# Aspects of $q$-oscillator quantum mechanics

S. Chaturvedi and V. Srinivasan
*School of Physics, University of Hyderabad, Hyderabad 500134, India*
(Received 28 November 1990)

We investigate some aspects of $q$ Heisenberg algebra. We show how su(2) and su(1,1) generators can be constructed in terms of the $q$ creation and annihilation operators. We also construct the coherent states for the $q$ oscillator and show that they can be obtained by the action of a displacement operator on the vacuum. For the multimode case with $q=0$, corresponding to the infinite statistics of Greenberg [Phys. Rev. Lett. **64**, 705 (1990)], we generalize our single-mode construction to obtain the corresponding coherent states. These states, which are eigenstates of the annihilation operators, interestingly, turn out to be degenerate owing to the noncommutativity of the displacement operators.

PACS number(s): 05.30.$-$d

## I. INTRODUCTION

Quantum-deformation algebras and groups [1] have attracted considerable attention in recent years particularly because of their relevance to certain models in field theory and statistical mechanics Quantum deformation of the Heisenberg algebra [2–10] and the su(2) and su(1,1) algebras have been extensively investigated [1,8–10]. The study of a particular deformation of the Heisenberg algebra—leading to infinite statistics—has been initiated by Greenberg [6] and investigated by Greenberg [11] and Mohapatra [12] with a view to seeking small violations of the Pauli exclusion principle in physical systems.

This work is a continuation of our earlier work [13] on the single-mode Heisenberg $q$ algebra

$$aa^\dagger - qa^\dagger a = 1 , \tag{1.1}$$

In particular, we construct an operator $A^\dagger$ which for the $q$ algebra is a truly canonically conjugate to $a$, i.e., it satisfies

$$[a, A^\dagger] = 1 , \tag{1.2}$$

$$[A^\dagger, N] = -A^\dagger , \tag{1.3}$$

where $N$ is the number operator for the $q$ commutation relations (1.1) and can be expressed in terms of $a$ and $A^\dagger$ as

$$N = A^\dagger a . \tag{1.4}$$

The operator $A^\dagger$ satisfying (1.2)–(1.4) enables us to carry over standard constructions of coherent states to the $q$ case. For the case of infinite statistics ($q=0$), we also discuss a multimode generalization of (1.2)–(1.4) and in turn construct multimode coherent states and pair coherent states. Further we construct and discuss nonlinear realizations of su(2) and su(1,1) algebras in terms of $a$ and $a^\dagger$, satisfying the $q$ commutator (1.1) in the same spirit as the work of Holstein and Primakoff [14] for the bosonic case.

## II. $q$ ALGEBRA

The $q$ Heisenberg algebra is

$$aa^\dagger - qa^\dagger a = 1 . \tag{2.1}$$

Introducing a number operator $N$ satisfying

$$[N,a] = -a , \quad [N,a^\dagger] = +a^\dagger , \tag{2.2}$$

and defining [1]

$$c = q^{-\lambda N/2} a , \quad c^\dagger = a^\dagger q^{-\lambda N/2} , \tag{2.3}$$

with $\lambda$ real, the algebra (2.1) can alternatively be cast into the form

$$cc^\dagger - q^{1-\lambda} c^\dagger c = q^{-\lambda N} , \tag{2.4}$$

particular cases of which corresponding to $\lambda=1$ and $\lambda=\frac{1}{2}$ are often used in the literature. In the present work we shall exclusively work with (2.1).

An explicit expression for $N$, in the normal-ordered form, constructed in our earlier work [13], is

$$N = \sum_{n=1}^{\infty} \frac{(1-q)^n}{(1-q^n)} a^{\dagger n} a^n . \tag{2.5}$$

The normalized eigenstates of $N$ are

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n_q!}} |0\rangle , \tag{2.6}$$

where

$$n_q \equiv \frac{(1-q^n)}{(1-q)} = 1 + q \cdots + q^{n-1} . \tag{2.7}$$

The action of $a$ and $a^\dagger$ on $|n\rangle$ is given by

$$a|n\rangle = \sqrt{n_q}\,|n-1\rangle , \quad a^\dagger|n\rangle = \sqrt{(n+1)_q}\,|n+1\rangle . \tag{2.8}$$

