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Interaction of two-dimensional localized solutions near a weakly inverted bifurcation

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We study the interaction of two-dimensional (2D) solutions as they arise for the envelope equations for a subcritical bifurcation to traveling waves. We show that these 2D localized solutions can collide and reemerge unchanged in size and shape after the collision in contrast to what is found in 2D soliton systems. Various other types of behavior arise as the impact parameter and the cross coupling between the waves are varied. We point out that these phenomena should be observable experimentally for hydrodynamic instabilities in anisotropic liquids such as nematic liquid crystals.

One of the most important discoveries in nonlinear science has been that of solitons by Zabusky and Kruskal [1]. They showed that for a one-dimensional evolution equation, the Korteweg–de Vries equation, localized solutions could collide and emerge unchanged in speed, size, and shape when compared to a time well before the collision. Thereafter, solitons have been found in many integrable systems including the nonlinear Schrödinger equation and the sine-Gordon equation [2–4]. All known integrable systems are Hamiltonian or purely dispersive and thus soliton behavior appears to be restricted to the domain of nondissipative systems or at best to systems for which a very small amount of dissipation is taken into account perturbatively.

Since for many physical systems dissipation is not just a small perturbation but plays an important role in the dynamics, in Refs. [5] and [6] the question was asked whether or not there are highly dissipative systems that share some properties with the solitonic systems, i.e., have localized solutions which collide and interpenetrate but which emerge unchanged in speed, size, and shape as compared to the state before the collision. It was shown [5,6] that, indeed, this behavior can occur in a system with both large *dissipation* and *dispersion:* a system of two coupled nonlinear envelope equations for a weakly inverted bifurcation [7,8].

One of the disappointing features of multidimensional integrable systems has been the lack of stable localized solutions, with solutions dispersing away as $t \to \infty$. This is in contrast to one-dimensional integrable systems where the solutions tend to form solitons. Recently, stable localized solutions have been found for the two-dimensional (2D) Davey-Stewartson equations [9–11]. Collisions of these solutions have been studied, and it has been found that, in general, there is an exchange of energy between the localized solutions and that the forms of the solutions are changed. Only for a special choice of spectral parameters are the forms preserved [11]. This is in contrast to one-dimensional solitons where, in general, the forms are preserved [4].

Many macroscopic physical systems are highly dissipative and not purely Hamiltonian. Therefore it is natural to investigate the interaction of localized solutions in multidimensional dissipative systems. In this paper we study the interaction of localized solutions occurring in a two-dimensional generalization of the system of amplitude equations studied in Refs. [5], [6], and [12]. In contrast to soliton systems where an increase from one to two dimensions greatly changes the behavior of solutions, we find in the dissipative system that the behavior of solutions is preserved in increasing the dimensions from one to two, with interpenetration of localized solutions in which the form is preserved, as well as complete annihilation and a transition from subcritical to supercritical. In addition, a new parameter—the impact parameter—emerges as a result of the increase in dimension. We find that whether the solutions annihilate or preserve their form can depend on the impact parameter and that the perpendicular distance between the trajectories after the collision can be different from that before the collision (the impact parameter).

The equations we study are

$$\partial_t A + v\partial_x A = \chi A + \gamma \nabla^2 A - \beta |A|^2 A - \delta |A|^4 A - \zeta |B|^2 A \tag{1a}$$

and

$$\partial_t B - v\partial_x B = \chi B + \gamma \nabla^2 B - \beta |B|^2 B - \delta |B|^4 B - \zeta |A|^2 B . \tag{1b}$$

The complex fields A and B correspond to right and left traveling waves, respectively ($v > 0$), and the coefficients γ, β, δ, and ζ are, in general, complex. Real and imaginary parts are denoted by the subscripts r and i, respectively.

