Relative randomness of quantum observables

Marek Kuś
*Fachbereich Physik, Universität–Gesamthochschule Essen, 4300 Essen, Federal Republic of Germany and Institute for Theoretical Physics, Polish Academy of Sciences, aleja Lotników 32/46, 02-668 Warsaw, Poland**

Karol Życzkowski
*Fachbereich Physik, Universität–Gesamthochschule Essen, 4300 Essen, Federal Republic of Germany and Instytut Fizyki, Uniwersytet Jagielloński, ulica Reymonta 4, 30-059 Kraków, Poland**
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We investigate statistics of matrix elements of observables, calculated in an eigenbasis of a quantum-mechanical evolution operator corresponding to a classically chaotic system. We propose a criterion that allows one to predict whether or not such statistics are faithful to random-matrix theory and define a random operator with respect to a given system.

The theory of random matrices [1] proved to be extremely useful in investigations of quantum systems that are chaotic in the classical limit. Spectra of Hamilton (or evolution) operators of such systems differ in cases of classically regular and chaotic motions. It is believed that for systems with fully developed chaos in the classical limit, (quasi)energies repel, whereas for classically regular cases one finds clustering of levels [2]. The appropriate distributions of the level spacings for chaotic systems coincide with the theoretical predictions from random-matrix-theory and correspond to so-called Gaussian ensembles in the case of autonomous systems and the circular ensembles for periodically driven systems. Depending on the symmetry properties of a system, one has to choose among orthogonal, unitary, and symplectic ensembles that give different degrees of level repulsion [2].

Recently, similar considerations were extended to the statistics of eigenvector components of chaotic systems [3]. The conclusion was reached that statistical properties of eigenvectors can also serve as a signature of chaos in quantum mechanics. There is an obvious advantage in this approach, at least from the computational point of view: The components of eigenvectors outnumber the eigenvalues and therefore give more reliable statistical ensembles. On the other hand, however, in contrast to eigenvalues, eigenvectors are defined relative to some basis while any statistical statement must be basis independent. For example, it is clear that if we choose (in an "unfortunate" way) as a basis the eigenbasis of the operator under investigation, then the components of eigenvectors bear no statistical information. Vaguely speaking, one faces the problem of defining an appropriate basis [4]. Let us consider a quantum system H with eigenvectors $|\phi_i\rangle$, the classical analog of which displays full-scale chaos. A basis $\{|k\rangle, k=1,\ldots,N\}$ might be called random [5] with respect to this system, if the eigenvector statistics (i.e., statistics of $|\langle\phi_i|k\rangle|^2$) confers to the predictions of random-matrix theory. Depending on the system's symmetry, the well-known χ^2_ν distribution with $\nu=1$, 2, or 4 should be applied [6].

The whole problem also may be looked upon from a slightly different point of view. Instead of asking questions about eigenvectors, one can investigate squares of absolutes values of matrix elements of some chosen operator (observable) between the (quasi)energy eigenstates of the system under consideration. The case is of particular interest inasmuch as such matrix elements define the transition strengths between eigenstates—quantities that can be measured experimentally. Henceforth we shall call an operator "random" with respect to a given system if the statistics of its matrix elements in the energy eigenbasis is well approximated by the χ^2_ν distribution resulting from the appropriate ensemble of random matrices. In other words, a random operator X produces from the energy eigenvectors $|\phi_i\rangle$ the random basis $X|\phi_i\rangle$. The problem of finding a random basis is now shifted to the one concerning the random operator. It is to be noted that an observable commuting with a Hamiltonian (or with the Floquet operator in the case of maps) is not random with respect to this operator: the distribution of its matrix elements gives no statistical information.

In this paper we want to give a simple criterion for finding random observables. First we want to stress that the problems of finding a random basis and a random operator are not equivalent in the following sense. Since an observable (being Hermitian) defines a basis consisting of its eigenvectors, one could expect that two commuting observables sharing the same eigenbasis should have similar distributions of their matrix elements between eigenstates of the investigated chaotic system. This is not the case since one also has to take into account the properties of eigenvalue spectra of both operators and, in particular, their degeneration. A simple example of such a case is presented in Fig. 1. We calculated the matrix elements of J_z and J_z^2 between the eigenstates $|\phi_i\rangle$ of the Floquet operator U for the kicked top [7]:

$$U=\exp\left[-i\frac{k}{2j}J_z^2\right]\exp\left[-i\frac{\beta}{2}J_y\right] , \tag{1}$$

where J_y and J_z are angular-momentum operators. The square of the total (conserved) angular momentum equals $j(j+1)$, which sets the dimension of the resulting matrices U, J_y, and J_z to $2j+1$. The parameters k and β were assigned the values 10.0 and 1.7, respectively, which ensures that in the classical limit $j\to\infty$ the system is fully chaotic. The figure shows histograms of $y=|\langle\phi_i|J_z^l|\phi_j\rangle|^2$ normalized [6] to fulfill $\langle y\rangle=1$, for $l=1$ [Fig. 1(a)] and 2 [Fig. 1(b)]. The obtained distributions display considerable differences. Whereas J_z gives the expected χ_1^2 result (denoted in the figure by a solid line), the distribution for the operator J_z^2 deviates significantly from the predicted curve.

