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Statistics of energy levels in integrable quantum systems

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We investigate various statistics of energy levels of integrable quantum systems with Hamiltonians $H=\frac{1}{2}(\mathbf{I}-\boldsymbol{\alpha})^2$ on the unit torus, with $\boldsymbol{\alpha}$ a parameter. We find strong numerical evidence, by using up to 10^9 levels, that for typical $\boldsymbol{\alpha}$, with respect to uniform distribution in the unit square, the *local* empirical statistics of the levels $E_{\mathbf{n}}=\frac{1}{2}(\mathbf{n}-\boldsymbol{\alpha})^2$, $\mathbf{n}\in\mathbb{Z}^2$, converge for large energies to a Poisson limit. The fluctuation of the total number of levels, $E_n<E$, scales like $E^{1/4}$ and its distribution converges to a non-Gaussian limit. The variance and skewness of this distribution can be computed analytically.

The nature of the distribution of quantal energy levels is a long-standing problem. There is much numerical and theoretical evidence that in the semiclassical or high-energy regime the levels of classically integrable systems typically follow Poisson statistics [1-5]. There are two types of statistics commonly studied: the distribution of nearest-neighbor spacings and the spectral rigidity $\Delta_3(L)$. The latter measures the mean-square fluctuations in the number of energy levels in an energy range containing on the average L levels. It was shown in [2] and later analyzed in [3] using semiclassical analysis of periodic orbits that $\Delta_3(L)$ is linear in L, as would be expected from Poisson levels, for small L, but then saturates.

In this paper we investigate numerically an integrable system related to those considered in [1-5]. We use up to 10^9 levels and sample over many values of a parameter appearing in the Hamiltonian. Our results lend strong support to the exponential nature of the nearest-neighbor spacings by substantially reducing the fluctuations observed earlier. We also find that the actual distribution of the number of energy levels in small intervals is Poisson.

Finally, we investigated the distribution of the properly scaled random variable which measures the rigidity for ranges of L where $\Delta_3(L)$ is no longer linear. We find both numerically and analytically that the distribution is not Gaussian.

Our choice of system for study was motivated by a recent result of Sinai [6] concerning the eigenvalues of the integrable Hamiltonian of a free particle, on a two-dimensional toroidal surface of revolution. Sinai relates this to the study of the points in the square lattice $\mathbb{Z}^2$ which fall in a narrow strip of fixed area surrounding a "typical" random closed curve in the plane $r=Rf(\theta)$. Sinai's results may be paraphrased as follows: For typical curves let $p_R(n)$ be the probability of having n lattice points in the strip, as R ranges over an interval $[c_1R,c_2R]$, $0<c_1<c_2$. Then $p_R(n)$ approaches, for large R, the Poisson form, $(\rho^n/n!)e^{-\rho}$, where ρ is the expected number of lattice points in the strip.

In Sinai's theorem the probability distribution which determines the concept of "typicality" of the curves requires an "infinite" number of random parameters to specify the function $f(\theta)$, cf. Major [7]. In the present work we examine curves with much less randomness, i.e., circles with center $\boldsymbol{\alpha}$ in the unit square. Using many statistical tests over wide energy ranges our results suggest (but of course do not prove) that Poisson statistics hold locally also for our model, while $\boldsymbol{\alpha}$-dependent behavior is found for the global distribution. This is consistent with the results in [1-5].

We consider the eigenvalues of an integrable system whose Hamiltonian is of second degree in the action variables I_1 and I_2 conjugate to angular variables θ_1 and θ_2 which live on the unit torus. We can write the Hamiltonian in the form

$$H=\tfrac{1}{2}(I_1-\alpha_1)^2+\tfrac{1}{2}(I_2-\alpha_2)^2\,, \tag{1}$$

with α_1 and α_2 some constants. The eigenvalues of this system will then be

$$E_{\mathbf{n}}(\boldsymbol{\alpha})=\frac{\hbar}{2}[(n_1-\alpha_1)^2+(n_2-\alpha_2)^2]\,, \tag{2}$$

where $\mathbf{n}\equiv(n_1,n_2)$ are coordinates of the vertices on the square lattice $\mathbb{Z}^2$. It is clear that mathematically the study of the statistics of energy levels becomes the study of the statistics of the number of lattice sites in domains defined by circles with center $\boldsymbol{\alpha}\equiv(\alpha_1,\alpha_2)$. For convenience, we set $\hbar/2=\pi$ so that the average level density is one. For a fixed $\boldsymbol{\alpha}$ these levels are just points on the positive real axis and we label them in increasing order as $E_i(\boldsymbol{\alpha})$, $i=1,2,\ldots$, including the degeneracies which can occur if α_1 and α_2 are rationally dependent.

