RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 44, NUMBER 5 1 SEPTEMBER 1991

Measure of the nonclassicality of nonclassical states

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(Received 30 January 1991; revised manuscript received 24 June 1991)

A continuous parameter introduced into the convolution transformation between P and Q functions leads to a measure of how nonclassical quantum states are with values ranging from 0 to 1: For photon-number states, the value is 1, the maximum possible. For squeezed vacuum states, it is a monotonically increasing function of the squeeze parameter with values varying from 0 to $\frac{1}{2}$. This measure is identical to the minimum number of thermal photon necessary to destroy whatever nonclassical effects existing in the quantum states.

In the coherent-state description of radiation fields, initiated by Glauber [1] and Sudarshan [2] in 1963, there are P and Q representations corresponding to normal and antinormal ordering, respectively, of the creation and annihilation operators. Their distribution, or quasidistribution, functions in the complex plane are related through the following convolution transformation [3]:

$$Q(z)=\frac{1}{\pi}\int d^2w\, e^{-|z-w|^2}P(w)\,, \tag{1}$$

where z and w are complex variables. Following Cahill and Glauber [3], but in a slightly different way, we introduce a continuous parameter τ and define a general distribution function as

$$R(z,\tau)=\frac{1}{\tau}\frac{1}{\pi}\int d^2w \exp\left(-\frac{1}{\tau}|z-w|^2\right)P(w)\,. \tag{2}$$

We shall call $R(z,\tau)$ the R function from now on. For the special cases of $\tau=0$, $\frac{1}{2}$, and 1, the R function is the same as P, W, and Q functions, respectively. In general, the R function is a continuous interpolation between P and Q functions.

It is well known that the P function for a coherent state is a δ function. It has been shown by Hillery [4] that the P functions of all the other pure states are more singular than the δ function. On the other hand, the Q function is actually the diagonal elements of the density operator:

$$Q(z)=\langle z|\rho|z\rangle\,; \tag{3}$$

hence, in contrast with the P function, it is a positive-definite regular function. Such a drastic contrast is, of course, due to the convolution transformation; we can think of convolution as a *moving average,* which is a *smoothness-increasing operation* [5].

Our primary motivation for introducing the τ parameter is to define a measure of how nonclassical quantum states are.

Radiation fields with certain characteristics that can be understood only by quantum-mechanical description are called nonclassical states [6]. The nonclassical nature of a quantum state can manifest itself in different ways. It is well known that photon antibunching [7] and sub-Poissonian distribution of photon numbers [8] are manifestations of the nonclassical photon statistics of single-mode radiations. Another manifestation occurs when the noise level of one quadrature component of a radiation field is below that of the vacuum; then it is called a squeezed state [9]. The studies on nonclassical properties of radiation fields in recent years have been more interested in whether they are nonclassical than how nonclassical they are. For the latter purpose, we need a precise definition for the nonclassical depth of a quantum state.

In 1979, Mandel [10] introduced a q parameter defined as

$$q\equiv(\langle n^{(2)}\rangle-\langle n\rangle^2)/\langle n\rangle\,, \tag{4}$$

where $\langle n^{(2)}\rangle$ is the second-order factorial moment of the photon-number distribution. A negative value of q means that the distribution is narrower than a Poisson distribution, hence called sub-Poissonian. We have a nonclassical photon state whenever the q parameter is negative. The photon-number distribution of a coherent state is Poissonian; hence its q parameter is 0. Therefore the coherent states can be considered as on the borderline between classical and nonclassical states. One attractive feature of this parameter is that its values are 0 for all coherent states and -1 for all Fock (photon-number) states. Since a Fock state has a photon-number distribution that is as narrow as it can be, we are inclined to think that a Fock state is as nonclassical as a quantum state can be. Then the nonclassical depth can be confined within a convenient range of -1–0, if we adopt q parameter as the measure. However, this definition cannot reflect faithfully the nonclassical nature of squeezed states; for example, the q values of squeezed vacuum states are always positive [11].