To characterize such situations in a more quantitative manner, we suggest the following reasoning. As previously mentioned, the observables commuting with U are not good for characterizing statistical properties of a chaotic system. This leads to the concept of measuring the degree of noncommutativity between U and an observable X by introducing

$$\mu_1=\frac{\|[U,X]\|^2}{\|X\|^2} , \tag{2}$$

and, more generally,

$$\mu_n=\frac{\|ad_U^n(X)\|^2}{\|X\|^2} , \tag{3}$$

where

$$\|X\|=\langle X|X\rangle^{1/2} \tag{4}$$

and

$$\langle X|Y\rangle=\mathrm{Tr}(XY^\dagger) \tag{5}$$

define the norm and scalar product in the space of Hermitian matrices representing observables. The $ad_U^n(X)$ are n-fold commutators,

$$ad_U^1(X)=[U,X] ,\quad ad_U^{n+1}(X)=[U,ad_U^n(X)] . \tag{6}$$

It is possible to rewrite Eq. (3) in the form

$$\mu_n=\begin{bmatrix}2n\\n\end{bmatrix}+\sum_{l=1}^{n}(-1)^l\begin{bmatrix}2n\\n-l\end{bmatrix}\frac{2\,\mathrm{Re}\langle X|X^{(l)}\rangle}{\|X\|^2} , \tag{7}$$

where

$$X^{(l)}=U^{\dagger l}XU^l . \tag{8}$$

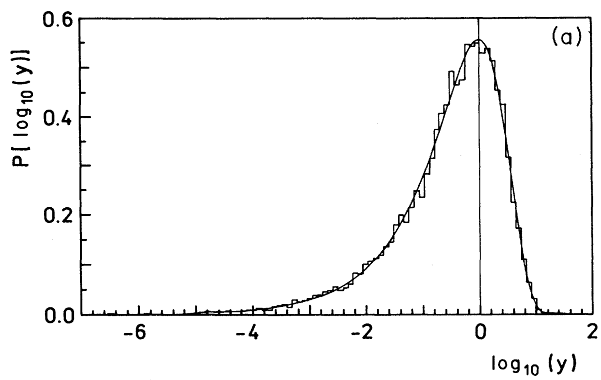
Observing that for unitary operator U

$$\|X\|=\|U^{\dagger n}XU^n\| , \tag{9}$$

and using Schwartz's inequality, one sees easily that

$$0\le\mu_n\le 2^{2n} . \tag{10}$$

Moreover, the minimal (maximal) value of μ_n is achieved if and only if U commutes (anticommutes) with X. In the latter case, one should not expect that the matrix ele-



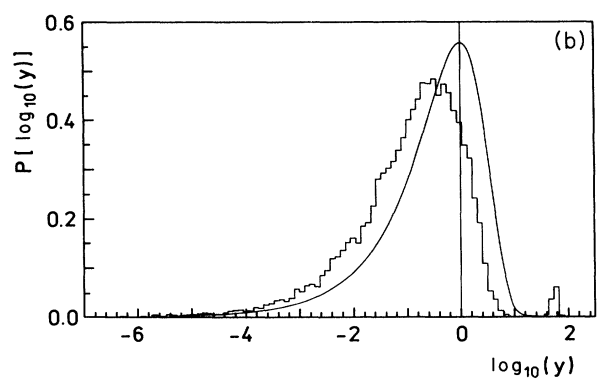


FIG. 1. Distribution of squares of absolute values of matrix elements for the operators (a) J_z and (b) J_z^2 in the Floquet basis of the kicked top. Solid lines correspond to the χ_ν^2 distribution provided by random-matrix theory. Total angular-momentum number $j=100$ and coupling constants $k=10.0$, $\beta=1.7$.