We present numerical studies of three types of statistics of these energy levels. They are (i) the level spacing ξ, (ii) the number η of levels in an interval of a given length, and (iii) the fluctuation ζ of the total number of levels below a given energy value:

$$\begin{aligned}
&\xi(\boldsymbol{\alpha},E)\equiv E_{i+1}-E_i,\ \text{when } E_i<E\le E_{i+1}\,,\\
&\eta(\boldsymbol{\alpha},E)\equiv\mathcal{N}\{E_i\in[E,E+c)\},\ c \text{ a parameter}\,,\\
&\zeta(\boldsymbol{\alpha},E)\equiv\frac{N(\boldsymbol{\alpha},E)-E}{[2(\pi E)^{1/2}]^{1/2}}\,,
\end{aligned} \tag{3}$$

where $N(\boldsymbol{\alpha},E)=\mathcal{N}\{E_i<E\}$, the number of levels below E, has an average over $\boldsymbol{\alpha}$ equal to E, and $2\sqrt{\pi E}$ is the length of the circle enclosing the levels. These quantities can be regarded as random variables: ξ and η measure *local* and ζ *global* statistical properties of the levels. We generate their statistics in three different ways by sampling uniformly over E in a given large interval keeping $\boldsymbol{\alpha}$ fixed, uniformly over $\boldsymbol{\alpha}$ in the unit square keeping E fixed, or over both $\boldsymbol{\alpha}$ and a large interval of E.

Our numerical data consists of three groups. The first group is designed to examine the statistics for a fixed $\boldsymbol{\alpha}$, sampling uniformly over a range of E. In this group, we randomly picked eight values of $\boldsymbol{\alpha}$ and numerically evaluated the distributions of ξ, η, and ζ over the ranges $[28\times10^6,\ 133\times10^6)$, $[2827\times10^6,\ 2975\times10^6)$, and $[282.74\times10^9,\ 283.12\times10^9)$. In the second group we have data for several fixed E ranging from 3×10^4 to 28×10^6; for each E the data is a result of sampling over up to 8×10^6 randomly picked $\boldsymbol{\alpha}$. The third group contains three subgroups of data taken by sampling over both $\boldsymbol{\alpha}$ and three ranges of E. These three subgroups of data are G_1 with $E\in[28.27\times10^6,\ 30.19\times10^6]$ and 4500 centers; G_2 with $E\in[2.8274\times10^9,\ 2.8293\times10^9]$ and 6700 centers; and G_3 with $E\in[282.7433\times10^9,\ 282.7490\times10^9]$ and 2300 centers. The above data were generated using double precision real numbers in computer programs. The round-off error is not a problem for these data except for the level spacing data in G_3. We, therefore, generated additional level spacing data G_3' for this energy range using quadrupole precision numbers.

LEVEL SPACING ξ

The values of spacings are divided into 200 bins that each has a probability $\frac{1}{200}$ according to the exponential distribution $\exp(-x)$. We find that for a typical $\boldsymbol{\alpha}$ with sampling over E or for a fixed E with sampling over $\boldsymbol{\alpha}$, the data match the exponential distribution quite well. In general data from energy ranges at higher E give a better fit. Figure 1 shows data which have been sampled over both E and $\boldsymbol{\alpha}$. The agreement with the exponential distribution is excellent. We also plot there the fluctuation of bin probability normalized by the standard deviation σ expected for a true random sampling of exponential distribution versus the bin position in the inset of Fig. 1. G_1 data show a small, but systematic deviation from exponential distribution. This presumably reflects the fact that we have not yet reached asymptopia. The fluctuation of the data at high-energy ranges (G_2) with smaller σ is comparable with that from randomly sampling.

