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# Exact scaling function of interface growth dynamics

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Strong-coupling properties of the Burgers–Kardar-Parisi-Zhang equation describing nonequilibrium interface growth are studied. A physical coupling constant is defined and related to the height-height correlation function for arbitrary substrate dimension. In 1+1 dimensions, the *exact* universal coupling constant and scaling function are computed using a mode-coupling theory. It is found that a finite surface tension is generated from zero, suggesting that macroscopic properties are not affected by the absence of microscopic surface tension.

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The problem of nonequilibrium interface growth has attracted much attention in the past several years [1]. Basic knowledge of the roughness of growing crystalline facets has obvious technological applications [2], while the scale-invariant aspect of surface morphologies share similarities to fractal landscapes in nature and continue to generate curiosity [3]. A widely accepted description of the macroscopic aspects of these growth processes is a simple Langevin equation due to Kardar, Parisi, and Zhang [4] (KPZ). Because the KPZ equation is the simplest nonlinear generalization of the ubiquitous diffusion equation, it also appears in many other problems of nonequilibrium dynamics such as the randomly-stirred fluid [5] (Burgers equation), dissipative transport [6] (the driven-diffusion equation), and flame-front propagation [7] (Kuramoto-Sivashinski equation). Through a simple transformation, the KPZ equation also describes the fluctuation of directed paths in random environment [8], a simplified spin-glass problem which contains much of the essential physics of disordered systems [9]. Needless to say, any advances in understanding the behavior of the KPZ equation may have a broad impact in both the fields of nonequilibrium dynamics and disordered systems.

Assuming that a coarse-grained interface of a $d$-dimensional substrate may be described by a height function $h(\mathbf{r},t)$ with $\mathbf{r}\in\mathbb{R}^d$, KPZ proposed that the following Langevin equation,

$$\frac{\partial h}{\partial t}=\nu\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2+\eta(\mathbf{r},t)\,, \qquad (1)$$

governs the macroscopic (large distance, long time) behavior of the interface. The first term in Eq. (1) represents surface tension which prefers a smooth surface. The second term originates from the tendency of the surface to (locally) grow normal to itself and is nonequilibrium in origin. The last term is a Langevin noise to mimic the stochastic nature of any growth process. In Eq. (1), the average growth velocity has been subtracted so that the noise has zero mean, i.e., $\langle\eta(\mathbf{r},t)\rangle=0$. The stochasticity is then described in the simplest case by an uncorrelated Gaussian noise with the second moment $\langle\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle=2D\delta^d(\mathbf{r}-\mathbf{r}')\delta(t-t')$, where $D$ characterizes the noise amplitude. Note that the continuum description (1) contains an *effective* length scale $a=(\nu^3/\lambda^2 D)^{1/(2-d)}$ but does not have an *explicit* microscopic cutoff. This description is valid as long as $a$ is large compared to the actual lattice cutoff $a_0$ present in a real system.

The subject of interest is the steady-state surface profile, which is characterized by the steady-state distribution function $P(\{h(\mathbf{r},t)\})$ (after sufficient time to insure the decay of any transient behaviors due to initial conditions). A somewhat less ambitious goal is the leading (second) moment of $P$, the truncated two-point correlation function

$$C(\mathbf{r},t)\equiv\langle[h(\mathbf{r}_0+\mathbf{r},t_0+t)-h(\mathbf{r}_0,t_0)]^2\rangle_{t_0\to\infty}\,. \qquad (2)$$

Since the equation of motion (1) is scale invariant in the limit of long distance and time (a consequence of a translational symmetry $h\to h+\text{const}$ in the growth direction) [4], the correlation function satisfies the scaling form $C(r,t)=r^{2\chi}F(t/r^z)$, where $r\equiv|\mathbf{r}|\gg a$, $\chi$ describes the scaling of the interface width, and $z$ characterizes the spread of surface disturbances. The scaling function $F$ has the usual limits to give the correlation function the following asymptotic scaling form [10],

