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Finite-temperature directed polymers in a random potential

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The phase transition of directed polymers in a random potential has been studied by both weak-disorder expansions and numerical simulations. In dimension d=2+1, no evidence for a finite-temperature phase transition was found. Instead, the crossover length t^* is very large at high temperature T and has been found to vary as $\ln t^* \sim T^2$. In d=3+1, transfer-matrix studies show the existence of a phase transition and some of the critical exponents have been determined.

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The directed polymer in a random potential represents one of the simplest problems involving disorder [1]. It is related to a wide variety of other apparently different systems such as the growth of an interface in the Eden model [2], ballistic aggregation [3], randomly stirred fluids [4], domain walls in the two-dimensional random bond Ising model [5], and Burgers's equation [6]. Furthermore it shares many common characteristics with the spin-glass problem. Most recent work concentrates on determining the exponents which characterize the low-temperature (strong-coupling) phase and looking for a phase transition between a high-temperature phase and a low-temperature phase [7]. In dimension d=2 there is no phase transition and only the strong-coupling phase exists. (We shall usually write d=2 as d=1+1 to indicate that there is one transverse and one longitudinal direction.) However, there is a controversy about the existence of a phase transition in d=2+1 [8–13]. Recent simulations on ballistic growth models [9, 10] have been interpreted in terms of a finite-temperature phase transition and this result was supported by other numerical simulations on directed polymers [11] and a driven solid-on-solid model [12]. However, a renormalization-group (RG) analysis of the related Kardar-Parisi-Zhang (KPZ) equation shows that the infinite-temperature fixed point is unstable implying no phase transition [14, 8]. Here, we study the directed polymer in a random potential by both weak-disorder expansions and numerical simulations. We find no evidence of a phase transition in d=2+1. Instead, there is an extremely large crossover length t^*, viz. $\ln t^* \sim T^2$ for high temperature T such that a polymer of length t follows a random walk in the transverse direction when $t \ll t^*$ but crosses over to the strong-coupling (zero-temperature) characteristics when $t \gg t^*$. In d=3+1, it is generally believed that there is a phase transition between high-temperature and low-temperature phases. We find the transition temperature T_c, and show that the fluctuations of free energy ΔF grow as $(\Delta F)^2 \sim \ln t$ at T_c. We also measure the longitudinal correlation length exponent $\phi \approx 4$, viz. $t^* \sim |T-T_c|^{-\phi}$, by finite-size scaling. In d=1+1, the crossover length t^* grows as $t^* \sim T^4$ at high temperature and, as expected, no phase transition was found.

Consider a directed polymer on a discrete "hyperpyramid" structure with random potential $\mu(\mathbf{x},t)$ assigned to each site $(\mathbf{x},t)$ where $\mathbf{x}$ is the (d-1)-dimensional transverse vector and t is the longitudinal length of the polymer in the direction for which no reverse step is allowed. The walk starts from $\mathbf{x}(t=0)=\mathbf{0}$ and its path is restricted by $|\mathbf{x}(t)-\mathbf{x}(t+1)| = 0$ or 1 . There is a bending energy γ against a transverse jump $|\mathbf{x}(t)-\mathbf{x}(t+1)|=1$. We shall study the partition function $Z(\mathbf{x},t)$ for the polymer ending at $(\mathbf{x},t)$, which can be obtained recursively. For example, in d=1+1

$$\begin{aligned} Z(x,t) = {} & Z(x,t-1)\exp\left(-\frac{\mu(x,t-1)}{T}\right) \\ & + Z(x-1,t-1)\exp\left(-\frac{\mu(x-1,t-1)+\gamma}{T}\right) \\ & + Z(x+1,t-1)\exp\left(-\frac{\mu(x+1,t-1)+\gamma}{T}\right) , \end{aligned} \tag{1}$$

where T is the temperature and the $\{\mu(\mathbf{x},t)\}$ are uncorrelated random variables which satisfy

$$\langle \mu(\mathbf{x},t)\mu(\mathbf{x}',t')\rangle = 2D\delta_{\mathbf{x},\mathbf{x}'}\delta_{t,t'} . \tag{2}$$

