Unifying stochastic Markov process and its transition probability density function

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A stochastic Markov process is defined which generalizes a class of processes previously considered by the author [Phys. Rev. A **42**, 4485 (1990)]. The enlarged class unifies such apparently unrelated processes as the Ornstein-Uhlenbeck process, the generalized Verhulst-Landau processes, the generalized Rayleigh process, the hyperbolic tangent processes, and many other cases of physico-mathematical interest. The Fokker-Planck equation associated with this unifying stochastic process is solved analytically for the transition probability density function, using a similar constructive solution method as in the author's previous work. The result is obtained as an eigenfunction expansion over a generally mixed spectrum. The discrete eigenfunctions are related (but not identical) to Jacobi polynomials. It is shown that the results for the above-mentioned processes are obtainable as limiting cases, and that some additional solvable equivalent Schrödinger problems can be defined.

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I. INTRODUCTION

A stochastic differential equation (SDE) of the Langevin type

$$\dot{x}(t)=f(x)+g(x)F(t) \tag{1}$$

formally describes a stochastic Markov process $\{x(t)\}$ if $F(t)$ is a (normalized) white-noise excitation, defined by

$$\langle F(t)\rangle=0 , \tag{2}$$

$$\langle F(t)F(t+\tau)\rangle=2\delta(\tau) . \tag{3}$$

The transition probability density function (PDF) $w(x,t|x_0)$ for $\{x(t)\}$ is the Green's-function solution of the Fokker-Planck equation (FPE) associated with (1):

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}(Bw)-\frac{\partial}{\partial x}(Aw) , \tag{4}$$

$$w(x,t=0|x_0)=\delta(x-x_0) , \tag{5}$$

and eventually subject to suitable boundary conditions.

The diffusion and drift coefficients in (4) are given, respectively by [2,3]

$$B(x)=g^2(x) ,\quad A(x)=f(x)+g\frac{\partial g}{\partial x} , \tag{6}$$

when the Stratonovich interpretation of (1) is accepted, considering $F(t)$ as a zero correlation-time limit of realistic continuous noise. For nonstationary processes, f and g in (1) and hence B and A in (6) may also depend on time.

Recently [1], the following four-parameter (SDE) has been considered,

$$\dot{x}(t)=\left[c-\frac{a}{2}\right]e^x+d+(ae^x+b)^{1/2}F(t) , \tag{7}$$

and the Green's function, or transition PDF, of the associated FPE,

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}[(ae^x+b)w]-\frac{\partial}{\partial x}[(ce^x+d)w] ,\quad x\in[-\infty,+\infty] \tag{8}$$

has been derived.

Some well-known physically important stochastic processes (e.g., the Toda process and the generalized Verhulst-Landau and Rayleigh processes) were shown to be retrievable from (7) by selecting particular sets of parameters a, b, c, d, and, eventually, a transformation $y(x)$.

In this paper a "maximal" generalization of (7) and (8) will be studied, in a sense that the FPE has the highest complexity compatible with the constructive solution method developed in [1]. Physically, this generalization will cause additional processes to belong to the enlarged class, which thus becomes unifying for a variety of well-known stochastic processes from physics and engineering sciences. Mathematically, a type of orthogonal eigenfunction spins off that is unifying for many well-known polynomials in mathematical physics.

II. UNIFYING STOCHASTIC PROCESS

As a generalization of (8), the following *stationary* FPE may be considered:

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}\left[\frac{ae^x+b}{fe^x+g}w\right]-\frac{\partial}{\partial x}\left[\frac{ce^x+d}{fe^x+g}w\right] ,\quad x\in[-\infty,+\infty] \tag{9}$$

where parameters a, b, f, and g are taken *non-negative* to have a valid diffusion equation. The introduction of the denominator (fe^x+g) in drift and diffusion causes these functions to become bounded over $[-\infty,+\infty]$ (except for cases where some parameters are zero) and does not invalidate the solution *method* of [1], in spite of the apparent complexity of (9).