$$C(r,t=0)=Ar^{2\chi},\quad C(r=0,t)=Bt^{2\chi/z}\,. \qquad (3)$$

Much of the work comes down to obtaining the numerical values of the two exponents $\chi$ and $z$. Due to a Galilean invariance [4], the exponents are linked by an identity $\chi+z=2$. In 1+1 dimensions ($d=1$), Eq. (1) also satisfies a fluctuation-dissipation theorem [11] (FDT) from which both exponents are obtained exactly, $\chi=\frac{1}{2}$ and $z=\frac{3}{2}$. A more systematic approach is via a dynamic renormalization group [12] (DRG). In a typical DRG analysis, one obtains the RG flow (or the beta function) of a dimensionless coupling constant which provides a measure of the effective nonlinearity present in the system. In this problem, the coupling constant is $g(b)=[\lambda^2(b)\times D(b)/\nu^3(b)]^{1/2}b^{(2-d)/2}$, where $b$ is the scale of coarse graining, and $\lambda(b)$, $D(b)$, and $\nu(b)$ are renormalized parameters. The scaling exponents are then evaluated at the fixed point of RG flows using the fixed point value of the coupling constant $g^*=g(b\to\infty)$. Physically, the universal coupling constant $g^*$ gives the crossover scale to asymptotic scaling behavior [13]. It is also related to the crossover point of the correlation function $C(\mathbf{r},t)$ as will be shown shortly. A systematic solution is possible if $g^*$ can be accessed perturbatively. However, such a scheme has not been found for the KPZ equation, and, instead, one encounters a strong-coupling fixed point: Naive RG calculations [4] failed to produce a controlled stable fixed point which describes the rough interface.

This problem persists even in d=1 where the exponents themselves are already well known. We reemphasize that the exact knowledge of exponents in 1+1 dimensions are due to "coincidences" (FDT and Galilean invariance), rather then the success of a systematic renormalization-group theory. For instance, it is not known whether the RG flow takes the system to finite or infinite coupling in d=1. Many numerical studies of interface growth models [1,14] are performed without any microscopic surface tension $\nu$, corresponding to an infinite bare coupling constant. Since the effective length scale $a \sim \nu^{1/3}$ in 1+1 dimensions, $\nu=0$ implies the breakdown of the continuum description (1). Therefore, one does not even know *a priori* whether the behavior of interfaces with zero and finite surface tension are controlled by the same fixed point. One possibility as recently suggested by Golubovic and Bruinsma [15] is that the interface may undergo a phase transition to some *unstable* region at a small but positive value of $\nu$.

In this paper we address the properties of the strong-coupling fixed point, e.g., the value of the universal coupling constant and the form of the scaling function. We start with a physical definition of the coupling constant and explain how it may be measured numerically or experimentally. We explore the connection to the dynamic correlation function (2) and derive the dimensionless form of the universal scaling function for *arbitrary spatial dimension d.* The procedure is explicitly demonstrated in d=1. There, a self-consistent mode-coupling theory becomes exact due to a FDT and Galilean invariance. From the numerical solution of the self-consistent equation, we obtain the *exact* universal scaling function and the dimensionless coupling constant. Furthermore, this method allows a nonperturbative analysis of the exact RG flow of various renormalized quantities, including the case of a zero microscopic surface tension where the usual (perturbative) methods breakdown.

We first propose a numerically useful definition of the universal coupling constant $g^*$. Since the main effect of the nonlinear term in Eq. (1) is to introduce an inclination-dependent growth rate, we can get a measure of the nonlinearity present by looking at the incremental growth velocity upon a small uniform tilt of the substrate,

$$\delta v(\nabla h) = \tfrac{1}{2}\lambda(\nabla h)^2 . \quad (4)$$

We would like to express the definition in terms of observables such as the measured asymptotic correlation function Eq. (3). For a large system of size $L$, we pick a pair of points a distance $b$ apart with $L \gg b \gg 1$. We uniformly tilt the surface by an angle $\theta_b = \{C(r=b,t=0)]^{1/2}/b$, according to the typical height fluctuation. Next, the resulting growth velocity increment needs to be put in terms of physical units. We choose to normalize it in units of $v_b = [C(r=0,t=\tau_b)]^{1/2}/\tau_b$, where $\tau_b$ is the associated correlation time. We can take $\tau_b$ to be the time scale at which $C(b,t)$ crosses over from being $b$ dependent and $t$ dependent. For correlation functions given by Eq. (3), this corresponds to $\tau_b = (A/B)^{z/2\chi} b^z$. Putting the above together, we give the following definition for the physical coupling constant,

$$g(b) \equiv \delta v(\theta_b)/v_b . \quad (5)$$

Universality of the KPZ equation and the growth processes it describes necessarily implies that the above quantity is universal. If we compute $g(b)$ defined in Eq. (5) using the equation of motion (1), we will find that it is precisely the dimensionless coupling constant used in DRG analysis [13].