We note that Eqs. (1) are of prototype character, since equations of this type [13] can be derived for systems with an intrinsic anisotropy, which show subcritical bifurcations to traveling waves. Fluid systems with an intrinsic anisotropy include, for example, nematic liquid crystals, which show short-range positional order, but for which the orientation of the molecules aligns on average spontaneously along a certain direction. For the onset of stationary electroconvection in nematic liquid crystals it has been shown [14] that the spatial derivative term in the resulting envelope equation can be cast into the form of a 2D Laplacian and the corresponding diffusion coefficient has been evaluated from the electrohydrodynamic equations for nematic liquid crystals [14]. A subcritical bifurcation to traveling waves occurring for hydrodynamic instabilities in nematic liquid crystals recently has been demonstrated experimentally for the case of electroconvection [15]. Furthermore, it has been predicted theoretically some time ago [16] that the onset of thermal convection in nematic liquid crystals can be via a subcritical oscillatory bifurcation.

In the following, we take χ and β_r negative so that the system is subcritical (sufficiently small perturbations damp while larger perturbations grow) and δ_r positive to cause saturation. The cross-coupling terms are responsible for interactions between the fields, $\zeta_r > 0$ corresponding to a stabilizing interaction and $\zeta_r < 0$ corresponding to a destabilizing interaction. These equations are solved numerically using a time-splitting method. The equations are split into their linear and nonlinear parts. The linear part is solved using Fourier transforms and the nonlinear part is evolved in time using the second-order Runge-Kutta method.

It has been shown [17] that (without the cross-coupling terms) these equations have stable radially symmetric localized solutions. Just as in the one-dimensional case, these particlelike solutions form from a wide range of initial conditions. Figure 1(a) shows two such solutions. The state was prepared by allowing the system to evolve with $v=0$ from two initial Gaussian forms. The final state is independent of the amplitudes, widths, and symmetry of the initial Gaussian forms for a wide range of amplitudes and widths. The particle states are then set in motion by taking v different from zero. Figures 1(b) and 1(c) show the two states interacting for $\zeta_r = 0.8$. The reduction in amplitude of the states during interaction is the result of the stabilizing cross coupling. Figure 1(d) shows the states well after the interaction. Comparing Fig. 1(d) with Fig. 1(a) we see that the form of the solution is preserved.

For larger stabilizing cross coupling the amplitudes will be reduced even more during interaction. If the reduction in amplitudes is sufficiently great, the amplitudes will be brought below the critical amplitude separating growth from decay (recall that the system is subcritical) and the solutions will annihilate. For example, for $\zeta_r = 1$ the solutions will annihilate in a head-on (impact parameter $b=0$) collision. Figure 2 shows a collision with $\zeta_r = 2$ and an impact parameter of $b=1$. We see that the solutions annihilate. For a much larger impact parameter, the solutions will not be sufficiently reduced in amplitude to cause an annihilation and the solutions will emerge from the interaction unchanged from the solutions before the collision. Figure 3 shows such an interaction for impact parameter $b=2$. Comparing the initial and final states, we see that the form of the solution is preserved as in the collision of Fig. 1. An interesting feature of this interaction, however, is that the perpendicular distance between the trajectories has increased as a result of the interaction [compare Fig. 3(c) with Fig. 3(a)].

If the cross coupling is destabilizing ($\zeta_r < 0$), the amplitudes will be increased as a result of the interaction. If the destabilizing cross coupling is not too large, the solu-