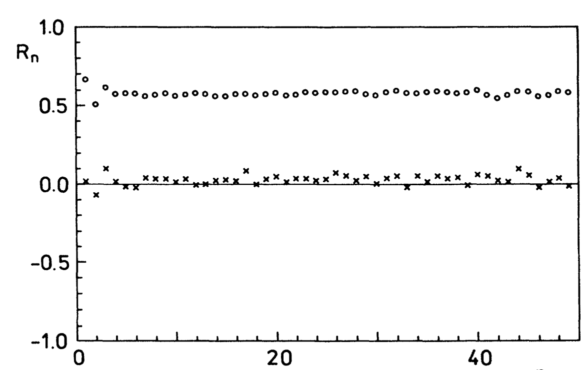


FIG. 2. Coefficients R_n for the operators J_z (asterisks) and J_z^2 (circles). Total angular-momentum number $j=50$, other parameters as in Fig. 1.

ments of X between the eigenstates of U are distributed randomly since anticommutation indeed requires strong correlations among them. In order to avoid effects corresponding to quantum recurrences, we shall restrict ourselves to some first coefficients, say $\{\mu_n, n=1,\ldots,N\}$.

It is convenient to invert the relation (7) between μ_n's and the iterations of the quantum map $X^{(l)}$ to obtain

$$R_n=1+\frac{n}{2}\sum_{l=1}^{n}(-1)^l\begin{bmatrix} n+l-1 \\ n-l \end{bmatrix}\frac{1}{l}\mu_l \;, \tag{11}$$

where

$$R_n=\frac{\mathrm{Re}\langle X|X^{(n)}\rangle}{\|X\|\,\|X^{(n)}\|} \;. \tag{12}$$

The extreme cases of commutation and anticommutation correspond now to $R_n=1$ and -1, respectively. It is thus reasonable to suppose that the most preferable case of statistical independence occurs in the middle between these values, i.e., for $R_n=0$. An operator X is random with respect to the system described by the evolution operator U if is orthogonal to its images $\langle X|U^{\dagger n}XU^n\rangle=0$.

As a support to this reasoning, we present in Fig. 2 the

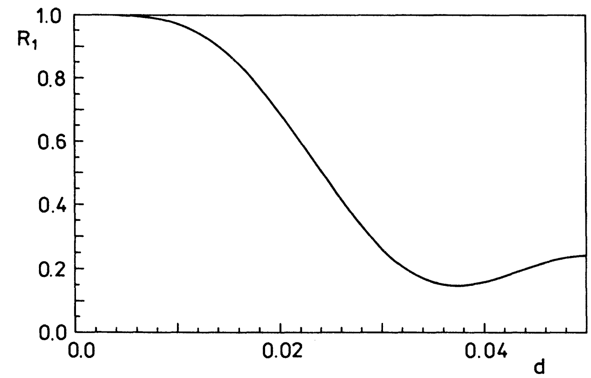


FIG. 4. Coefficient R_1 for the operator $V=\exp(-idJ_z)$ as a function of the parameter d.

relevant results for the operators J_z and J_z^2 and U given by Eq. (1) with $j=50$. The values of the coefficients R_n for both operators are nearly constant as functions of n and j, giving remarkably different results for J_z and J_z^2. For the former, which exhibits nice statistical properties according to the Porter-Thomas distribution with respect to U, the values of R_n fluctuate around 0, whereas for the latter the coefficients R_n prefer values close to 0.5.

The observed saturation of the coefficients R_n with the iteration number n allows us to argue that a random operator conserves its statistical properties during time evolution. In other words, a random operator X and its image $X^{(l)}$ have the same statistics of matrix elements.

Actually, the whole argument can be extended to include operators which are not observables. It is of some interest to know how the coefficients R_n change when the degree of commutativity changes gradually. In Fig. 3 we present statistics of matrix elements for the unitary operator $V=exp(-idJ_z)$ in the eigenbasis of U for $j=100$ and two different values of the parameter d. Two parities of eigenstates can be treated separately [7], so one has to deal with two matrices of dimension N equal to j and $j+1$. The matrix elements are normalized as

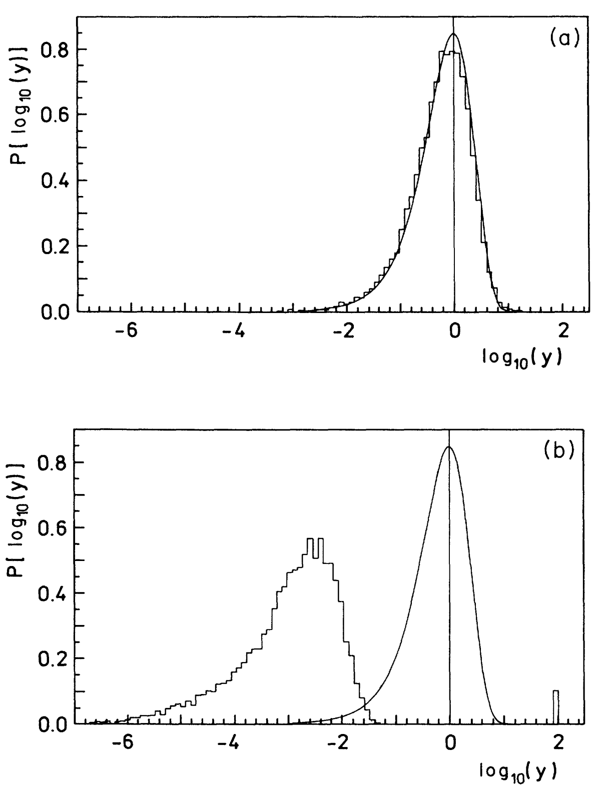


FIG. 3. Distribution of squares of absolute values of matrix elements for the operator $V=\exp(-idJ_z)$ in the Floquet basis of the kicked top for (a) $d=0.04$ and (b) $d=0.001$; system parameters as in Fig. 1. Solid lines correspond to the predictions of random-matrix theory.