NUMBER OF LEVELS η IN AN INTERVAL OF A FIXED LENGTH c

We considered several values of c. Our data indicate that for a typical $\boldsymbol{\alpha}$ distribution in E, or for a fixed E the distribution in $\boldsymbol{\alpha}$, approaches the corresponding Poisson distribution. When sampled over both $\boldsymbol{\alpha}$ and E the agreement becomes even better. We illustrate our findings using the data sampled over both $\boldsymbol{\alpha}$ and E for $c=2$. We plot in Fig. 2 the data as well as the corresponding Poisson distribution. To examine them more carefully, we plot the

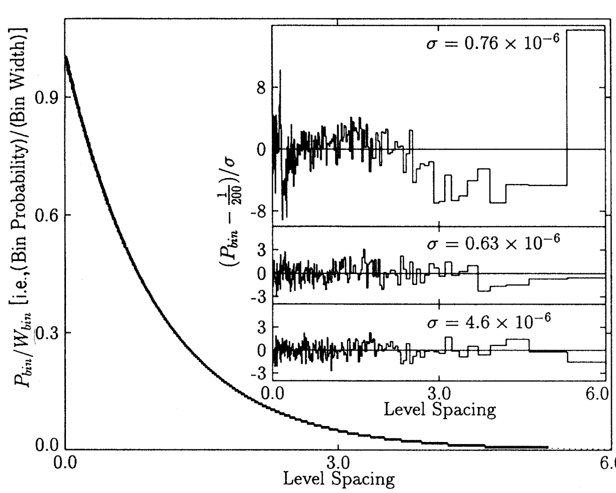

FIG. 1. Distribution of level spacings. Shown are histograms for G_1, G_2, and G_3' data. The dotted line is the exponential distribution. The insets are the relative errors of the distribution for the same data groups (top) G_1, (middle) G_2, and (bottom) G_3', normalized by σ, the standard deviation from random sampling.

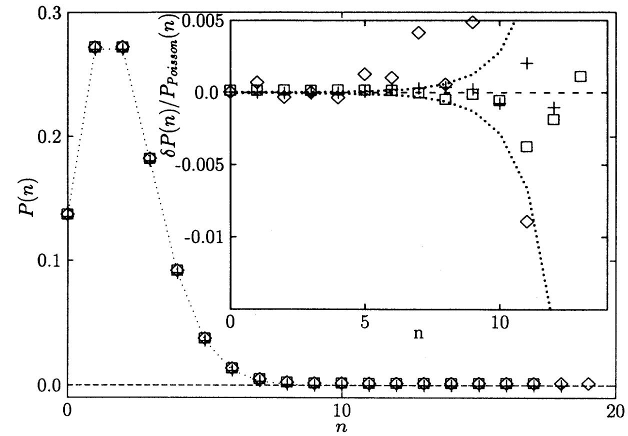

FIG. 2. Distribution of η for $c=2$. The dotted line connects the Poisson distribution. The symbols are ◇ for G_1 data; + for G_2 data; □ for G_3 data. Inset: The relative error with the Poisson distribution for the same data. The dotted lines are $\pm 2\sigma$ margin for random sampling.

relative difference between our data and the Poisson distribution in the inset of Fig. 2. This figure shows that for small n the fit is very good and as the energy range becomes higher the good fit extends to larger n.

FLUCTUATION ζ OF NUMBER OF LEVELS IN A CIRCLE

The fluctuations in $N(\boldsymbol{\alpha},E)$ for $\boldsymbol{\alpha}=0$ is a classic problem in number theory. Kendall [8] was the first to consider the problem with random center $\boldsymbol{\alpha}$. He showed that the variance of $N(\boldsymbol{\alpha},E)-E$ is of order $E^{1/2}$ for typical $\boldsymbol{\alpha}$. This can be understood intuitively by noting that the fluctuations in $N(\boldsymbol{\alpha},E)$ are due solely to the "randomness" in the location of lattice sites near the boundary which is of length $2\sqrt{E\pi}$. Kendall found that $\langle\zeta^2\rangle$ averaged over both $\boldsymbol{\alpha}$ and *all* values of E is equal to $(0.676497\ldots)^2/(2\pi)=0.07283\ldots$. Higher-order moments can also be computed. Dyson [9] found that the skewness $\langle\zeta^3\rangle/\langle\zeta^2\rangle^{3/2}=-0.179\ldots$. In our simulation we sampled ζ over both $\boldsymbol{\alpha}$ and energy. We found a variance of 0.072 ± 0.001 and a skewness of -0.20 ± 0.03 in good agreement with the above results.