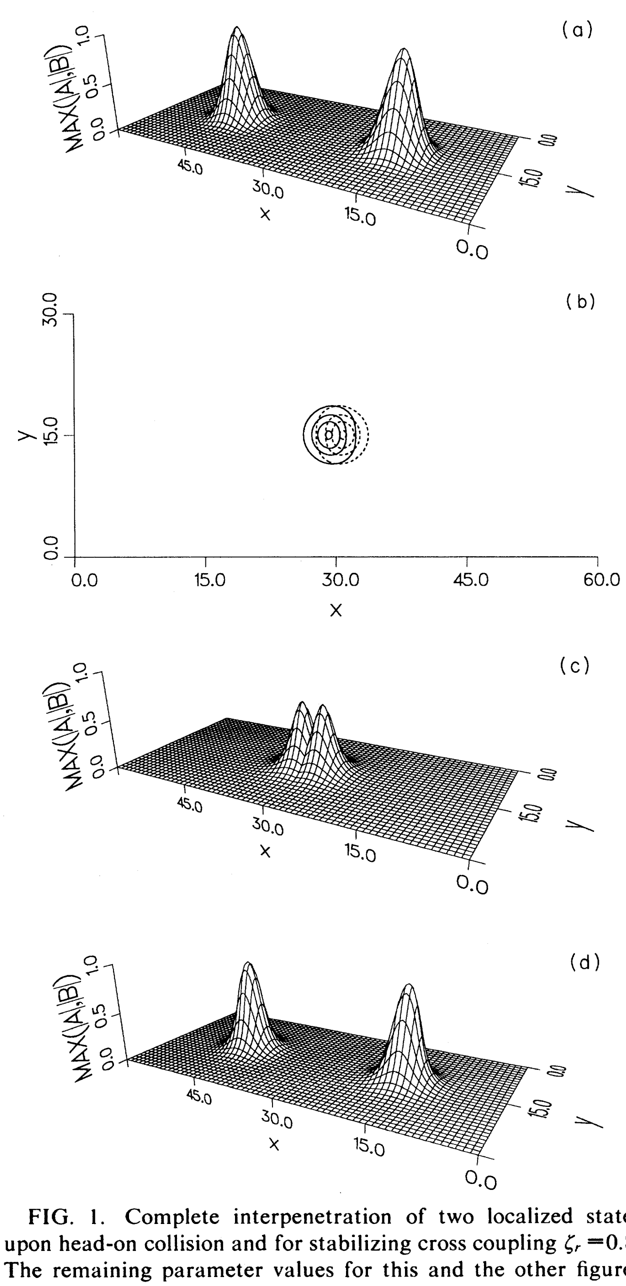

FIG. 1. Complete interpenetration of two localized states upon head-on collision and for stabilizing cross coupling $\zeta_r = 0.8$. The remaining parameter values for this and the other figures are $\chi = -0.1$, $\beta_r = -3$, $\beta_i = -1$, $\gamma_r = 1$, $\gamma_i = 0$, $\delta_r = 2.75$, $\delta_i = -1$, $\zeta_i = 0$, and $v = 1$. (a) The two localized states before the interaction; (b) contour plot during the interaction; (c) 3D plot during the interaction, and (d) well after the interaction.

tions will interact and emerge from the interaction unchanged. For example, this occurs for a head-on collision for $\zeta_r = -0.5$ and for impact parameter $b=4$ for $\zeta_r = -1$. For much larger destabilizing cross coupling the interaction will cause the system to make a transition from subcritical to absolutely unstable and the interaction will spread with time in all directions. Figure 4 shows such an interaction. The initial state is the same as that of Fig. 3(a), but with impact parameter $b=3$ instead of $b=2$. We see that the interaction region grows as a result of the destabilizing cross coupling. The interaction region will continue growing with time until the entire system is filled.

If the group velocity v is sufficiently small, it is possible for the interaction to cause the states to stop propagating and to form a stationary compound object. Figure 5 shows such a compound object for $v=0.1$ and $\zeta_r=1$. Such a compound state has also been found in the one-dimensional equations [6].

In this paper we have studied interactions of particlelike solutions for a two-dimensional generalization of a system of coupled envelope equations for a weakly inverted bifurcation [7,8]. We find that behavior occurring in one spatial dimension [5,6] also occurs in two. This is in contrast to soliton systems for which an increase in the dimensions of the system greatly changes the behavior of the solutions [4,9–11]. We found interactions which preserve the form of the solutions (a behavior which occurs with one-dimensional solitons), as well as interactions which cause complete annihilation and interactions which cause a transition from subcritical to supercritical. We have also found behavior which cannot occur in a one-dimensional system such as a change in the perpendicular distance between trajectories as a result of the interaction.

