-ip)e^{-i\pi\bar{N}^-(E)}\}\,, \tag{10}$$

whose zeros as a function of p are located exactly at the momenta p_n, and thus the condition $\xi^-(p)=0$ constitutes an *exact quantization rule* for the quantum energy levels $E_n=p_n^2+\frac{1}{4}$ of our billiard system.

For an evaluation of product (8) the multiplicities g_n and j_n are needed as an input. Recently, Schleicher [13] has calculated the first primitive hyperbolic conjugacy classes of SL(2,$\mathbb{Z}$), i.e., g_n, for $3\le n\le 32767$. The multiplicity j_n, i.e., the number of conjugacy classes of involution elements [8], has been calculated by us for $n=m^2+2$, $1\le m\le 32000$. In Fig. 1 we show $\xi^-(p)$ for $8.5\le p\le 60$, where the triangles mark the momenta p_n ($n=1,\ldots,73$) which are taken from Hejhal [9]. The solid lines have been obtained by evaluating (8), taking into account all periodic orbits with length $l_\gamma\le l_{n_{\max}}=19.360\ldots$, $n_{\max}=16000$, while the dotted lines correspond to the rather low cutoff $n_{\max}=12$. The solid lines show the expected oscillations, exhibiting zeros which are in very good agreement with the true eigenvalues. The dotted curves are similar, but the agreement with the true zeros becomes worse at higher energies. It is easy to see that all the information on the nontrivial zeros of $Z^-(s)$ is already contained in the $k=0$ term in Eq. (8), which might be called (the reciprocal of) a Ruelle-type ζ function. We have repeated the above calculations, keeping only the $k=0$ term. As expected, the results are nearly identical to the curves shown in Fig. 1. We conclude that it is indeed possible to cross the entropy barrier, as conjectured some time ago [11], and that our quantization condition based on (10) works also from a practical point of view.

To obtain a better understanding of the convergence properties, we transform [14] the Euler product (8) into the Dirichlet series

