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Mobility of spiral waves

L.M. Pismen
Department of Chemical Engineering, Technion, Haifa 32000, Israel

A.A. Nepomnyashchy
Department of Mathematics, Technion, Haifa 32000, Israel
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We consider interacting rotating spiral waves (vortices) under conditions when the asymptotic wave number is small and apply consistent approximations near the vortex core and in the outer region and asymptotic matching to derive a mobility relationship that connects the propagation velocity of a vortex with relevant characteristics of the extrinsic phase field in its vicinity. The results are applied to the problem of the existence of bound states of a pair of interacting vortices of the opposite charge. It is found that interaction decays exponentially with separation and remains attractive at all distances.

Rotating spiral waves are one of the best-known nonequilibrium structures, observed in numerous experiments with the Belousov-Zhabotinsky reaction [1,2], and in other systems, such as in recent studies of catalytic reactions [3]. Numerically, spiral waves can be modeled in a number of ways, including systems with separated scales of the FitzHugh-Nagumo type [4] and discrete analogs [5]. The most suitable model for analytic studies of nonequilibrium wave patterns, including rotating spiral waves, is the complex Ginzburg-Landau equation (CGLE)

$$\partial_t u = (1+i\eta)\nabla^2 u + u - (1+i\nu)|u|^2 u. \qquad (1)$$

This equation arises as the amplitude equation in the vicinity of the Hopf bifurcation in spatially extended systems [1], and therefore is generic for active media displaying wave patterns.

The ubiquity of spiral waves is due to their topological stability: the circulation of phase $\theta = \arg u$ around any contour γ enclosing the core of the spiral is constant and quantized to $2\pi n$:

$$\oint_\gamma \nabla\theta \cdot d\boldsymbol{l} = 2n\pi. \qquad (2)$$

Therefore an n-armed spiral can be referred to as a topological defect with the charge n. Only spirals of unit charge are topologically stable. In the following, we shall call spiral waves, interchangeably, vortices and defects.

Far from the vortex core, CGLE can be replaced by a transformed phase equation incorporating the asymptotic relationship between the real amplitude $\rho = |u|$ and the local wave number. We call this the *far-field* equation. The far field is singular at the vortex core. Such singularities can be considered as a type of "particles" whose interaction is mediated by the far field. Of course, this approximation is valid only in the case when the defects are well separated, so that the distance between their cores is much larger than the latter's characteristic scale (taken as unity). Other locations where the wave number changes rapidly and the far field is singular are grain boundaries, formed by collision of waves emanating from different centers.

Solutions of Eq.(1) in the form of stationary isolated spiral waves were constructed by Hagan [6], who combined numerical solution in the core region with matching to the far field. The problem of motion and interaction of spiral waves recently attracted wide interest. Numerical simulation [7, 8] showed that the attractive interaction of oppositely charged vortices changes to repulsion at larger distances. Further simulations [9] indicated still another change of sign, that would cause formation of stable bound pairs. Formation of pairs was predicted also by analytical studies [9, 10] which, however, were based on mutually incompatible approaches and led to contradictory results.

The analytical theory of interaction of well-separated spiral waves can take as a "ground state" some combination of Hagan's solutions. Rica and Tirapegui [11] took as a zero approximation a simple superposition of Hagan's spirals. A more sophisticated choice by Elphick and Meron [10] was the *product* of Hagan's solutions. Both *Ansätze* reduce in the far field to a superposition of phases corresponding to isolated spiral waves. This choice is unfortunate, as it disregards the screening of the far field and formation of grain boundaries, and therefore grossly exaggerates the strength of the interaction.

The influence of grain boundaries was shown to be crucial in a related study of interaction of a spiral wave with a boundary by Biktashev [12]; his approach, calling for the solution of an infinite chain of equations for Fourier components is, however, practically unworkable even in the simplest cases of interest. Aranson, Kramer, and Weber [9] found it possible to account for the effect of screening without incorporating grain boundaries explicitly. These authors observed that the far-field equation is the Burgers equation, and can be linearized through the Hopf-Cole transformation [13]. Since the superposition principle holds for the resulting linear equation, one can take as a zeroth-order approximation a superposition of

Hagan's solutions for free vortices. This will be a *logarithmic* rather than a linear superposition. The method is applicable only when the asymptotic wave number is small, since only then the far-field equation is indeed of the Burgers type. Even then, however, there remains a more subtle obstacle that renders the superposition principle inapplicable. The point is that the topological circulation condition (2), which is linear in θ, becomes *nonlinear* after the Hopf-Cole transformation. Therefore the logarithmic superposition of Hagan's solutions, taken as the zero approximation in Ref. [9], does not satisfy the circulation condition and defines, as a matter of fact, a multivalued function.