We now relate the universal coupling constant more directly to the correlation function $C(\mathbf{r},t)$. Using Eq. (5) and the exponent identity, the temporal correlation function in Eq. (3) is rewritten as

$$C(0,t) = (2g^*)^{-2\chi/z} [\lambda A^{1/\chi} t]^{2\chi/z} ,$$

where $A$ is the amplitude of the equal-time correlation function in (3). This gives the following form of the full correlation function,

$$C(r,t) = A r^{2\chi} F(\lambda \sqrt{A}\, t/r^z) , \quad (6)$$

which is valid in *arbitrary substrate dimension d.* The argument of the scaling function is now dimensionless, and the scaling function itself is universal. It has the asymptotic form $F(\xi \to 0) = 1$ and $F(\xi \to \infty) = (\xi/2g^*)^{2\chi/z}$. Here we see that $g^*$ plays the role of the crossover scale between the correlation function's space-dependent and time-dependent regimes. It may also be expressed in terms of a universal ratio of the amplitudes of Eq. (3) as

$$g^* = \frac{\lambda}{2} \left[ \frac{A}{B^{z/2}} \right]^{1/\chi} . \quad (7)$$

The above relation can be readily used to find the universal values of phase transition and strong-coupling fixed points in numerical studies of the KPZ equation above 2+1 dimensions. We also note that although our analysis is carried out in the context of the KPZ equation, the connection between $g^*$ and the crossover scale of the dynamic correlation function is a much more general result for systems obeying dynamic scaling of the form Eq. (3). [Though of course $g^*$ would generally take on a form more complicated than Eq. (7) in the absence of an exponent identity.]

In the remainder of this paper, we make the above discussion concrete by explicitly solving the scaling function in d=1. As already discussed, the (1+1)-dimensional growth problem is greatly simplified due to a FDT and Galilean invariance. In this case, one can prove that a self-consistent mode-coupling theory is exact in the macroscopic limit [13]. The self-consistent equations for the correlation function $C(k,\omega)$ and the response function $G(k,\omega)$ in Fourier space are

$$C(k,\omega)=C_0(k,\omega)+2|G(k,\omega)|^2\left(\frac{\lambda}{2}\right)^2\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}d\mu k_+^2 k_-^2 C(k_+,\omega_+)C(k_-,\omega_-)\,,$$
$$G^{-1}(k,\omega)=G_0^{-1}(k,\omega)+4\left(\frac{\lambda}{2}\right)^2\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}d\mu\, k_+^2 k k_- C(k_+,\omega_+)G(k_-,\omega_-)\,, \tag{8}$$

where $k_\pm=k/2\pm q$, $\omega_\pm=\omega/2\pm\mu$, and $C_0$ and $G_0$ are the bare functions with $\lambda=0$.

The mode-coupling equations have been previously used [6,16] to obtain the scaling exponents $\chi$ and $z$. Here we show that it can be used to obtain the entire scaling function [17]. To solve these equations, we define $G(k,\omega)=[\nu(k,\omega)k^2-i\omega]^{-1}$, $C(k,\omega)=2D(k,\omega)|G(k,\omega)|^2$, and look for solution of the form $D(k,\omega)=\lambda(D_0/\nu_0)^{3/2}\times k^{-1/2}f(\hat{\omega})$, and $\nu(k,\omega)=\lambda(D_0/\nu_0)^{1/2}k^{-1/2}f(\hat{\omega})$, where $\hat{\omega}=\omega/\lambda(D_0/\nu_0)^{1/2}k^{3/2}$ is a dimensionless variable, and $D_0$ and $\nu_0$ are related to the bare parameters. In the macroscopic limit $k,\omega\rightarrow 0$, the bare terms $G_0,C_0$ can be neglected. We find that the self-consistent equations become reduced to a one-variable integral equation that is straightforwardly solved numerically. From the solution, we find the truncated correlation function in real space to be

$$C(r,t)=ArF(\lambda\sqrt{A}t/r^{3/2})\,, \tag{9}$$

where $F(\xi)$ is the universal scaling function and is shown in Fig. 1. The dimensionless argument of $F$ has the form demanded by Eq. (6) with $z=\frac{3}{2}$. The dimensionless coupling constant can be read off from the crossover point of $F(\xi)$ (see Fig. 1). We obtain $g^*=0.87$. This result can be checked more precisely in simulations by directly looking at the scaling amplitudes. Our work thus predicts that if $C(r,t=0)=Ar$, then $C(r=0,t)=0.69(\lambda A^2 t)^{2/3}$, with $\lambda$ defined by Eq. (4). The numerical error is $\pm 2\%$. Note that the above result is valid for very large systems in *steady state.* Transient behaviors [14] such as the growth of interfacial width starting from flat initial conditions may be more complicated but can be computed in the same spirit.

