Para-Bose oscillator as a deformed Bose oscillator

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We show that a single para-Bose oscillator may be regarded as a deformed Bose oscillator. We construct a nonlinear realization of the single-mode para-Bose algebra in terms of a single boson. This is in contrast to the Green decomposition that expresses a single para-Bose oscillator in terms of p anticommuting bosons. We also construct an operator canonically conjugate to the para-Bose annihilation operator that permits us to carry over familiar constructions to the para-Bosonic case.

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Quantum deformation algebras and groups [1–14] have attracted considerable attention in recent years. In particular, deformations of the Heisenberg algebra [2–14] and those of su(2) and su(1,1) [1,8–10] algebras have been extensively investigated. The particular deformation of the Heisenberg algebra that has been considered in the literature is

$$aa^\dagger - qa^\dagger a = 1 , \tag{1}$$

$$[a,\mathcal{N}]=a, \quad [a^\dagger,\mathcal{N}]=-a^\dagger . \tag{2}$$

The q commutator (1) may equivalently be written as

$$[a,a^\dagger]=g(\mathcal{N}) , \tag{3}$$

where, for $q\neq 0$,

$$g(\mathcal{N})=q^{\mathcal{N}} , \tag{4}$$

and, for $q=0$,

$$g(\mathcal{N})=\theta(1-\mathcal{N}) , \tag{5}$$

where $\theta(x)=1$ for $x>0$ and 0 for $x\leq 0$. With $g(\mathcal{N})$ given by (4) and (5), one may regard the commutation relations (2) and (3) as the definition of the q-Heisenberg algebra. This transcription of the q-Heisenberg algebra suggests more general possibilities corresponding to choices of $g(\mathcal{N})$ other than those given above. Indeed, we show that the single-mode para-Bose algebra may also be viewed as a distortion of the Heisenberg algebra.

A single-mode para-Bose system [15–19] is characterized by the commutation relations

$$[a,N]=a, \quad [a^\dagger,N]=-a^\dagger , \tag{6}$$

where

$$N=\tfrac{1}{2}\{a^\dagger,a\} . \tag{7}$$

The vacuum state is assumed to satisfy

$$a|0\rangle=0 , \tag{8}$$

$$aa^\dagger|0\rangle=p|0\rangle , \tag{9}$$

where p is the order of the para-Bose system. From the algebra (6) and (7) and the conditions (8) and (9) on $|0\rangle$, it follows that

$$\mathcal{N}=\tfrac{1}{2}\{a^\dagger,a\}-p/2 , \tag{10}$$

which has eigenvalues $n=0,1,2,\ldots$. The corresponding eigenstates may be obtained by repeated applications of $a^\dagger$ on $|0\rangle$. The normalized eigenstates are given by

$$|2n\rangle=\frac{(a^\dagger)^{2n}}{[2^n n!p(p+2)\cdots(p+2n-2)]^{1/2}}|0\rangle , \tag{11}$$

$$|2n+1\rangle=\frac{(a^\dagger)^{2n+1}}{[2^n n!p(p+2)\cdots(p+2n)]^{1/2}}|0\rangle . \tag{12}$$

The action of the creation and annihilation operators on these states is given by

$$a^\dagger|2n\rangle=\sqrt{(2n+p)}|2n+1\rangle ,$$
$$a^\dagger|2n+1\rangle=\sqrt{(2n+1)}|2n+2\rangle ,$$
$$a|2n\rangle=\sqrt{(2n)}|2n-1\rangle ,$$
$$a|2n+1\rangle=\sqrt{(2n+p)}|2n\rangle . \tag{13}$$

We may rewrite these relations as

$$a|n\rangle=\sqrt{f(n)}|n-1\rangle , \tag{14}$$

$$a^\dagger|n\rangle=\sqrt{f(n+1)}|n+1\rangle , \tag{15}$$

where

$$f(n)=n+\tfrac{1}{2}[1-(-1)^n](p-1) . \tag{16}$$

From (14) and (15) it follows that

$$aa^\dagger=f(\mathcal{N}+1) , \tag{17}$$

$$a^\dagger a=f(\mathcal{N}) , \tag{18}$$

and hence

$$[a,a^\dagger]=g(\mathcal{N}) , \tag{19}$$

where

$$g(\mathcal{N})=f(\mathcal{N}+1)-f(\mathcal{N})=1+(-1)^{\mathcal{N}}(p-1) . \tag{20}$$

The commutators of a and $a^\dagger$ with the number operator $\mathcal{N}$ are, of course, the usual ones,

$$[a,\mathcal{N}]=a, \quad [a^\dagger,\mathcal{N}]=-a^\dagger . \tag{21}$$

The para-Bose commutation relations (6) and (7) may therefore be replaced by (19). This fact has also been noted by Mukunda *et al.* [18]. This observation permits us, as in the case of q deformations [4,5,13,14], to express the para-Boson annihilation and creation operators a and $a^\dagger$ in terms of bosonic annihilation and creation operators b and $b^\dagger$ as

$$a=\left[\frac{f(b^\dagger b+1)}{b^\dagger b+1}\right]^{1/2} b \; , \tag{22a}$$

$$a^\dagger=b^\dagger\left[\frac{f(b^\dagger b+1)}{b^\dagger b+1}\right]^{1/2} . \tag{22b}$$