It follows from (2.8) that

$$a^\dagger a|n\rangle = n_q |n\rangle , \tag{2.9}$$

 ©1991 The American Physical Society

which written as an operator relation gives

$$a^\dagger a=\frac{(1-q^N)}{(1-q)}\ , \tag{2.10}$$

leading to the following useful expression for $N$ as a function $a^\dagger a$:

$$N=\frac{\ln[1-(1-q)a^\dagger a]}{\ln q}=\frac{\ln[a,a^\dagger]}{\ln q}\ . \tag{2.11}$$

This expression for $N$ has been discussed by many authors [2,4,5, 8–12]. It, however, becomes singular in the limit $q\to 0$ limit. The normal-ordered expression (2.5), in contrast, smoothly goes over to the expression given by Greenberg [6] in this limit. Incidentally, taking expectation values of $N$ given by (2.5) between the number states one obtains an interesting identity

$$\begin{aligned} S_m &\equiv \frac{(1-q^m)}{(1-q)}+\frac{(1-q^m)(1-q^{m-1})}{(1-q^2)}+\cdots \\ &\quad +\frac{(1-q^m)(\cdots)\cdots(1-q)}{(1-q^m)} \\ &=m\ , \end{aligned} \tag{2.12}$$

which can easily be proved by showing that $S_{m+1}-S_m=1$.

The relation (2.11) permits us to write the commutator $[a,a^\dagger]$ as

$$[a,a^\dagger]=q^N\ . \tag{2.13}$$

From (2.5) it follows that the number operator $N$ can be written as

$$N=a^\dagger X a\ , \tag{2.14}$$

where

$$X=\sum_{n=1}^{\infty}\frac{(1-q)^n}{(1-q^n)}a^{\dagger n-1}a^{n-1}\ . \tag{2.15}$$

If we define

$$A^\dagger=a^\dagger X\ ,\quad A=Xa\ , \tag{2.16}$$

then $N$ may be written as

$$N=A^\dagger a=a^\dagger A\ . \tag{2.17}$$

Further it is easily verified that

$$[a,A^\dagger]=1\ ,\quad [A,a^\dagger]=1\ , \tag{2.18}$$

and

$$[N,A^\dagger]=A^\dagger\ ,\quad [A,N]=A\ . \tag{2.19}$$

$A^\dagger$ ($A$) is thus a truly canonically conjugate to $a$ ($a^\dagger$).

Using (2.16) and (2.17) we may rewrite (2.18) as

$$aa^\dagger X=N+1\ , \tag{2.20}$$

which gives us $X$ as a function of $a^\dagger a$

$$X=(1+qa^\dagger a)^{-1}(N+1)\ . \tag{2.21}$$

Further, since $X$ is a function only of $a^\dagger a$, it follows that

$$[X^\gamma a,a^\dagger X^{1-\gamma}]=1\ . \tag{2.22}$$

A particularly convenient choice is $\gamma=\frac{1}{2}$ which gives

$$[B,B^\dagger]=1\ ,\quad \mathcal{N}\equiv B^\dagger B=N\ , \tag{2.23}$$

where

$$B=\sqrt{X}a\ . \tag{2.24}$$

We have thus constructed bosonic creation and annihilation operators in terms of those of a $q$ oscillator. Using (2.10) and (2.21) we may rewrite (2.24) as

$$B=\left[\frac{(1-q)(N+1)}{(1-q^{N+1})}\right]^{1/2}a\ . \tag{2.25}$$

Further, using

$$N=\mathcal{N}\ , \tag{2.26}$$

and inverting (2.25), we obtain [4,5,15]

$$a=\left[\frac{(1-q^{\mathcal{N}+1})}{(1-q)(\mathcal{N}+1)}\right]^{1/2}B\ . \tag{2.27}$$

In the special case of $q=0$, one obtains a relation

$$a=(1+B^\dagger B)^{-1/2}b\ , \tag{2.28}$$

well known in the quantum-optics literature, defining the phase operators for a single boson [16].