To help resolve the contradictions, we shall consider slowly moving vortices under conditions when the asymptotic wave number is small, and apply consistent approximations near the vortex core and in the outer region combined with asymptotic matching to derive mobility relationships that connect the stationary velocity with certain characteristics of the external phase field. We shall also attempt to construct a bound state of a pair of vortices of the opposite charge $n = \pm 1$ by solving far-field equations in the same approximation.

We are looking for stationary solutions in the comoving frame associated with a vortex propagating with a constant velocity $v \ll 1$ along some direction that we further take as the y axis. The basic equation (1) is reduced by rescaling the real amplitude, coordinates, and speed and applying the gauge transformation

$$\theta \to \theta - \omega t + \frac{1}{2}\eta \mathbf{v} \cdot \mathbf{x} \tag{3}$$

to the standard form

$$v\rho_y + \nabla^2 \rho + (1 - |\nabla\theta|^2 - \rho^2)\rho = 0, \tag{4}$$

$$v\theta_y + \nabla^2\theta + 2\rho^{-1}\boldsymbol{\nabla}\rho \cdot \boldsymbol{\nabla}\theta - q(1 - k^2 - \rho^2) = 0, \tag{5}$$

where $q = (\eta - \nu)/(1 + \nu\eta)$ and k is the asymptotic wave number connected with the frequency ω by the relationship

$$\omega = \frac{\eta - q(1-k^2)}{1 + q\eta(1-k^2)}. \tag{6}$$

$O(v^2)$ corrections are omitted in the above expressions.

We shall look for solutions to Eqs. (4) and (5) in the form of a series in the small parameter v:

$$\begin{aligned} \theta &= n\phi + \psi_0(r) + v\,\mathrm{Re}[e^{i\phi}\psi_1(r)] + \cdots, \\ \rho &= \rho_0(r) + v\,\mathrm{Re}[e^{i\phi}\rho_1(r)] + \cdots, \end{aligned} \tag{7}$$

where r and ϕ are polar coordinates. In the zeroth order, Hagan's [6] symmetric solutions $\psi_0(r)$, and $\rho_0(r)$, are recovered. At $q \ll 1$, the derivative $\psi_0'(r) = O(q)$. As a consequence, the first-order equations split into two pairs of equations for the real and imaginary parts of ρ_1 and ψ_1. In the leading order $O(q^0)$, ψ_1 is real while ρ_1 is imaginary. The first-order equations are identical then to the respective pair of equations in the real case [14]. For the purpose of matching, we need only the asymptotic expression for ψ at large r, that is written, combining results from Refs. [6,14] and omitting a superfluous constant phase, as

$$\psi(r) = q\left(C\ln r + \frac{1}{2}\ln^2 r\right) + vr(\cos\phi)\left(B - \frac{1}{2}\ln r\right), \tag{8}$$

with numerical constants $C = -0.098$ and $B = 0.309$.

Far from the core, the real amplitude is slaved to the phase. The far-field equation obtained by using the asymptotic relation $\rho^2 = 1 - |\nabla\theta|^2$ in Eq. (5) reduces it at $|q| \ll 1$ (when the asymptotic wave number is small) to an equation of the Burgers type . Assuming $\nabla\theta = O(\epsilon)$, $v = O(\epsilon)$, where $\epsilon \ll q$ we obtain in the leading $O(\epsilon^2)$ order

$$v\theta_y + \nabla^2\theta - q(|\nabla\theta|^2 - k^2) = 0. \tag{9}$$

As we shall see below, the vortex velocity is indeed of $O(kq^2)$, which justifies neglecting terms quadratic in v.

Equation (9) can be linearized via the Hopf-Cole transformation $\theta = -q^{-1}\ln F$ and further simplified by setting $F = e^{-vy/2}G$ and stretching the coordinate by the factor kq. The resulting equation is

$$\nabla^2 G - G = 0. \tag{10}$$

The rescaled radial coordinate will be denoted as $s = kqr$.

In view of the circulation condition (2), G is not a single-valued function. For an isolated vortex, we can replace G by a single-valued function by setting $G = e^{-nq\phi}H(s)$. In polar coordinates, the equation of H reads

$$H'' + s^{-1}H' - [1 + (q/s)^2]H = 0. \tag{11}$$

The solution near a moving vortex is expressed by a combination of Bessel harmonics with complex indices:

$$H = K_{iq}(s) + \mathrm{Re}[D_0 I_{iq}(s) + D_1 I_{1+iq}(s)e^{i\phi}], \tag{12}$$

where $D_0 = \bar{D}_0 + i\tilde{D}_0$, $D_1 = \bar{D}_1 + iq\tilde{D}_1$ are complex coefficients. The scaling of the imaginary part of D_1 implied by the above definition is justified *a posteriori* by matching conditions defining the value of this coefficient required to ensure steady motion with the prescribed speed. The same conditions will justify scaling the velocity as $v = kq^2 w$, so that $w = O(1)$ corresponds to $|D_1| = O(1)$.