If we consider all the walks starting from $(\mathbf{0},0)$, we can define the total partition function $Z(t)$ as

$$Z(t) = \sum_{\mathbf{x}} Z(\mathbf{x},t) \tag{3}$$

and the free energy $F(t)$ as $F(t) = -T\ln Z(t)$. In considering the polymer at finite temperature, we take the symbol $\langle A\rangle$ as the thermal average of a quantity A by

$$\langle A\rangle \equiv \frac{\sum_{\mathbf{x}} A Z(\mathbf{x},t)}{\sum_{\mathbf{x}} Z(\mathbf{x},t)} \tag{4}$$

and $\overline{A}$ as the sample average of A.

There are two commonly studied quantities in the directed polymer problem. One is the free-energy fluctuation $\Delta F(t)$:

$$\Delta F(t) = [\overline{F^2(t)} - \overline{F(t)}^2]^{1/2} . \qquad (5)$$

The other is a measure of the transverse fluctuation of the polymer such as $\overline{\langle \mathbf{x}\rangle^2}$ or $\overline{\langle \mathbf{x}^2\rangle}$. These quantities are characterized by the scaling exponents ω and z defined as

$$\Delta F(t) \sim t^{\omega} , \quad \overline{\langle \mathbf{x}^2\rangle} \sim t^{2/z} . \qquad (6)$$

There is a scaling relation $2/z - \omega = 1$ in the low-temperature phase [15]. The values of z and ω in dimensions higher than d=1+1 are under debate [7, 16].

At infinite temperature the disorder plays no role: the polymer follows a random walk with z=2 and $\langle \mathbf{x}\rangle$ is zero by symmetry. At low temperature, the walk is attracted by the local potentials such that it becomes superdiffusive. To see the transition between these two phases, we define a dimensionless quantity

$$g(T,t) \equiv \overline{\langle \mathbf{x}\rangle^2}/\,\overline{\langle \mathbf{x}^2\rangle} , \qquad (7)$$

which depends on T and t. It is similar to Binder's g factor defined in spin systems [17]. At $T = \infty$, $g = 0$. On the other hand, when $T = 0$, $\langle \mathbf{x}^2\rangle = \langle \mathbf{x}\rangle^2$ and g is unity. If there is a phase transition at T_c, the variation of g near T_c can be written in the finite-size scaling form

$$g = g^*(t^{1/\phi}(T - T_c)) , \qquad (8)$$

where ϕ is a longitudinal correlation length exponent and g^* is a scaling function. This implies that g is independent of the length t at T_c and the longitudinal correlation length t^* diverges as $t^* \sim |T - T_c|^{-\phi}$.

At high temperature, the calculation of g is straightforward and to first order in D/T^2 gives

$$g(T,t) = \frac{2\alpha}{(5-d)(8\pi)^{(d-1)/2}T^2}\,t^{(3-d)/2} , \qquad (9)$$

where $\alpha = D[e^{\gamma/T} + 2(d-1)]^{(d-1)/2}$. Eq. (9) indicates that $d = 2 + 1$ is a critical dimension. Below the critical dimension, g increases with t for fixed T showing no stable high-temperature phase. For $d > 2 + 1$, since g approaches zero with t at high temperature, there exists a stable high-temperature phase. At d=2+1, the value of g is independent of length t in the first-order calculation. If we add the second-order correction, g becomes

$$g(T,t) \sim \frac{D}{T^2}\left(1 + C\frac{D\ln(t/t_0)}{T^2}\right) , \quad d = 2 + 1 \qquad (10)$$

where C is a positive constant and t_0 is a cutoff length. Since g grows logarithmically with t, the infinite-temperature fixed point is unstable, and the crossover length t^* to the low-temperature phase is very big, $\ln t^* \sim T^2$. The same conclusion has effectively been reached in the RG analysis of the KPZ equation [18].