In 1987, Hillery [12] gave a definition for the "nonclassical distance" of a quantum state in terms of the trace norm of the difference between the density operator of the quantum state and that of the nearest classical state. This measure can be applied to squeezed states as well as to nonclassical photon statistics. Unfortunately, it is very difficult to find the nearest classical state in practical calculations; so, usually, the best one can do is to determine the upper and lower bounds of this measure.

It is well known in the quantum optics community that the origin of all nonclassical effects is that the P functions of quantum states are singular and not positive definite. The smoothing effect of the convolution transformation of Eq. (2) is enhanced as τ increases. If τ is large enough so that the R function becomes acceptable as a classical distribution function, i.e., it is a positive definite regular function, then we say that the smoothing operation is complete. Let C denote the set of all the τ that will complete the smoothing of the P function of a quantum state and let the greatest lower bound, or infimum, of all the τ in C be denoted by

$$\tau_m \equiv \inf_{\tau \in C} (\tau) . \tag{5}$$

We propose to adopt τ_m as the nonclassical depth of the quantum state.

According to this definition, we have $\tau_m = 0$ for an arbitrary coherent state $|\alpha\rangle$, since its P function is of the form of a δ function, $\pi\delta^2(z-\alpha)$. On the other hand, for $\tau=1$ we have $R(z,1)=Q(z)$, which is always acceptable as a classical distribution function for any quantum state; hence 1 is an upper bound for τ_m. Therefore, we can specify the range of τ_m to be

$$0 \leq \tau_m \leq 1 . \tag{6}$$

Following Mehta [13], we shall use the following formal expression for the P function in terms of the density operator ρ and the coherent states $|u\rangle$ and $|-u\rangle$ as

$$P(w) = e^{|w|^2} \frac{1}{\pi} \int d^2u \langle -u|\rho|u\rangle e^{|u|^2} \exp(wu^* - w^*u) , \tag{7}$$

where the integral may not exist. We need not to be concerned about this point, however, since our real interest is in the convolution transform of $P(w)$. Substituting Eq. (7) into Eq. (2), assuming the order of the integrations to be exchangeable, and carrying out the integration with respect to w, we obtain

$$R(z,\tau) = \frac{1}{1-\tau} \exp\left[\frac{1}{1-\tau}|z|^2\right] \frac{1}{\pi} \int d^2u \langle -u|\rho|u\rangle \exp\left[-\frac{1}{1-\tau}[(2\tau-1)|u|^2 + (z^*u - zu^*)]\right] . \tag{8}$$

We can adjust the value of τ to insure not only that $R(z,\tau)$ exists as an ordinary function but also that it is positive definite. The greatest lower bound τ_m of the acceptable τ is, by our definition, the nonclassical depth of the quantum state.

We now try to determine the nonclassical depths for two of the most familiar types of nonclassical radiation states: the photon-number (Fock) states and the squeezed vacuum states.

The density matrix of the photon-number state $|n\rangle$ can be expressed as $\rho_n = |n\rangle\langle n|$. So we have

$$\langle -u|\rho_n|u\rangle = \langle -u|n\rangle\langle n|u\rangle = \frac{(-1)^n}{n!}|u|^{2n} e^{-|u|^2} . \tag{9}$$

Using this expression in Eq. (8), we obtain

$$R_n(z,\tau) = \frac{1}{\tau}\left[-\frac{1-\tau}{\tau}\right]^n \exp\left[-\frac{|z|^2}{\tau}\right] \times L_n\left[\frac{|z|^2}{\tau(1-\tau)}\right] , \tag{10}$$

where L_n is the Laguerre polynomial.

We list the explicit expressions for the first few $R_n(z,\tau)$ as follows:

$$R_0(z,\tau) = \frac{1}{\tau}\exp\left[-\frac{|z|^2}{\tau}\right] , \tag{11}$$

$$R_1(z,\tau) = \frac{1}{\tau}\exp\left[-\frac{|z|^2}{\tau}\right]\left[\frac{1}{\tau^2}|z|^2 - \frac{1-\tau}{\tau}\right] , \tag{12}$$

$$R_2(z,\tau) = \frac{1}{\tau}\exp\left[-\frac{|z|^2}{\tau}\right]\left[\frac{1}{2\tau^4}|z|^4 - \frac{2(1-\tau)}{\tau^3}|z|^2 + \left[\frac{1-\tau}{\tau}\right]^2\right] , \tag{13}$$

