Infinite statistics and the relation to a phase operator in quantum optics

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(Received 24 April 1991)

A representation of the oscillators for infinite statistics in terms of bosons is given. The relation to the Susskind-Glogower operator is also discussed.

PACS number(s): 05.30.−d

In a recent paper Greenberg [1] has given a simple example of infinite statistics using Hegstrom's suggestion. He has also discussed the statistical mechanics of particles obeying infinite statistics. In this Brief Report I demonstrate how the single-particle operators corresponding to infinite statistics can be represented in terms of bosonic operators. I also show the relation between the operators of infinite statistics and the phase operator used to characterize the radiation fields.

Let us first consider a single-particle situation. Let b and $b^\dagger$ be the annihilation and creation operators. For infinite statistics b and $b^\dagger$ satisfy the relations

$$bb^\dagger=1\ ,\quad b^\dagger b\neq 1\ . \tag{1}$$

Let a and $a^\dagger$ be the annihilation and creation operators satisfying the Bosonic commutation relation $[a,a^\dagger]=1$. We now represent the operator b in terms of a as follows.

$$b=(1+a^\dagger a)^{-1/2}a\ ,\quad b^\dagger=a^\dagger(1+a^\dagger a)^{-1/2}\ . \tag{2}$$

It can be checked that the operators defined by (2) satisfy the algebra of infinite statistics. The Fock space representation [2] of the operators b and $b^\dagger$ is easily found from (2),

$$\begin{aligned} b|n\rangle&=(1+a^\dagger a)^{-1/2}a|n\rangle\\ &=(1+a^\dagger a)^{-1/2}\sqrt{n}\,|n-1\rangle=|n-1\rangle\ , \end{aligned} \tag{3}$$

where $|n\rangle$ is an eigenstate of $a^\dagger a$. Thus one can show that

$$\begin{aligned} &b|n\rangle=|n-1\rangle(1-\delta_{n0})\ ,\quad b^\dagger|n\rangle=|n+1\rangle\ ,\\ &b^\dagger b|n\rangle=|n\rangle(1-\delta_{n0})\ . \end{aligned} \tag{4}$$

Thus in the space of the Fock states of $a^\dagger a$ the operators b and $b^\dagger$ also act as annihilation and creation operators [1]. The number operator N as constructed by Greenberg holds,

$$N=\sum_{n=1}^{\infty}b^{\dagger n}b^{n}\ ,\quad N|m\rangle=m|m\rangle\ . \tag{5}$$

Thus the number operator for the infinite statistics is the same as the number operator $a^\dagger a$, i.e.,

$$\sum_n b^{\dagger n}b^{n}=a^\dagger a=N\ . \tag{6}$$

Recent papers [2,3] also introduce the "coherent states" of the operator b defined by

$$b|z\rangle=z|z\rangle\ ,\quad |z\rangle=\sqrt{1-|z|^2}\sum_{m=0}^{\infty}z^m|m\rangle\ ,\quad |z|<1\ . \tag{7}$$

These states lead to a number distribution given by

$$p(n)=|z|^{2n}(1-|z|^2)=\frac{(\langle n\rangle)^n}{(1+\langle n\rangle)^{n+1}}\ , \tag{8}$$

where $\langle n\rangle$ is the mean of the number operator (not $b^\dagger b$)

$$\langle n\rangle=|z|^2/(1-|z|^2)\ . \tag{9}$$

It is interesting to note that the *number operator* has a *Bose-Einstein distribution* rather than a Poisson distribution as is the case for the coherent states of the usual boson annihilation operator.

We next note that the operator algebra (1) is realized by the phase operator $e^{i\hat\phi}$ in quantum optics. Susskind and Glogower [4] introduced the phase operator by the relation

$$e^{i\hat\phi}=(aa^\dagger)^{-1/2}a\ , \tag{10}$$

which is the same as the b operator as given by Eq. (2). Thus we find that the phase operator which is introduced to study the phase characteristics of the radiation fields [5] gives us a realization of the b operator of the infinite statistics.

For many particles, we can satisfy (1) and (2) for each particle and thus we use

$$\begin{aligned} &b_jb_j^\dagger=1\ ,\quad b_j=(1+a_j^\dagger a_j)^{-1/2}a_j\quad \forall j\ ,\\ &a_ja_l^\dagger=0 \text{ if } j\neq l\ . \end{aligned} \tag{11}$$

The latter condition is necessary as we have to have $b_jb_l^\dagger=\delta_{jl}$ so as to generate infinite statistics. Note that if one were to use

$$[a_j,a_l^\dagger]=0\quad \text{for } j\neq l\ , \tag{12}$$

then the grand partition function of a system of free particles will be the same as that for a system of particles obeying Bose statistics.

Finally we consider the time evolution of the Jaynes-Cummings Hamiltonian where the bosonic oscillator has been replaced by an oscillator obeying algebra of infinite statistics, i.e., we consider the Hamiltonian

$$H=\hbar\omega_0S^z+\hbar N\omega_0+\hbar g(S^+b+S^-b^\dagger)\ . \tag{13}$$

Here N is defined by (5). Assuming that the atom was initially in the excited state then the calculations show that the probability $p(t)$ of finding the atom in the excited state is

$$p(t)=\sum_{n=0}^{\infty} p_n \cos^2(gt) . \tag{14}$$

Here p_n gives the initial distribution of the b oscillator. On using the normalization of the density matrix for the b oscillator, we get

$$p(t)=\cos^2(gt) . \tag{15}$$

It is remarkable that the excitation probability is always periodic irrespective of the initial state of the b oscillator. This is in contrast to the well-known result [5], for the standard Jaynes-Cummings model which, depending on the initial state of the field, exhibits collapse and revival of the Rabi oscillations. Note further that the frequency of oscillation depends on the coupling constant g and not on the number of excitations in the b oscillator. These differences can be traced back to the n independence of the matrix element $\langle n+1|b^\dagger|n\rangle$.

The author is grateful to V. Srinivasan for discussions on the subject of infinite statistics.

[1] O. W. Greenberg, Phys. Rev. Lett. **64**, 705 (1990); for generalization see S. Chaturvedi, A. K. Kapoor, R. Sandhya, V. Srinivasan, and R. Simon, Phys. Rev. A **43**, 4555 (1991).

[2] I have now learned from O. W. Greenberg that boson representations of a q oscillator have been given by Freund and Nambu [as quoted by O. W. Greenberg, in *Proceedings of the Workshop on Quantum Groups,* edited by T. Curtright, D. Fairlie, and C. Zachos (World Scientific, Singapore, 1990), p. 166].

[3] J. H. Shapiro, S. R. Shepard, and N. C. Wong, in *Coherence and Quantum Optics VI,* edited by J. H. Eberly, L. Mandel, and E. Wolf (Plenum, New York, 1989), p. 1077.

[4] L. Susskind and J. Glogower, Phys. **1**, 49 (1964).

[5] This is also in contrast to the results for the Jaynes-Cummings model with bosons replaced by q oscillators [M. Chaichian, D. Ellinas, and P. Kulish, Phys. Rev. Lett. **65**, 980 (1990)].