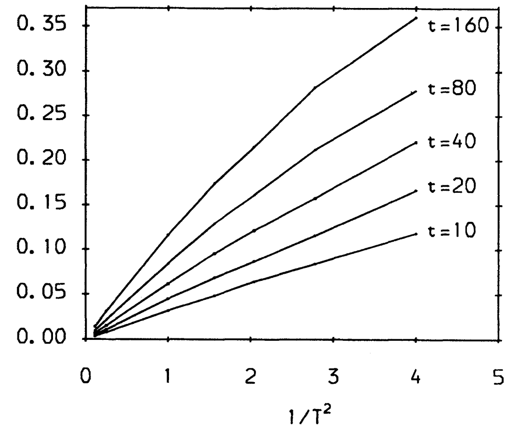

FIG. 1. $g(T,t)$ as a function of $1/T^2$ for polymer length t=10, 20, 40, 80, and 160 in d=1+1.

From the transfer-matrix method as described in Eq. (1), we also calculated g numerically for systems of length t=200 (d=1+1), 100 (d=2+1) and 32 (d=3+1). Uniformly distributed random numbers in the interval $(-\frac{1}{2}, \frac{1}{2})$ were used for the $\{\mu(\mathbf{x},t)\}$ and 20 000 different random configurations were collected to take a sample average, with γ=0.5. In d=1+1, $g(T,t)$ is given in Fig. 1 as a function of $1/T^2$ for fixed t=10, 20, 40, 80, and 160. Since the curves for different lengths do not cross each other, we infer that there is no phase transition. At high temperature, the curves are very straight and their slopes are $0.0099\sqrt{t}$ in good agreement with $0.0096\sqrt{t}$ from Eq. (9) with $\gamma \ll T$. This indicates that the crossover length $t^*(T)$ grows as T^4. This is consistent with the renormalization-group analysis of the KPZ equation but disagrees with previous numerical results [12]. To check this further, we assume a scaling form $g = f(\sqrt{t/t^*(T)})$ in d=1+1 and adjust $t^*(T)$ to collapse the data. As shown in Fig. 2, the data collapse is reasonably good for a wide range of temperatures. From the plot of $\ln t^*$ versus $\ln T$, we get a slope $\phi = 4.0 \pm 0.2$ in good agreement with $t^*(T) \sim T^4$.

In d=2+1, $g(T,t)$ versus $1/T^2$ is shown in Fig. 3 for t=6, 12, 24, 48, and 96. The slope s of $g \approx s/T^2$ in

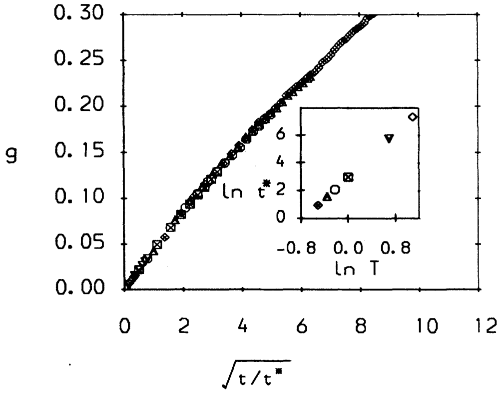

FIG. 2. Scaling plot of $g(T,t)$ as a function of $\sqrt{t/t^*(T)}$ in d=1+1 for T=0.6, 0.7, 0.8, 1.0, 2.0, and 3.0. The inset shows a plot of $\ln t^*(T)$ vs $\ln T$ whose slope is 4.0 ± 0.2.