The eigenstates (2.6) of $N$ may be expressed in terms of $A^\dagger$ or $B^\dagger$ as

$$|n\rangle=\sqrt{n_q!}\frac{(A^\dagger)^n}{n!}|0\rangle\ , \tag{2.29}$$

$$|n\rangle=\frac{(B^\dagger)^n}{\sqrt{n!}}|0\rangle\ . \tag{2.30}$$

## III. REALIZATIONS OF su(2) AND su(1,1) ALGEBRAS IN TERMS OF A q OSCILLATOR

We now construct Hermitian and non-Hermitian realizations of the su(2) algebra

$$[J_z,J_\pm]=\pm J_\pm\ ,\quad [J_+,J_-]=2J_z\ , \tag{3.1}$$

and those of su(1,1) algebra

$$[K_z,K_\pm]=\pm K_\pm\ ,\quad [K_-,K_+]=2K_z \tag{3.2}$$

in terms of a $q$ oscillator. To do this we set

$$J_+=a^\dagger G(N)\ ,\quad J_-=G(N)a\ ,\quad J_z=H(N) \tag{3.3}$$

and fix the functional forms of $G$ and $H$ by applying the commutation relations (3.1) on the number states of the $q$ oscillator. This gives

$$J_+=a^\dagger\sqrt{X}\sqrt{(2\alpha-N)}\ ,\quad J_-=\sqrt{(2\alpha-N)}\sqrt{X}a\ , \tag{3.4a}$$

$$J_z=N-\alpha\ , \tag{3.4b}$$

where $\alpha$ is a real number.

Following essentially the same analysis as above, we obtain

$$K_- = \sqrt{(2\beta+N)}\sqrt{X}a \ , \quad K_+ = a^\dagger\sqrt{X}\sqrt{(2\beta+N)} \ , \tag{3.5a}$$

$$K_z = \beta + N \ . \tag{3.5b}$$

Using (2.24) and the fact that $\mathcal{N}=N$, we obtain

$$J_+ = B^\dagger\sqrt{(2\alpha-\mathcal{N})} \ , \quad J_- = \sqrt{(2\alpha-\mathcal{N})}B \ , \tag{3.6a}$$

$$J_z = \mathcal{N} - \alpha \ , \tag{3.6b}$$

and

$$K_- = \sqrt{(2\beta+\mathcal{N})}B \ , \quad K_+ = B^\dagger\sqrt{(2\beta+\mathcal{N})} \ , \tag{3.7a}$$

$$K_z = \beta + \mathcal{N} \ , \tag{3.7b}$$

which are, respectively, the Holstein-Primakoff [14] realizations of su(2) and su(1,1) in terms of a boson.

In the $q=0$ case, choosing $\beta=\frac{1}{2}$, and using the fact that in this case $X=1+N$, we obtain a rather simple looking realization of su(1,1) generators

$$K_- = (N+1)a \ , \quad K_+ = a^\dagger(N+1) \ , \quad K_z = N + \tfrac{1}{2} \ , \tag{3.8}$$

which on using (2.2) may also be written as

$$K_- = aN \ , \quad K_+ = Na^\dagger \ , \quad K_z = N + \tfrac{1}{2} \ . \tag{3.9}$$

The validity of this special realization of su(1,1) may also be directly verified by making use of the fact that, owing to $aa^\dagger = 1$, we have

$$N^2 = \sum_{n=1}^{\infty} (2n-1)a^{\dagger n}a^n \ . \tag{3.10}$$

We now turn our attention to non-Hermitian realizations of su(2) and su(1,1) in terms of the $q$ annihilation and creation operators. actually, our construction of the operator $A^\dagger$ given by (2.18), which is canonically conjugate to $a$ provides us with a powerful method for turning known bosonic realizations such as (3.6) and (3.7) into $q$-oscillator realizations by simply making the replacements

$$B \to a \ , \quad B^\dagger \to A^\dagger \ . \tag{3.11}$$

In the case of su(1,1), for instance, there is a well-known bosonic realization used in the construction of squeezed coherent states [17]

$$K_+ = \tfrac{1}{2}B^{\dagger 2} \ , \quad K_- = \tfrac{1}{2}B^2 \ , \quad K_z = \tfrac{1}{4}(2\mathcal{N}+1) \ . \tag{3.12}$$

Using (3.12) we obtain its $q$ counterpart as

$$K_+ = \tfrac{1}{2}A^{\dagger 2} \ , \quad K_- = \tfrac{1}{2}a^2 \ , \quad K_z = \tfrac{1}{4}(2N+1) \ , \tag{3.13}$$

which has been used to construct "squeezed" coherent states for the $q$ oscillator in analogy with the bosonic ones [13].