etc. From Eq. (11) we see that $R_0(z,\tau)$ is positive definite whenever τ is positive, so we have $\tau_m=0$ for the vacuum state. From Eqs. (12) and (13), etc., we see that, for $n>0$ and $0<\tau<1$, the $R_n(z,\tau)$ are always regular functions, but they are not positive definite, since each of them has n real positive roots. However, the $R_n(z,\lambda)$ are positive definite for $\tau \geq 1$; So we have $\tau_m=1$, which reconfirms our belief that the photon-number states are the most nonclassical quantum states.

As a by-product of our calculation, we have also obtained a new expression, as far as we know, for the P function of the photon-number state $|n\rangle$ as

$$P_n(z) = \lim_{\tau\to 0}\frac{1}{\tau}\left[-\frac{1-\tau}{\tau}\right]^n \exp\left[-\frac{|z|^2}{\tau}\right] \times L_n\left[\frac{|z|^2}{\tau(1-\tau)}\right] . \tag{14}$$

It is interesting to point out that expression similar to Eq. (10) has been derived by Lachs [14] for the distribution of photon number in the superposition of a coherent state $|z\rangle$ and the thermal radiation as

$$\mathrm{Prob}(n) = \frac{1}{1+\langle n_{\mathrm{th}}\rangle}\left[\frac{\langle n_{\mathrm{th}}\rangle}{1+\langle n_{\mathrm{th}}\rangle}\right]^n \times \exp\left[\frac{-|z|^2}{1+\langle n_{\mathrm{th}}\rangle}\right] L_n\left[\frac{-|z|^2}{\langle n_{\mathrm{th}}\rangle^2+\langle n_{\mathrm{th}}\rangle}\right] , \tag{15}$$

where $\langle n_{\mathrm{th}}\rangle \equiv (e^{\hbar\omega/kT}-1)^{-1}$ is the *average photon number* in the thermal radiation. The two expressions will be identical if we make the following correspondence:

$$\tau \to 1+\langle n_{\mathrm{th}}\rangle . \tag{16}$$

This correspondence is by no means accidental; the photon-number distribution is calculated according to the formula

$$\mathrm{Prob}(n) = \frac{1}{\pi}\int d^2\alpha P(\alpha)|\langle\alpha|n\rangle|^2 , \tag{17}$$

where

$$P(\alpha)=\frac{1}{\langle n_{\rm th}\rangle}\exp(-|\alpha-z|^2/\langle n_{\rm th}\rangle) \tag{18}$$

is the P function of the superposed state and $|\langle\alpha|n\rangle|^2$ is the Q function of the photon-number state.

The squeezed vacuum states are generated from the vacuum state by the well-known squeeze operator [15]

$$S(\zeta)\equiv\exp(\tfrac{1}{2}\zeta a^\dagger a^\dagger-\tfrac{1}{2}\zeta^* aa)\,, \tag{19}$$

where $\zeta\equiv re^{i\theta}$ is a complex parameter. So a squeezed vacuum can be identified by ζ and expressed as

$$|\zeta\rangle\equiv S(\zeta)|0\rangle$$
$$=\sqrt{\mathrm{sech}r}\sum_{n=0}^{\infty}\frac{1}{2^n n!}e^{ni\theta}(\tanh r)^n\sqrt{(2n)!}|2n\rangle\,. \tag{20}$$

Then we have

$$\langle -u|\rho(\zeta)|u\rangle=(\mathrm{sech}r)\exp\{-|u|^2+\tfrac{1}{2}(\tanh r)\times[e^{i\theta}(u^*)^2+e^{-i\theta}(u)^2]\}\,. \tag{21}$$

Substitution of Eq. (21) into Eq. (8) gives the Gaussian function

$$R_\zeta(z,\tau)=\frac{(\mathrm{sech}r)}{\sqrt{D}}\exp\left[-\frac{1}{D}(ax^2+2bxy+cx^2)\right] \tag{22}$$

with

$$\begin{aligned} a&=\tau+(1-\tau)\tanh^2 r-\cos\theta\tanh r\,,\\ b&=\sin\theta\tanh r\,,\\ c&=\tau+(1-\tau)\tanh^2 r+\cos\theta\tanh r\,,\\ D&=\tau^2-(1-\tau)^2\tanh^2 r\,. \end{aligned} \tag{23}$$