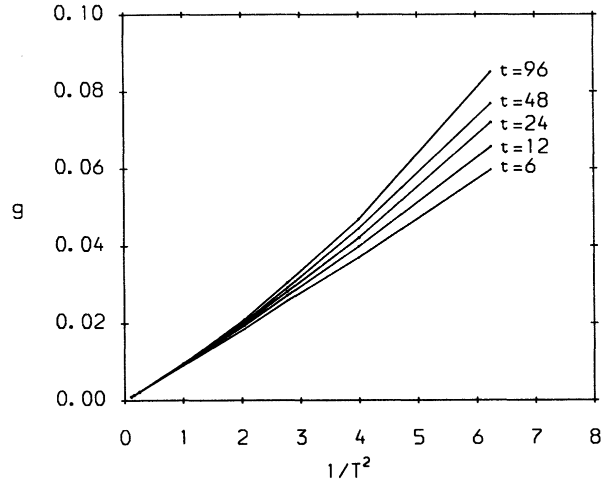

FIG. 3. $g(T,t)$ as a function of $1/T^2$ for length t=6, 12, 24, 48, and 96 in d=2+1.

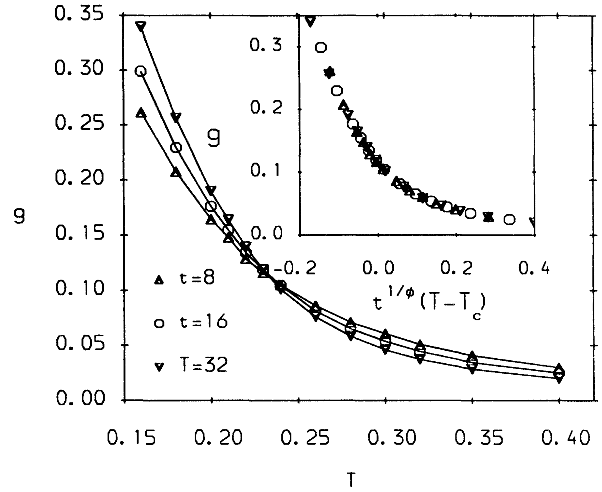

FIG. 5. $g(T,t)$ as a function of T for t=8, 16, and 32 in d=3+1. The inset shows a scaling plot of g vs $(T-T_c)t^{1/\phi}$, with $T_c = 0.23$ and $\phi = 4.0$.

the high-temperature regime is 0.0086 which is in good agreement with the value 0.0083 of Eq. (9) with $\gamma \ll T$. One may argue that there is a phase transition, because all the curves for the different lengths seem to meet each other near T=1 and then stick together up to infinite temperature with $g(T=1) \neq g(T \to \infty)$. However, our data show that $g(T=1,t)$ is increasing very slowly with t, being consistent with Eq. (10) with the $(\ln t)/T^4$ term included. Figure 4 shows the data collapse of $T^2 g$ versus $[\ln(t/t_0)]/T^2$ where t_0 is a cutoff length the value of which is around 3.7 for the best data collapse. We conclude therefore that there is no phase transition. Instead, the crossover length t^* is extremely large, following $\ln t^* \sim T^2$.

This large crossover length in $d = 2+1$ probably accounts for the discrepancy between our results and those of Derrida and Golinelli [11], who interpreted their data in $d = 2+1$ in terms of a phase transition, on the basis that the size dependence of the data at the higher temperatures was consistent with the behavior expected in the weak-coupling phase. In our view, however, it is very likely that, due to the existence of the large crossover length from weak- to strong-coupling behavior, the higher-temperature data had not yet accessed (for the system widths studied) the strong coupling regime. Actually, the same is true of our data, as is clear from Fig. 3 where the values of g are much smaller than unity for all temperatures and times displayed: in the strong-coupling phase $g \to 1$ for $t \to \infty$. From Fig. 3, however, it is clear that g is systematically increasing with time for nearly all temperatures studied, indicating that these temperatures are certainly in the strong coupling phase. Furthermore, at the highest temperatures, where no trend is discernible in Fig. 3, the data are well described (see the scaling plot of Fig. 4) by the weak-disorder expansion (10), which it-