To examine the distribution of ζ, we divided the values of ζ into bins of size $\frac{1}{100}$. The curves from different ranges of energy with the same $\boldsymbol{\alpha}$ collapse essentially on a common curve that depends strongly on $\boldsymbol{\alpha}$. Figure 3 shows the distributions of ζ for two randomly selected centers. When data are sampled over both $\boldsymbol{\alpha}$ and E the distribution looks symmetric near the center—despite the negativity of the third moment.

CONCLUDING REMARKS

Our improved statistics suggest that the large variance in the ξ statistics of [3] are due to either not using enough levels at high energy or having a Hamiltonian with less

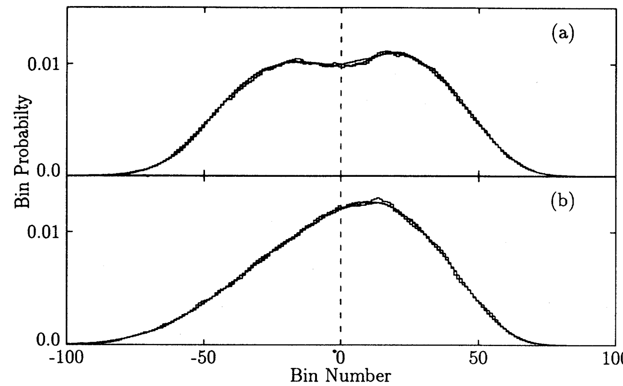

FIG. 3. Distribution of ζ for fixed centers. (a) and (b) are for two "typical" $\boldsymbol{\alpha}$'s (0.4685..., 0.01814...), (0.3437..., 0.4304...), and each contains three histograms for three energy ranges.

randomness. It should also be clear that there is no conflict between the slow growth of fluctuations in $N(\boldsymbol{\alpha},E)$ and the strictly Poisson nature of the level statistics as $E\to\infty$. Thus the average of ζ^2 over $\boldsymbol{\alpha}$ is given by

$$\langle\zeta^2\rangle(E)=\frac{1}{2\sqrt{\pi E}}\left[\int_0^E\int_0^E u(s_1,s_2)ds_2ds_2+E\right], \qquad (4)$$

where $[u(s_1,s_2)+1]ds_1ds_2$ is the average (over $\boldsymbol{\alpha}$) of the number of pairs of distinct levels in ds_1ds_2 (remember our average density is one). For a pure Poisson $u(s_1,s_2)=0$, while in our case $(1/E)\int_0^E\langle\zeta^2\rangle(s)ds\to a^2/(2\pi)$. This, however, does not prevent $u(s_1+E,\, s_2+E)$ from going to zero as $E\to\infty$ which is required for an approach to local Poisson. The large E behavior of $u(s_1+E,\, s_2+E)$ is an open interesting question. So is the nature of the asymptotic distribution of ζ.

The problem considered here can be generalized in various ways. We describe some further results in [10] (see also [11–13]). We only note the following very recent results: Heath-Brown [14] has proven the existence of the limiting distribution for $\zeta(0,E)$ and Beck [15] has proven the central limit theorem for a rectangular domain oriented at "very" irrational angles with its center $\boldsymbol{\alpha}$ distributed as before. The appropriate scaling of ζ found here is the square root of $\ln E$, in agreement with Berry's arguments for nonintegrable systems [3]. Beck has also shown that the statistics of lattice sites lying in an irrationally oriented thin hyperbolic needle with a random origin $\boldsymbol{\alpha}$ converge to that of a Poisson set of points.

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