$$Z^-(s)=1+\sum_{n=1}^{\infty}A_ne^{-sL_n}\,. \tag{11}$$

Here the sum runs over all pseudo-orbits [15] whose

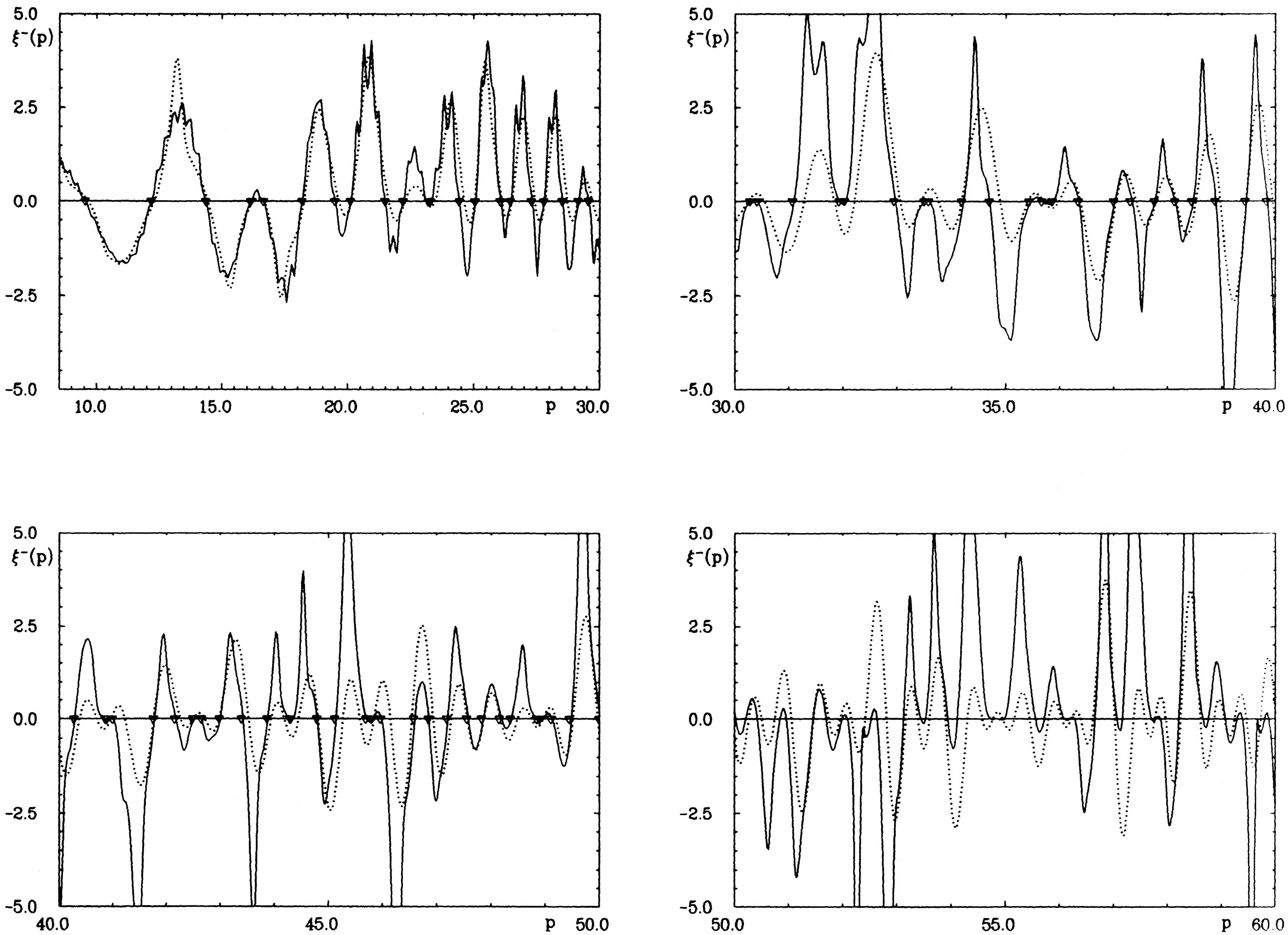

FIG. 1. The function $\xi^-(p)$ for $8.5 \le p \le 60$. The product (8) has been cut off at $n_{\max}=16000$ (solid lines) and $n_{\max}=12$ (dotted lines). The triangles mark the 73 momenta computed by Hejhal [9].

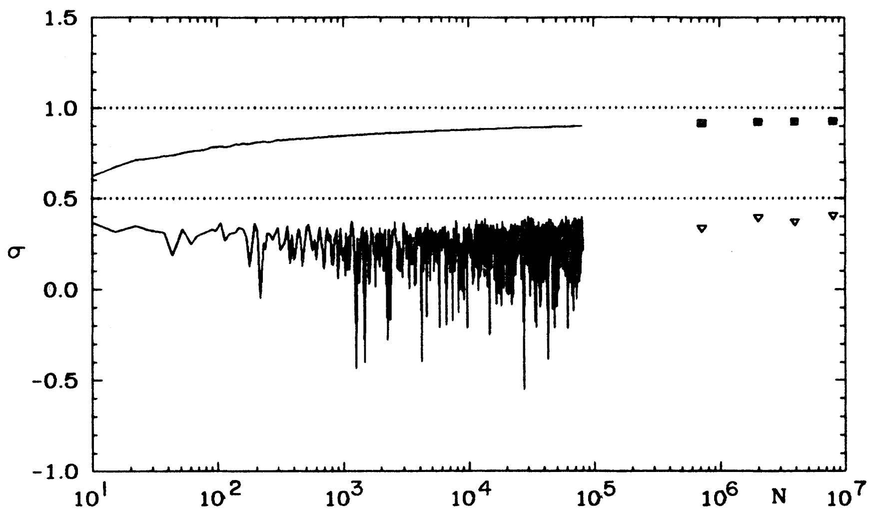

FIG. 2. The sequences in Eq. (13) for σ_a (upper solid symbols) and σ_c (lower open symbols).

lengths L_n ($L_n \le L_{n+1}$) are given by $L_n = \sum_{i=1}^{k} \alpha_i l_{\gamma_i}$ ($k \ge 1$, $\alpha_i \in \mathbb{N}$). For the coefficients A_n one obtains

$$A_n = \prod_{i=1}^{k} \frac{(-1)^{\alpha_i} \chi_{\gamma_i}^{(\alpha_i/2)(\alpha_i+1)} e^{-(\alpha_i/2)(\alpha_i-1)l_{\gamma_i}}}{\prod_{r=1}^{\alpha_i} \left(1 - \chi_{\gamma_i}^{r} e^{-r l_{\gamma_i}}\right)} . \qquad (12)$$

The abscissas of absolute convergence σ_a and of convergence σ_c of the Dirichlet series (11) are determined by

$$\sigma_a = \overline{\lim_{N\to\infty}} \frac{1}{L_N} \ln \sum_{n=1}^{N} |A_n| , \quad \sigma_c = \overline{\lim_{N\to\infty}} \frac{1}{L_N} \ln \left| \sum_{n=1}^{N} A_n \right| . \qquad (13)$$

In Fig. 2 we have plotted the two sequences which appear in Eq. (13) for $N = 10$ to 8×10^6, where the last figure is identical to the number of all pseudo-orbits with length $L_n \le 17.034\ldots$. The solid symbols, which belong to σ_a, seem to approach 1 for $N\to\infty$ in agreement with the exact value $\sigma_a = 1$ for the entropy barrier. The lower open symbols lie below $\frac{1}{2}$, which may serve to indicate that $\sigma_c < \frac{1}{2}$ and thus that the Dirichlet series (11) converges for $\mathrm{Re}\, s \ge \frac{1}{2}$, i.e., on the critical line. Of course, we cannot exclude the possibility that the true limit is $\sigma_c = \frac{1}{2}$, for example, in which case the series (11) would converge only for $\mathrm{Re}\, s \ge \frac{1}{2} + \epsilon$, $\epsilon > 0$. In any case, Fig. 2 strongly suggests that (11) converges for $\frac{1}{2} < \mathrm{Re}\, s < 1$, which implies that we can cross the entropy barrier. We have also calculated $\xi^-(p)$ by inserting the Dirichlet series (11) into (10) using a cutoff $L_{\max} = 17.034\ldots$ for the pseudolengths. The result is practically indistinguishable from the solid curves shown in Fig. 1.

Finally, let us note that one can estimate from Fig. 1 the eigenvalues in the range $50 \le p \le 60$, which is beyond Hejhal's computation [9].

We would like to thank Jens Bolte for useful discussions. This work was supported by the Deutsche Forschungsgemeinschaft under Contract No. DFG-Ste 241/4-3.

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