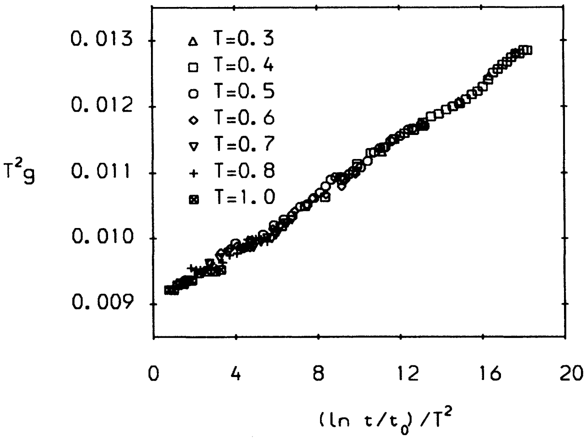

FIG. 4. Scaling plot of $T^2 g(T,t)$ as a function of $[\ln(t/t_0)]/T^2$ for $10 \le t \le 100$ in d=2+1. The cutoff length $t_0 \approx 3.7$ is chosen for the best data collapse.

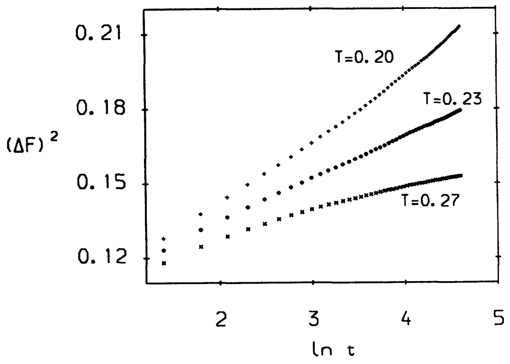

FIG. 6. $(\Delta F)^2$ as a function of $\ln t$ for different temperatures in d=3+1. For $T = T_c$ (≈ 0.23), $(\Delta F)^2$ grows as $\ln t$.

self indicates flow to strong coupling at very large times, in agreement with the renormalization-group predictions [5,8]. By contrast we note that no finite-size scaling plot was attempted in Ref. [11] for $d = 2 + 1$.

In d=3+1, $g(T,t)$ is calculated for t=8, 16, and 32 as shown in Fig. 5 for $\gamma = 0.5$ and uniform $\mu(\mathbf{x},t)$. Near $T_c = 0.23 \pm 0.01$, all the curves cross each other implying that there is a phase transition at this temperature. From the finite-size scaling plot g versus $(T-T_c)t^{1/\phi}$ we get a nice data collapse with $\phi = 4.0 \pm 0.7$. This longitudinal correlation exponent ϕ is inconsistent with a previous numerical result of 6.7 [11]. If $z(T_c) = 2$, then the one-loop RG result from the KPZ equation is $\phi = 2/(d-3) + O(1)$ [8]. As shown in Fig. 6, the free-energy fluctuation $(\Delta F)^2$ grows as $(\Delta F)^2 \sim \ln t$ at T_c. The same behavior is observed in the surface width of a growth model [8]. Above T_c, $(\Delta F)^2$ seems to saturate as $t \longrightarrow \infty$. However, the measured transverse displacement fluctuation $\overline{\langle \mathbf{x}^2 \rangle}$ is proportional to t both above T_c and at T_c. Since the relation $\overline{\langle \mathbf{x}^2 \rangle - \langle \mathbf{x} \rangle^2} \sim t$ is exact without any logarithmic corrections [19], and $g(T_c)$ is a constant, $\overline{\langle \mathbf{x}^2 \rangle}$ should be proportional to t at T_c. This behavior is consistent with $\omega(T_c) = 0$ and $z(T_c) = 2$ [20]. The data for g at high temperature $T = 0.4 > T_c$ shows a $1/\sqrt{t}$ behavior for large t in agreement with Eq. (9).

In summary, we have studied the dimensionless quantity g both by a weak-noise expansion and by transfer-matrix methods for directed polymers in a random potential. No phase transition is found in d=2+1. Instead, there is an enormously large crossover length t^*, viz. $\ln t^* \sim T^2$. We suspect that this is the origin of previous claims for a phase transition in d=2+1. The exponents ω and z and the longitudinal correlation length exponent ϕ have been determined for the phase transition in d=3+1.

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