This completes the description of the system's asymptotic scaling behavior at the RG fixed point. The mode-coupling theory is next extended to include the *approach* to fixed point starting from a microscopic theory. In accordance with the spirit of the renormalization group, we only allow a portion of modes to interact by including a cutoff factor $\exp[-(qL)^{-2}]$ in the integrands of the self-consistent equations. This also approximates studying finite systems of size $L$. The correlation function and response function are now explicitly $L$ dependent. They may be described in terms of

$$D_L(k,\omega)\sim\nu_L(k,\omega)\sim L^{1/2}f(\hat{\omega},kL)\,, \tag{10}$$

where $f$ now satisfies a two-variable integral equation. The exact *functional* RG of $D(k,\omega)$ and $\nu(k,\omega)$ can be obtained from the full solution of the integral equation; the flow behavior of $D$ and $\nu$ are recovered at $k,\omega=0$. The asymptotic form $D_L\sim\nu_L\sim L^{1/2}$ is, of course, the expected one given the exponents $\chi$ and $z$, and can be directly measured [18]. Here we want to emphasize that the self-consistent equations provide a connection between the microscopic and macroscopic (renormalized) theory.

The foregoing analysis is valid as long as the microscopic surface tension $\nu$ is finite. As already mentioned, the continuum equation of motion (1) breaks down in the absence of surface tension, and one does not know whether systems with zero and finite surface tension are controlled by the same fixed point. To understand this better, we take a closer look at the RG flow in the vicinity of $\nu=0$. We use a discrete description by imposing a lattice cutoff $a_0$. The self-consistent equations are then integrated to the edge of the first Brillouin zone $(2\pi/a_0)$ and take on the boundary conditions $\nu_{b=a_0}(k,\omega)=0$ and $D_{b=a_0}(k,\omega)=D$, where $b$ is the scale of coarse graining. At the onset of flow, we can hold $D_b$ constant and solve for $\nu_b$. We find that a surface tension of the order $\nu_b\sim(\lambda^2 D a_0)^{1/3}$ is generated upon a very small amount of coarse graining $[\ln(b/a_0)\sim 1]$. This indicates that a finite surface tension is immediately generated starting from zero [19]. Hence interfaces with or without bare surface tension are indeed controlled by the *same* RG fixed point and therefore share the same universal behaviors. This conclusion is in direct contradiction with the suggestion of a phase transition [15] at small $\nu$. Our result can be explicitly checked numerically by comparing the values of the universal fixed point $g^*$ for models with zero and finite microscopic surface tension.

Finally, we note that if systems with zero and finite bare $\nu$ flow to the same fixed point, then the basin of attraction of the fixed point must also include systems with a small but *negative* bare $\nu$. Equation (1) with $\nu<0$ is essentially the Kuromoto-Sivashinski (KS) equation [7] with noise. If we take the hypothesis [20] that an effective noise can be generated by the deterministic KS equation (since random initial conditions are amplified by linear instabilities), then our result suggests that the deterministic KS

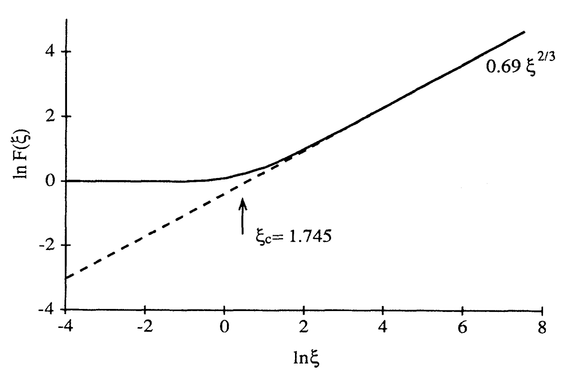


FIG. 1. The exact universal scaling function $F(\xi)$. The dashed line indicates the asymptotic scaling form: $F(\xi\rightarrow\infty)=0.69\xi^{2/3}$. The crossover point is $\xi_c=2g^*=1.74$. Numerical error is $\pm 2\%$.

equation can also flow to the fixed point describing the noisy KPZ equation. Numerical evidences in support of such possibilities can be found in Ref. [21].