Equation (9) may even further be generalized (see Sec. IV) by the introduction of additional parameters, but for convenience and *without affecting generality* (see Secs. IV A and IV B), the following *minimal parametric* version of (9) will be selected for the subsequent analysis (new x, t, α, β, and γ):

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}\left[\frac{e^x+\alpha}{\alpha e^x+1}w\right]-\frac{\partial}{\partial x}\left[\frac{2\beta e^x+2\alpha\gamma}{\alpha e^x+1}w\right],$$
$$x\in[-\infty,+\infty],\quad 0\le\alpha\le 1 . \tag{10}$$

The Langevin SDE for the stochastic process $\{x(t)\}$ associated with (10) is [see (1) and (6)]

$$\dot{x}(t)=\frac{2\alpha\beta e^{2x}+[(\alpha^2-1)/2+2\beta+2\alpha^2\gamma]e^x+2\alpha\gamma}{(\alpha e^x+1)^2}+\left[\frac{e^x+\alpha}{\alpha e^x+1}\right]^{1/2}F(t) , \tag{11}$$

from which it can be anticipated that $\{x(t)\}$ will be stable in probability if

$$\beta<0 ,\quad \gamma>0 . \tag{12}$$

These also are sufficient conditions for the existence of a *stable deterministic* equilibrium point $f(x_e)=0$ at

$$x_e=\ln\left\{-\left[\frac{\alpha^2-1}{8\alpha\beta}+\frac{1}{2\alpha}+\frac{\alpha\gamma}{2\beta}\right]+\left[\left[\frac{\alpha^2-1}{8\alpha\beta}+\frac{1}{2\alpha}+\frac{\alpha\gamma}{2\beta}\right]-\frac{\alpha}{\beta}\right]^{1/2}\right\} . \tag{13}$$

Conditions (12) will be accepted henceforth as a working hypothesis.

For $\alpha=1$, a *constant diffusion* subclass of (11) is obtained for which the SDE may be written as

$$\dot{x}(t)=(\beta+\gamma)+(\beta-\gamma)\tanh(x/2)+F(t) , \tag{14}$$

representing a generalization (by addition of the constant drift $\beta+\gamma$) of the tanh process studied by Wong [4] and recently reconsidered by Jauslin [5]. This already demonstrates that the enlarged class of stochastic processes defined by the "unifying" SDE (11) contains new physically important members.

III. SOLUTION OF THE FOKKER-PLANCK EQUATION

The FPE (10)

$$\frac{\partial w(x,t)}{\partial t}=\frac{\partial^2}{\partial x^2}\left[\frac{e^x+\alpha}{\alpha e^x+1}w\right]-\frac{\partial}{\partial x}\left[\frac{2\beta e^x+2\alpha\gamma}{\alpha e^x+1}w\right],$$
$$x\in[-\infty,+\infty],\quad 0\le\alpha\le 1,\quad \beta<0,\gamma>0 , \tag{15}$$

subject to the initial condition

$$w(x,0|x_0)=\delta(x-x_0) \tag{16}$$

and to natural boundary conditions

$$w(\pm\infty,t|x_0)=0,\quad \frac{\partial w}{\partial x}=0 \ \text{ for } x=\pm\infty \tag{17}$$

will now be solved for the transition PDF $w(x,t|x_0)$, which completely characterizes the stochastic Markov process $\{x(t)\}$.

The solution method [1] will simultaneously produce all components of the eigenfunction expansion

$$w(x,t|x_0)=w_s(x)\sum_k \varphi_k(x)\varphi_k(x_0)e^{-\lambda_k t} , \tag{18}$$

where the summation is symbolic, eventually including a continuous part of the spectrum as well.