It will be most interesting to see whether the phenome-

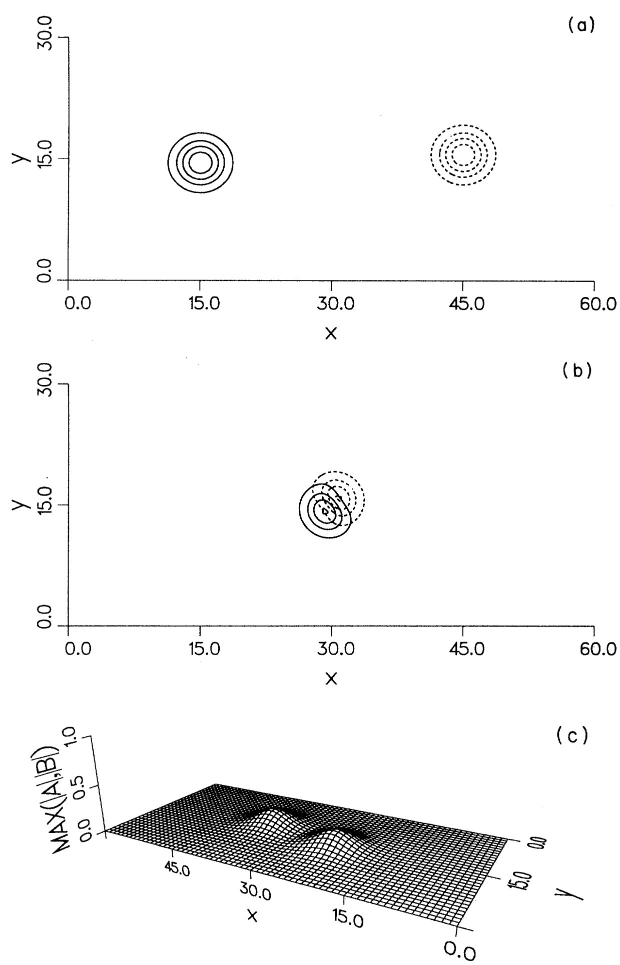

FIG. 2. Complete annihilation of two localized states for impact parameter $b=1$ and stabilizing cross coupling $\zeta_r=2$. (a) Contour plot well before the interaction; (b) contour plot during the interaction; (c) 3D plot of the decaying remnants after the interaction.

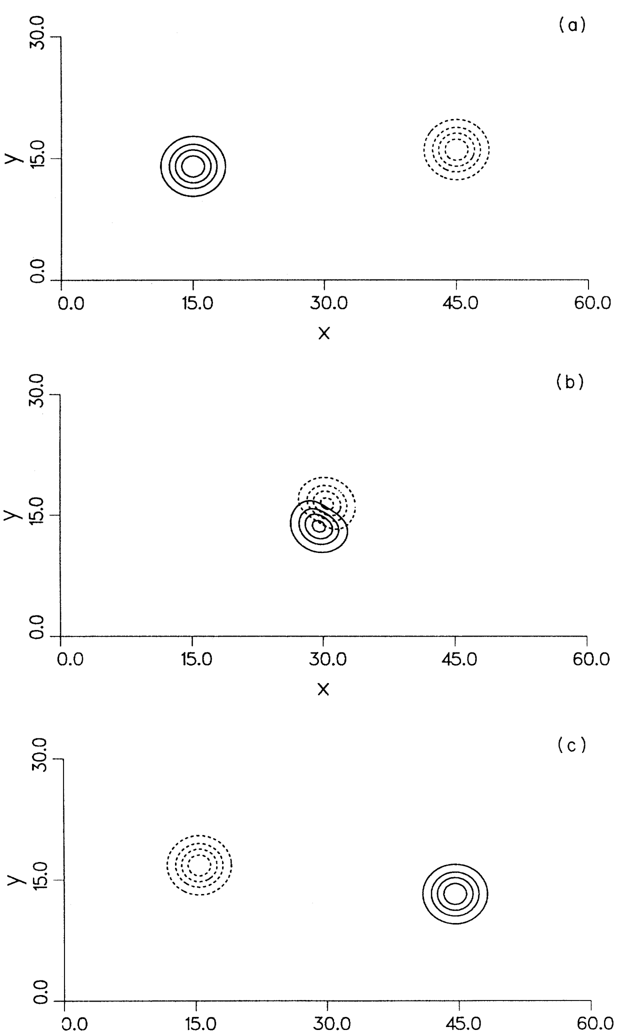

FIG. 3. Interaction of two localized states for larger impact parameter ($b=2$) shown as contour plots. Note that the perpendicular distance between the trajectories is larger after than before the interaction. (a) Initial state; (b) interaction; (c) final state after interaction.