## IV. COHERENT STATES

The eigenstates

$$a|\lambda\rangle = \lambda|\lambda\rangle \tag{4.1}$$

of the annihilation operator of a $q$ oscillator have been constructed by expanding $|\lambda\rangle$ in terms of the number states $|n\rangle$ and solving the recursion relations for the coefficients. This gives [4,5]

$$|\lambda\rangle = [e_q(|\lambda|^2)]^{-1/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n_q!}}|n\rangle \ , \tag{4.2}$$

where $e_q(x)$ is the $q$ generalization of the exponential function

$$e_q(x) = \sum_{n=0}^{\infty} \frac{(x)^n}{n_q!} \ . \tag{4.3}$$

Further, in analogy with the bosonic coherent states, $|\lambda\rangle$ may also be written as

$$|\lambda\rangle = [e_q(|\lambda|^2)]^{-1/2} e_q(\lambda a^\dagger)|0\rangle \ . \tag{4.4}$$

The operator $A^\dagger$ satisfying (2.18) constructed above enables us to rewrite (4.4) more simply as

$$|\lambda\rangle = [e_q(|\lambda|^2)]^{-1/2} e^{\lambda A^\dagger}|0\rangle \ , \tag{4.5}$$

thereby replacing the $q$ exponential of $\lambda a^\dagger$ by the ordinary exponential of $\lambda A^\dagger$. The form of $|\lambda\rangle$ thus becomes completely analogous to that in the bosonic case.

For the multimode generalization of the $q=0$ case, i.e., for the case of infinite statistics of Greenberg [6]

$$a_i a_j^\dagger = \delta_{ij} \ , \tag{4.6}$$

we had constructed, in our earlier work operators $A_j^\dagger$ which satisfy

$$[a_i, A_j^\dagger] = \delta_{ij} \ . \tag{4.7}$$

These operators are given by

$$A_i^\dagger = a_i^\dagger + \sum_k a_k^\dagger a_i^\dagger a_k + \sum_{k,l} a_k^\dagger a_l^\dagger a_i^\dagger a_l a_k + \cdots \ . \tag{4.8}$$

They, however, do not commute with each other.

$$[A_i^\dagger, A_j^\dagger] \neq 0 \ . \tag{4.9}$$

These operators enable us to construct multimode coherent states $|\lambda\rangle$ satisfying

$$a_i|\lambda\rangle = \lambda_i|\lambda\rangle \ , \tag{4.10}$$

and are given by

$$|\lambda\rangle = \left[\prod_i e_q(|\lambda_i|^2)\right]^{-1/2} \prod_i \exp(\lambda_i A_i^\dagger)|0\rangle \ . \tag{4.11}$$

Note that since $A_i^\dagger$ do not commute with each other, each ordering in the product of exponentials that occurs on the right-hand side leads to a different state. All these states, however, correspond to the eigenvalue $\lambda_i$ of $a_i$ and therefore constitute a degenerate set of solutions of the eigenvalue equation (4.11).

Finally, we consider the pair coherent states [18,19] for two modes $a_1$ and $a_2$ obeying (4.6). In analogy with the bosonic case, we define pair coherent states $|\zeta,p\rangle$ as the simultaneous eigenstates of the operators $a_1 a_2$ and $N_1 - N_2$,

$$a_1 a_2|\zeta,p\rangle=\zeta|\zeta,p\rangle\ , \tag{4.12a}$$

$$(N_1-N_2)|\zeta,p\rangle=p|\zeta,p\rangle\ , \tag{4.12b}$$

where

$$N_i=a_i^\dagger a_i+\sum_k a_k^\dagger a_i^\dagger a_i a_k+\sum_{k,l} a_k^\dagger a_l^\dagger a_i^\dagger a_i a_l a_k+\cdots \tag{4.13}$$

are the number operators for the two modes. The pair coherent states $|\zeta,0\rangle$ corresponding to the zero eigenvalue of the number difference operator $N_1-N_2$ can easily be constructed by noting that the algebra $a_i a_j^\dagger=\delta_{ij}$ implies that $b\equiv a_1 a_2$ satisfies

$$bb^\dagger=1\ . \tag{4.14}$$

The eigenstates of $b$ satisfying (4.14) have already been constructed. Hence the pair coherent states $|\zeta,0\rangle$ are given by

$$\begin{aligned}|\zeta,0\rangle&=(1-|\zeta|^2)^{1/2}\sum_{n=0}^{\infty}(\zeta b^\dagger)^n|0,0\rangle\\&=(1-|\zeta|^2)^{1/2}\sum_{n=0}^{\infty}(\zeta a_2^\dagger a_1^\dagger)^n|0,0\rangle\ .\end{aligned} \tag{4.15}$$