The following Laplace and Fourier transforms are defined:

$$\bar{\theta}(z,p)=\int_0^\infty dt\, e^{-pt}\theta(z,t)=\int_0^\infty dt\, e^{-pt}\int_{-\infty}^\infty dx\, e^{zx}w(x,t|x_0) ,$$
$$z=i\omega,\quad \omega\in[-\infty,+\infty] , \tag{19}$$

$$\bar{\eta}(z,p)=\int_0^\infty dt\, e^{-pt}\int_{-\infty}^\infty dx\, e^{zx}\frac{w(x,t|x_0)}{\alpha e^x+1} ; \tag{20}$$

i.e., $\bar{\theta}$ is the Laplace (L) transformed characteristic function of the stochastic process $\{x(t)\}$ and $\bar{\eta}$ is the Laplace-Fourier (LF) transform of the function

$$W(x,t|x_0)=\frac{w(x,t|x_0)}{\alpha e^x+1} , \tag{21}$$

such that

$$\bar{\theta}(z,p)=\alpha\bar{\eta}(z+1,p)+\bar{\eta}(z,p) . \tag{22}$$

Substitution of (21) into (15) and LF transforming the resulting equation yields, with (16) and (17), a *functional recurrence equation* for $\bar{\eta}(z,p)$:

$$(\alpha p-z^2-2\beta z)\bar{\eta}(z+1,p)+(p-\alpha z^2-2\alpha\gamma z)\bar{\eta}(z,p)=e^{zx_0} . \tag{23}$$

Defining now the functions

$$p_0(z)=\alpha(z^2+2\gamma z) , \tag{24}$$

$$q_0(z)=z^2+2\beta z , \tag{25}$$

and the "shift" operator Δ by

$$\Delta^k G_0(z)=\Delta(\Delta^{k-1}G_0)=G_0(z+k)=G_k(z) , \tag{26}$$

for any function $G_0(z)$, (23) can be written as an operator equation:

$$[(p-p_0)+(\alpha p-q_0)\Delta]\bar{\eta}(z,p)=e^{zx_0} , \tag{27}$$

which formally inverts to

$$\bar{\eta}(z,p)=\left[1+\frac{\alpha p-q_0}{p-p_0}\Delta\right]^{-1}\frac{e^{zx_0}}{p-p_0}$$
$$=\sum_{k=0}^{\infty}\left[-\frac{\alpha p-q_0}{p-p_0}\Delta\right]^{k}\frac{e^{zx_0}}{p-p_0}$$
$$=\frac{e^{zx_0}}{p-p_0}\sum_{k=0}^{\infty}(-1)^k\prod_{j=1}^{k}\left[\frac{\alpha p-q_{j-1}}{p-p_j}\right]e^{kx_0} \quad (28)$$
$$=\frac{e^{zx_0}}{p-p_0}\psi(z,p,x_0)\ , \quad (29)$$

with

$$p_k(z)=\Delta^k p_0(z)=p_0(z+k)\ ,\quad q_k(z)=q_0(z+k)\ . \quad (30)$$

Using now the factorizations

$$p-p_0(z)=p-\alpha(z^2+2\gamma z)=-\alpha[z-u^+(p)][z-u^-(p)]\ , \quad (31)$$
$$\alpha p-q_0(z)=\alpha p-(z^2+2\beta z)=-[z-v^+(p)][z-v^-(p)]\ , \quad (32)$$

with

$$u^{\pm}(p)=-\gamma\pm\left[\gamma^2+\frac{p}{\alpha}\right]^{1/2},\quad \mathrm{Re}\left[\left[\gamma^2+\frac{p}{\alpha}\right]^{1/2}\right]\geq 0 \quad (33)$$

$$v^{\pm}(p)=-\beta\pm(\beta^2+\alpha p)^{1/2},\quad \mathrm{Re}[(\beta^2+\alpha p)^{1/2}]\geq 0 \quad (34)$$

it is easily seen that ψ in (29) may be written as a generalized hypergeometric expression:

$$\psi(z,p,x_0)={}_3F_2(z-v^+,z-v^-,1;z-u^++1,z-u^-+1;-(e^{x_0}/\alpha))\ . \quad (35)$$