The expression for the single-valued part of the phase function $\psi(r,\phi) = \frac{1}{2}kqwy + H(kqr,\phi)$ suitable for matching with the inner solution is obtained using the asymptotic forms of Bessel functions with a complex index at $s \ll 1$ that contains trigonometric functions of $q\ln(\frac{1}{2}qkr)$. To enable matching with the inner solution (8), the argument should be close to $(-\pi/2)\,\mathrm{sgn}\,q$. The expression for the radially symmetric part, obtained by expanding to $O(q)$ near this point and omitting irrelevant constant terms, is

$$\frac{1}{2}\ln^2 r + \ln r\left(\frac{\pi}{2|q|} + \ln\frac{kq}{2} + \gamma - \bar{D}_0\right), \tag{13}$$

where $\gamma = 0.577...$ is the Euler constant. Matching with the radially symmetric terms in (8) defines the asymptotic wave number [6]

$$k = \frac{2}{q}\exp\left(-\frac{\pi}{2|q|} - \gamma + \bar{D}_0 + C\right). \tag{14}$$

The angle-dependent part of (12) should be expanded to $O(q^2)$. Using (14) we obtain

$$-\frac{kq^2r}{2}\{[\tilde{D}_1+\bar{D}_1(C+\bar{D}_0-1+\ln r)]\cos\phi+(w-\tilde{D}_1)\sin\phi\}. \tag{15}$$

Matching the coefficients at $\ln r\,(\cos\phi)$ gives

$$\bar{D}_1 = w. \tag{16}$$

The same relation cancels the coefficient at $\sin\phi$. Finally, matching the coefficients at $\cos\phi$ defines $\tilde{D}_1$:

$$\tilde{D}_1 = -w(2B + C + \bar{D}_0 - 1). \tag{17}$$

Equations (16) and (17) represent the principal result of the matching procedure. Taken together, they can be viewed as a general mobility relationship that connects the velocity the vortex with relevant characteristics of the extrinsic part of the phase field prevalent just outside the vortex core, i.e., in a belt where both the outer asymptotic values of the inner solution and the inner asymptotic values of the far field are applicable. The structure of the mobility relationship is, however, not as straightforward as, say, in the case $q = 0$ [14]. The role of a driving force is played not just by an extrinsic phase gradient but by the projection D of the extrinsic field on the first complex index converging Bessel harmonic $I_{1+iq}e^{i\phi}$. Equations (16) and (17) define both the absolute value and the argument of D that would cause the motion with the prescribed speed and in the prescribed direction. The projection on the symmetric zeroth harmonic $I_{iq}(s)$ affects the motion indirectly through a change of the prevailing asymptotic wave number.

Finding the mobility relationship alone is still far less than sufficient for understanding the vortex dynamics. The main difficulty still rests in the solution of the far-field equation that defines the residual extrinsic field attenuated by screening before it reaches the vicinity of the core. We shall now attempt to construct a solution corresponding to a bound vortex-antivortex pair, and use Eqs.(16) and (17) as conditions of existence of this bound state.

To construct an approximate solution to the linearized equation (10) at small q, we shall take as the zeroth-order approximation the superposition of two vortices that was viewed as the exact solution in Ref. [9], and add to it a $O(q)$ correction to satisfy the circulation condition to this order. Thus, we adopt the *Ansatz*

$$G = e^{-q\phi}K_{iq}(s) + e^{-q\phi'}K_{iq}(s') + q\tilde{G} + O(q^2), \tag{18}$$

where $s' = (s^2 + R^2 - 2Rs\cos\phi)^{1/2}$ and ϕ' is the angle measured *clockwise* around the point $(R,0)$.