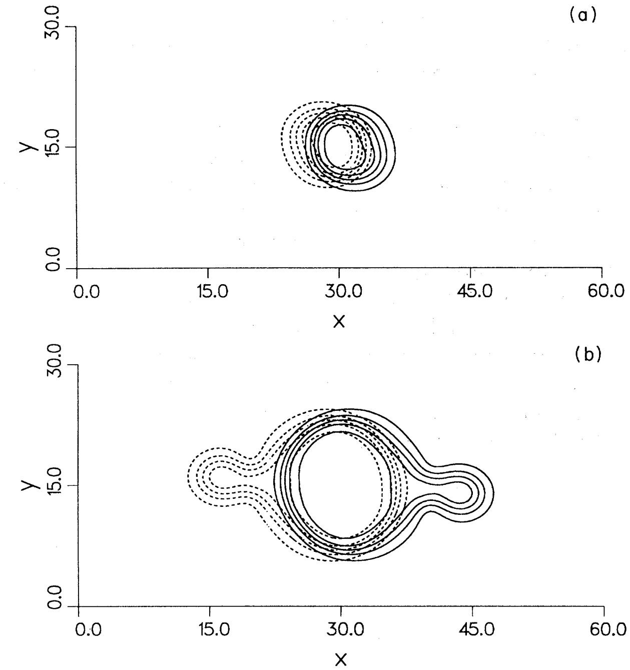

FIG. 4. Transition from subcritical to absolutely unstable supercritical for destabilizing cross coupling $\zeta_r=-1$. The initial state is chosen as in Fig. 3(a) except for the impact parameter ($b=3$ in the present figure). (a) Interaction, spreading has started; (b) spreading continues and the pattern eventually fills the whole cell.

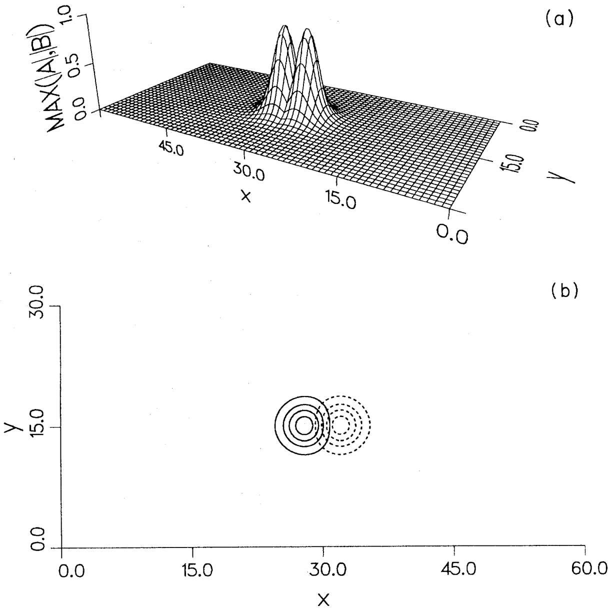

FIG. 5. Stationary compound state as a result of a head-on collision for stabilizing cross coupling $\zeta_r=1$ and sufficiently small group velocity ($v=0.1$). (a) 3D plot; (b) contour plot.

na predicted here will be observed in electroconvection or thermal convection in nematic liquid crystals or in another hydrodynamic instability in an anisotropic fluid. This would then allow for a detailed comparison of the presented results with the collision behavior of two-dimensional particlelike solutions in physical systems having dissipation and dispersion.

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