## V. SUPERSYMMETRIC $q$ OSCILLATOR

The results given in Sec. II permit an easy construction of a supersymmetric $q$ oscillator. Consider the Hamiltonian

$$H=\omega(N+b^\dagger b)\ , \tag{5.1}$$

where $b$ and $b^\dagger$ are fermion creation and annihilation operators and $N$ is the number operator (2.7). For a $q$ oscillator. We assume that the creation and annihilation operators of the $q$ oscillator commute with $b$ and $b^\dagger$. As noted above, $N$ can be written as

$$N=B^\dagger B\ , \tag{5.2}$$

where $B$ and $B^\dagger$ given by (2.26) or (2.27) satisfy bosonic commutation relations. If we now define

$$Q=B^\dagger b\ ,\quad Q^\dagger=b^\dagger B\ , \tag{5.3}$$

then it is easily seen that

$$H=\omega\{Q,Q^\dagger\}\ ,\quad [Q,H]=0\ ,\quad \{Q,Q\}=0\ , \tag{5.4}$$

which is the usual supersymmetry algebra.

## VI. CONCLUSIONS

We have investigated some aspects of $q$ Heisenberg algebra. We have shown how one can construct su(2) and su(1,1) generators in terms of the $q$ creation and annihilation operators. An important construction in our work is that of the operator $A^\dagger$ which is canonically conjugate to $a$. This enables us to convert known bosonic realizations of su(1,1) and su(2) algebras into those in terms of $q$ creation and annihilation operators. It also enables us to express coherent states of $q$ oscillators as having been obtained by the action of a displacement operator on the vacuum state. For the case of infinite statistics of Greenberg, we have generalized our single-mode construction to obtain the corresponding coherent states. These states, which are simultaneous eigenstates of the annihilation operators, interestingly, turn out to be degenerate owing to the noncommutativity of the displacement operators for each mode.

## ACKNOWLEDGMENT

We are grateful to Ashok Kapoor for numerous discussions.

[1] C. Zachos, in *Symmetries in Science*, edited by V. B. Gruber (Plenum, New York, in press). This work contains an exhaustive list of references on $q$ algebras and their applications.
[2] V. Kuryshkin, Ann. Fond. Louis de Broglie **5**, 111 (1980).
[3] M. Schmutz, Physica A **101**, 1 (1980).
[4] G. Brodimas, A. Jannussis, D. Sourlas, and V. Zisis, Lett. Nuovo Cimento **30**, 123 (1981).
[5] G. Brodimas, A. Jannussis, D. Sourlas, V. Zisis, and P. Poupoulos, Lett. Nuovo Cimento **31**, 177 (1981).
[6] O. W. Greenberg, Phys. Rev. Lett. **64**, 705 (1990).
[7] A. B. Govorkov, Thoer. Math. Phys. **54**, 234 (1983).
[8] L. C. Biedenharn, J. Phys. A **22**, L873 (1989).
[9] A. J. Macfarlane, J. Phys. A **22**, 4581 (1989).
[10] M. Chaichian and P. Kulsih, Phys. Lett. B **234**, 72 (1990).
[11] O. W. Greenberg, *Proceedings of the Argonne Workshop on Quantum Groups*, edited by T. Curtright, D. Fairlie, and C. Zachos (World Scientific, Singapore, 1990), pp. 166–180; Phys. Rev. D **43**, 4111 (1991).
[12] R. N. Mohapatra, Phys. Lett. B **242**, 407 (1990).
[13] S. Chaturvedi, A. K. Kapoor, R. Sandhya, V. Srinivasan, and R. Simon, Phys. Rev. A **43**, 4555 (1991).
[14] T. Holstein and R. Primakoff, Phys. Rev. **58**, 1098 (1940).
[15] P. G. O. Freund and Y. Nambu (unpublished).
[16] G. S. Agarwal (unpublished).
[17] H. P. Yuen, Phys. Rev. A **13**, 4 (1976).
[18] D. Bhaumik, K. Bhaumik, and B. Dutta-Roy, J. Phys. A **9**, 1507 (1976).
[19] G. S. Agarwal, J. Opt. Soc. Am. B **5**, 1940 (1988).