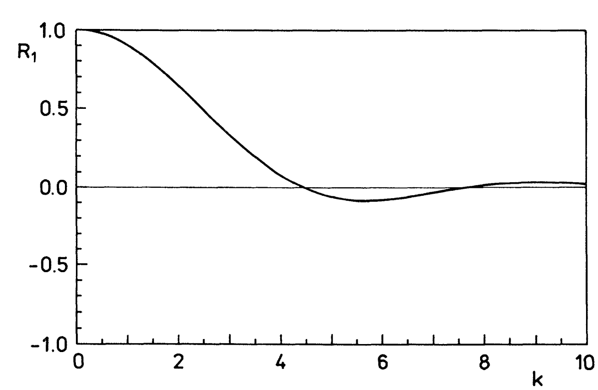


FIG. 5. Coefficient R_1 for the operator J_z calculated for different realizations of the kicked top in the transition region between regularity ($k\leq 4$) and chaos ($k\geq 4$); $j=50$, $\beta=1.7$.

$\sum_{m=1}^{N}|V_{ml}|^2=N$ so that $\langle|V_{ml}|^2\rangle=1$. For large values of d, the statistics is very well approximated by the expected χ_ν^2 curve ($\nu=2$ in this case since operator V is unitary), but differs significantly from the predictions of random-matrix theory for smaller d, where V is close to the unity operator. Note the peak of distribution at $\log_{10}y=2$ (i.e., at $y=j$) in Fig. 3(b), which is characteristic of the unity operator, under this normalization rule. This change of matrix-element distributions is accompanied by the dependence of the coefficient R_1 on the parameter d, as shown in Fig. 4. For those values of d for which the coefficient R_1 takes values smaller than 1.0, the operator $V(d)$ indeed gives the expected distribution of its matrix elements.

One can also address the question of how the coefficients R_n calculated for some observable vary when the system itself undergoes a transition between regular and chaotic regimes. In the case of the kicked top, where we have at our disposal two parameters k and β [Eq. (1)], we can easily investigate such a transition. While for $k=0$ the system is classically integrable, for higher values of this parameter chaotic behavior can be observed [7]. The resulting dependence on k of the coefficient R_1 for the operator J_z is presented in Fig. 5. Similar results can be obtained for R_n, $n>1$. It is worth stressing that a change of R_1 from 1 for $k=0$ to approximately 0 for $k>4$ is completed exactly in the transition region between regular and chaotic regimes [7]. This observation suggests a possibility of using coefficients R_n as indicators of such transitions.

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*Permanent address.

[1] See, e.g., M. L. Mehta, *Random Matrices* (Academic, New York, 1967), or T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981); for distributions of eigenvector components, consult especially early references of J. F. McDonald, J. Math. Phys. **13**, 1399 (1972).

[2] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, in press).

[3] F. M. Izrailev, Phys. Lett. **125A**, 250 (1987); J. Brickmann, Y. M. Engel, and R. D. Levine, Chem. Phys. Lett. **137**, 441 (1987); M. Kuś, J. Mostowski, and F. Haake, J. Phys. A **21**, L1073; Y. Alhassid and M. Feingold, Phys. Rev. A **39**, 374 (1989); A. Shudo and T. Matsushita, *ibid.* **39**, 282 (1989); **41**, 1912 (1990).

[4] A. Peres, Phys. Rev. A **30**, 504 (1984); in *Proceedings of the International Conference on Quantum Chaos, Trieste, 1990* (World Scientific, Singapore, in press).

[5] W. K. Wootters, Found. Phys. **20**, 1365 (1990).

[6] F. Haake and K. Życzkowski, Phys. Rev. A **42**, 1013 (1990); K. Życzkowski, J. Phys. A **23**, 4427 (1990).

[7] H. Frahm and H. J. Mikeska, Z. Phys. B **65**, 249 (1986); F. Haake, M. Kuś, and R. Scharf, *ibid.* **65**, 381 (1987); K. Nakamura, Y. Okazaki, and A. R Bishop, Phys. Rev. Lett. **57**, 5 (1986).