For $R_\zeta(z,\tau)$ to be normalizable we must have

$$ac-b^2>0,\ D>0\,. \tag{24}$$

Both conditions lead to the same conclusion that

$$\tau_m=\tanh r/(1+\tanh r)\,. \tag{25}$$

This nonclassical depth can be expressed as a function of the squeeze parameter, $s\equiv e^r$, as follows:

$$\tau_m(s)=(s^2-1)/2s^2\,. \tag{26}$$

From Eq. (26) we see that τ_m is a monotonically increasing function of s; it varies from 0 to $\frac{1}{2}$ as s varies from 1 to ∞.

Exactly the same result is obtained when we extend the study to more general squeezed states of the Stoler [15] type and of the Yuen [16] type; the latter are also known as two-photon coherent states.

On the other hand, we consider the superposition of two quantum states with $P_1(z)$ and $P_2(z)$ as their P functions. According to Glauber [17] the P function for the superposed state is the convolution product of $P_1(z)$ and $P_2(z)$; namely,

$$P_{\rm su}(z)=\frac{1}{\pi}\int d^2z\,P_1(z-w)P_2(w)\,. \tag{27}$$

It is well known that the P function for a single-mode thermal radiation is [17]

$$P_{\rm th}(z)=\frac{1}{\langle n_{\rm th}\rangle}\exp(-|z|^2/\langle n_{\rm th}\rangle)\,. \tag{28}$$

We now consider the superposition of the thermal radiation with an arbitrary state of single-mode radiation with $P(z)$ as its P function. Then the P function of this quantum state with thermal noise can be expressed as

$$P_{\rm su}(z)=\frac{1}{\langle n_{\rm th}\rangle}\frac{1}{\pi}\int d^2w\exp(-|z-w|^2/\langle n_{\rm th}\rangle)P(w)\,. \tag{29}$$

Comparing Eqs. (1) and (29) we see that the superposed P function, $P_{\rm su}(z)$, is identical to the Q function when $\langle n_{\rm th}\rangle=1$. The implication of this coincidence can be stated as follows: One thermal photon is always sufficient to destroy whatever nonclassical effects any single-mode radiation might have.

The R function for the superposed state of Eq. (29) can be obtained as

$$R_{\rm su}(z,\tau)=\frac{1}{\tau+\langle n_{\rm th}\rangle}\frac{1}{\pi}\int d^2w\exp\left[\frac{-|z-w|^2}{\tau+\langle n_{\rm th}\rangle}\right]P(w)\,. \tag{30}$$

So the nonclassical depth of a quantum state with and without thermal noise, respectively, are related by

$$\tau_m^{\rm th}=\tau_m-\langle n_{\rm th}\rangle\,; \tag{31}$$

which means that the reduction in the nonclassical depth of a quantum state with thermal noise is exactly equal to the number of thermal photon present. This also gives the following physical meaning to the nonclassical depth we have defined previously: The nonclassical depth of a quantum state is the minimum number of thermal photons necessary to destroy any of its nonclassical characteristics.

We have previously calculated the nonclassical depth of a Fock state to be exactly 1, so it takes one thermal photon to ruin the nonclassical nature of a Fock state. We have also calculated the nonclassical depth of a squeezed state to vary from 0 to $\frac{1}{2}$ as s varies from 1 to ∞; so it never takes more than $\frac{1}{2}$ of a thermal photon to ruin a squeezed state.

In conclusion, we have introduced a definition for the nonclassical depths of quantum states of radiation; and we have tested it for the examples of photon-number states and squeezed vacuum states with satisfactory results. We have also given a physical interpretation of the measure as the thermal photon number it takes to spoil the nonclassical characteristics of the quantum states.

This work was supported by the U.S. Navy, Office of Naval Research, under Grant No. N00014-89-J-1050.

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