The number operator $\mathcal{N}$ for the para-Boson expressed in terms of a boson is

$$\mathcal{N}=b^\dagger b \; . \tag{23}$$

In view of the relation (23), we may invert the relations (22) to express the bosonic annihilation and creation operators b and $b^\dagger$ in terms of those for a para-Boson

$$b=\left[\frac{\mathcal{N}+1}{f(\mathcal{N}+1)}\right]^{1/2} a \; , \tag{24}$$

$$b^\dagger=a^\dagger\left[\frac{\mathcal{N}+1}{f(\mathcal{N}+1)}\right]^{1/2} . \tag{25}$$

These relations enable us to construct an operator $A^\dagger$ canonically conjugate to a, satisfying

$$[a,A^\dagger]=1 \; , \tag{26}$$

$$[A^\dagger,\mathcal{N}]=-A^\dagger \; , \tag{27}$$

as was done by us for the case of q deformations [13,14]. From (23)–(25) it follows that the number operator $\mathcal{N}$ can be written as

$$\mathcal{N}=a^\dagger\frac{\mathcal{N}+1}{f(\mathcal{N}+1)}a \; . \tag{28}$$

Defining

$$A^\dagger=a^\dagger\frac{\mathcal{N}+1}{f(\mathcal{N}+1)} \; , \tag{29}$$

and using (17) and (28), it is easy to see that $A^\dagger$ given by (29) indeed satisfies (26). Equation (27) is a simple consequence of (21). The conjugate of the relation (26) gives

$$[A,a^\dagger]=1 \; , \tag{30}$$

implying that the operator $a^\dagger$ is canonically conjugate to A.

Having constructed the operator $A^\dagger$, the construction of the para-Bose coherent states, the eigenstates of the para-Bose annihilation operator a,

$$a|z\rangle=a|z\rangle \; , \tag{31}$$

is immediate. They are given by

$$|z\rangle=C(|z|)\exp(zA^\dagger)|0\rangle \; . \tag{32}$$

Calculation of the normalization constant $C(|z|)$,

$$C(|z|)=[\langle 0|\exp(z^*A)\exp(zA^\dagger)|0\rangle]^{-1/2} \; , \tag{33}$$

is most conveniently done by using (25) and (23) to express $A^\dagger$ in terms of bosonic operators as

$$A^\dagger=b^\dagger[h(b^\dagger b)]^{1/2} \; , \tag{34}$$

where

$$h(b^\dagger b)=\frac{(b^\dagger b+1)}{f(b^\dagger b+1)} \; . \tag{35}$$

This gives

$$\exp(zA^\dagger)|0\rangle=\sum_{n=0}^{\infty}\frac{z^n}{n!}\{b^\dagger[h(b^\dagger b)]^{1/2}\}^n|0\rangle \; ,$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\left[\prod_{m=1}^{n}\sqrt{h(m-1)}\right]|n\rangle \; , \tag{36}$$

and hence

$$\langle 0|\exp(z^*A)\exp(zA^\dagger)|0\rangle=\sum_{n=0}^{\infty}\frac{(zz^*)^n}{n!}\left[\prod_{m=1}^{n}h(m-1)\right] . \tag{37}$$

From (35), it follows that

$$h(2m)=\frac{2m+1}{2m+p}, \quad h(2m+1)=1 \; . \tag{38}$$

Using (38) in (37), we finally obtain

$$C(|z|)=[2^{p/2-1}\Gamma(p/2)]^{1/2}[F(|z|^2)]^{-1/2} \; , \tag{39}$$

where

$$F(z)=z^{1-p/2}[(I_{p/2-1}(z)+I_{p/2}(z)] \; . \tag{40}$$

Para-Bose coherent states have also been constructed directly by Mukunda *et al.* [18] and by Sharma, Mehta, and Sudarshan [19]. Their result is

$$|z\rangle=C(|z|)F(za^\dagger)|0\rangle \; , \tag{41}$$

where $C(|z|)$ and $F(z)$ are given above. It is indeed gratifying to see that our construction of the operator $A^\dagger$ canonically conjugates to enable us to express the operator appearing on the right-hand side of (33), involving modified Bessel functions by a single exponential as in the Bose case.

To conclude, we have shown that a para-Bose oscillator may be regarded as a deformed Bose oscillator. We have constructed a nonlinear realization of a para-Boson in terms of a single boson in the spirit of the work of Holstein and Primakoff [20]. This is to be contrasted with the Green decomposition [15], which expresses a para-Boson of order p as a linear combination of p mutually anticommuting Bosons. Further, we note that all the constructions given above perform verbatim for any single-mode system satisfying (19) and (21).

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