The jump condition for a pair of defects can be introduced in a number of ways. We choose to make cuts on the x axis connecting the centers of the defects along the rays $x < 0$ and $x > R$. The exact jump condition for the vortex at the origin and the antivortex at $(R,0)$ is

$$G(x,\,0+) = \begin{cases} G(x,\,0-)e^{-2\pi q} \text{ at } x < 0 \\ G(x,\,0-)e^{2\pi q} \text{ at } x > R. \end{cases} \tag{19}$$

With the *Ansatz* (18), the inconsistency in the jump condition near the origin is caused exclusively by the other defect. Expanding (19) in q yields the jump condition for $\tilde{G}$:

$$\tilde{G}(x,\,0+) - \tilde{G}(x,\,0-) = -2\pi[\,\Theta(-x)K_0(R-x) - \Theta(x-R)K_0(x)], \tag{20}$$

where $\Theta(x)$ is the Heaviside step function. The index of the Bessel function has been changed to zero, since $K_{iq}(s) - K_0(s) = O(q)$, unless the argument is small. The solution of the problem (10) and (20) is presented via an appropriate Green's function (taking account of the symmetry with respect to the axis $x = R/2$) as

$$\tilde{G}(x,\,y) = -2\pi\int_{-\infty}^{0}\mathcal{G}(x,y,x')K_0(R-x')dx'. \tag{21}$$

The Green's function $\mathcal{G}(x,y,x')$ satisfies

$$\begin{aligned} &\nabla^2\mathcal{G} - \mathcal{G} = 0,\\ &\mathcal{G}(x,0+,x') - \mathcal{G}(x,0-,x') = \delta(x-x'),\\ &\mathcal{G}_y(x,0+,x') - \mathcal{G}_y(x,0-,x') = 0,\\ &\mathcal{G}_x(R/2,y,x') = 0. \end{aligned} \tag{22}$$

The resulting expression for the $\tilde{G}$ reads

$$\tilde{G}(x,y) = -y\int_0^\infty dx'\,K_0(R+x')\left(\frac{K_1\left(\sqrt{(x+x')^2+y^2}\right)}{\sqrt{(x+x')^2+y^2}} + \frac{K_1\left(\sqrt{(R+x'-x)^2+y^2}\right)}{\sqrt{(R+x'-x)^2+y^2}}\right). \tag{23}$$

The single-valued function $H(s,\phi)$ can be now written as

$$H = K_{iq}(s) + e^{q(\phi-\phi')}K_{iq}(s') + q\tilde{G}(s,\phi). \tag{24}$$

Expanding the second term in s and q, and computing the asymptotic forms of the correction $\tilde{G}$ at $s \to 0$ we see that the terms containing the angle ϕ are compensated exactly, and (24) is indeed single-valued to $O(q^2)$.

The required projections on Bessel harmonics are obtained by comparing the $s \to 0$ limit of (24) with the expression (12) using the asymptotic forms of Bessel functions at $s \ll 1$, $|q \ln(\frac{1}{2}qkr)| \ll 1$. The result is

$$D_0 = K_0(R), \;\; \bar{D}_1 = 2K_1(R), \tag{25}$$
$$\tilde{D}_1 = 2K_1(R)[1 - \gamma + \ln 2 + R^{-1}\chi(R) + \mathcal{I}],$$

where $\chi(R) = K_0(R)/K_1(R)$, and $\mathcal{I}K_1(R)$ is the regular part of the $s \to 0$ limit of the integral in (23). This result, combined with Eqs. (16) and (17) confirms the conjecture in Ref. [9] on an exponential decay of vortex interaction, in contrast to a power decay in Ref. [11].

The integral $\mathcal{I}$ is most conveniently evaluated by extracting the singular part of the integral in (23) at $y = 0$, $x \to 0+$. In this way, we obtain

$$\mathcal{I} = \gamma - \ln \chi(R) + \mathcal{I}_1 + \mathcal{I}_2,$$

$$\mathcal{I}_1 = \lim_{x \to 0} \int_x^\infty d\xi \frac{1}{\xi}\left[\frac{K_0(R+\xi-x)K_1(\xi)}{K_1(R)} - \frac{\chi(R)}{\xi}\exp\left(-\frac{\xi - x}{\chi(R)}\right)\right], \tag{26}$$

$$\mathcal{I}_2 = \int_0^\infty d\xi \frac{K_0(R+\xi)K_1(R+\xi)}{(R+\xi)K_1(R)}.$$

Using (25), (16), and (17) we obtain the condition for separation of bound vortices

$$R^{-1}\chi(R) - \ln\chi(R) + \mathcal{I}_1 + \mathcal{I}_2 + K_0(R) + \ln 2 + 2B + C = 0. \tag{27}$$

Computation shows that the left-hand side (lhs) of this equation is always positive, and, consequently, the vortices attract at all distances, and bound states never form. No direct comparison can be made with results of numerical computations [7–9], that were all carried out at $q = O(1)$. Our results are, however, consistent with observations of Bodenschatz, Weber, Kramer, and co-workers [8], indicating that the distance where the attraction of oppositely charged vortices changes to repulsion diverges as the asymptotic wave number decreases.

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