In summary, we have analyzed the steady-state behavior of growth dynamics described by the KPZ equation. We provide a physical definition of the dimensionless coupling constant $g^*$. We then relate $g^*$ to the dynamic correlation function and give the dimensionless form of the universal scaling function for *arbitrary substrate dimension d.* We solve this function explicitly in $d=1$ using an exact mode-coupling theory. The predicted value of the universal coupling constant is $g^*=0.87$. The theory is extended to provide a link between the microscopic and macroscopic growth dynamics. We find that a finite surface tension can be generated from zero, suggesting that interfaces with and without microscopic surface tension exhibit the same macroscopic behavior (rather than being separated by a phase transition as suggested in Ref. [15]). We also observe the logical implication of our result to the behavior of the Kuromoto-Sivashinski equation. This knowledge should help towards developing a deeper understanding of many subtle issues encountered in the studies of nonequilibrium dynamics and disordered systems, particularly in the active and controversial field of directed random paths.

*Note added.* After this work was completed, we received from Amar and Family a related numerical study of the scaling functions for the *width* fluctuation of $(1+1)$-dimensional interfaces. They reported values of a universal amplitude ratio $R$ ranging from $R=3.5\pm0.2$ from simulation of three different models to $R\approx3.2$ from a direct simulation of the KPZ equation. For a similar (i.e., global) scaling function, our theory yields $R=3.6\pm0.1$, in good agreement with the numerical result of Amar and Family. Our result is also superior to the one ($R\approx3.9$) obtained by Amar and Family from integration of the *uncontrolled* one-loop RG equation as expected.

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[1] For a recent review, see, J. Krug and H. Spohn, in *Solids Far From Equilibrium: Growth, Morphology and Defects,* edited by C. Godriche (Cambridge Univ. Press, New York, 1990).

[2] See, e.g., H. van Beijeren and I. Nolden, in *Structure and Dynamics of Surfaces II,* edited by Schommers and P. von Blakenhagen (Springer-Verlag, Berlin, 1987). See also T. Hwa, M. Kardar, and M. Paczuski, Phys. Rev. Lett. **66**, 441 (1991), for a discussion of the nonequilibrium aspect.

[3] B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).

[4] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); E. Medina, T. Hwa, M. Kardar, and Y.-C. Zhang, Phys. Rev. A **39**, 3053 (1989).

[5] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A **16**, 732 (1977).

[6] H. van Beijeren, R. Kutner, and H. Spohn, Phys. Rev. Lett. **54**, 2026 (1985); H. K. Janssen and B. Schmittmann, Z. Phys. B **63**, 517 (1986).

[7] Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. **55**, 356 (1977); G. I. Sivashinsky, Acta Astronaut. **6**, 569 (1979), and references therein.

[8] D. A. Huse, C. L. Henley, and D. S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985); M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).

[9] For a recent review, see D. S. Fisher and D. A. Huse (unpublished). See also G. Parisi, J. Phys. (Paris) **51**, 1595 (1990); M. Mezard, *ibid.* **51**, 1831 (1990).

[10] For brevity, we use the notation $C(r,t=0)$ and $C(r=0,t)$ for the asymptotic scaling forms $C(r\rightarrow\infty,t)$ and $C(r,t\rightarrow\infty)$ throughout the text.

[11] U. Dekker and F. Haake, Phys. Rev. A **11**, 2043 (1975).

[12] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977), and references therein.

[13] E. Frey, T. Hwa, and D. S. Fisher (unpublished).

[14] See, e.g., F. Family, Physica A **168**, 561 (1990), and references therein.

[15] L. Golubovic and R. Bruinsma, Phys. Rev. Lett. **66**, 321 (1991).

[16] J. Krug, Phys. Rev. A **36**, 5465 (1987).

[17] This method can also be used to obtain the exact scaling function for the driven-diffusive system [6] in any dimension.

[18] D. E. Wolf and L.-H. Tang, Phys. Rev. Lett. **65**, 1591 (1990).

[19] We thank T. Nattermann for bringing to our attention an earlier suggestion of this result. See T. Nattermann and W. Renz, Phys. Rev. B **38**, 5184 (1988).

[20] V. Yakhot, Phys. Rev. A **24**, 642 (1981).

[21] S. Zalesky, Physica D **34**, 417 (1989), and references therein.