Considered as a function of p, $\bar{\eta}(z,p)$ has an infinity of simple poles at

$$p=p_k(z)=p_0(z+k)=\alpha[(z+k+\gamma)^2-\gamma^2]\ ,\quad k=0,1,2\ldots \quad (36)$$

and

$$\lim_{p\to\infty}\bar{\eta}(z,p)=0\ . \quad (37)$$

Hence an alternative representation of $\bar{\eta}(z,p)$ is given by the partial fraction expansion [6]

$$\bar{\eta}(z,p)=\sum_{k=0}^{\infty}\frac{r_k(z)}{p-p_k(z)}\ . \quad (38)$$

Substitution of (38) in (27), multiplying by $(p-p_k)$, and taking the limit $p\to p_k$ shows the residues to be recurrent:

$$r_k(z)=-\left[\frac{\alpha p_k-q_0}{p_k-p_0}\right]r_{k-1}(z+1),\quad k>0\ . \quad (39)$$

With $r_0(z)$ following from (29),

$$r_0(z)=\lim_{p\to p_0}(p-p_0)\bar{\eta}(z,p)=e^{zx_0}\psi(z,p_0(z),x_0)\ , \quad (40)$$

one has

$$r_k(z)=(-1)^k\prod_{m=0}^{k-1}\left[\frac{\alpha p_k-q_m}{p_k-p_m}\right]\times e^{(z+k)x_0}\psi(z+k,p_k(z),x_0)\ . \quad (41)$$

From (33) and (34) it follows that

$$u^+(p_k(z))=z+k\ ,\quad u^-(p_k(z))=-z-k-2\gamma\ ,\quad v^{\pm}(p_k(z))=v_k^{\pm}(z)=v_0^{\pm}(z+k)=-\beta\pm[\alpha^2(z+k+\gamma)^2+\beta^2-\alpha^2\gamma^2]^{1/2} \quad (42)$$

such that (41) can be written as

$$r_k(z)=e^{zx_0}\left[\frac{e^{x_0}}{\alpha}\right]^k\frac{1}{k!}\frac{\Gamma(2z+k+2\gamma)\Gamma(z+k-v_k^+)\Gamma(z+k-v_k^-)}{\Gamma(2z+2k+2\gamma)\Gamma(z-v_k^+)\Gamma(z-v_k^-)}\times{}_2F_1\left[z+k-v_k^+,z+k-v_k^-;2z+2k+2\gamma+1;-\frac{e^{x_0}}{\alpha}\right], \quad (43)$$

where the ${}_2F_1$ function follows from contraction of ${}_3F_2$ [see (35)] by equality of an upper and a lower parameter:

$$z+k-u^+(p_k(z))+1=1\ . \quad (44)$$

Laplace inversion of (38) gives the time-dependent F transform of the function $W(x,t|x_0)$ [Eq. (21)]:

$$\eta(z,t)=e^{zx_0}\sum_{k=0}^{\infty}e^{p_k(z)t}\left[\frac{e^{x_0}}{\alpha}\right]^k\frac{1}{k!}\frac{\Gamma(2z+k+2\gamma)\Gamma(z+k-v_k^+)\Gamma(z+k-v_k^-)}{\Gamma(2z+2k+2\gamma)\Gamma(z-v_k^+)\Gamma(z-v_k^-)}\times{}_2F_1(z+k-v_k^+,z+k-v_k^-;2z+2k+2\gamma+1;-e^{x_0}/\alpha)\ . \quad (45)$$

Replacing the above summation in k by a contour integral in a complex variable s, around the poles,

$$s=s_k=k, \quad k=0,1,2,\ldots, \tag{46}$$

of the summator $\Gamma(-s)$, one obtains

$$\eta(z,t)=\frac{e^{zx_0}}{2\pi i}\int_{c_1} ds\, e^{p_0(z+s)t}\left[-\frac{e^{x_0}}{\alpha}\right]^s \frac{\Gamma(2z+s+2\gamma)}{\Gamma(2z+2s+2\gamma)}$$
$$\times\frac{\Gamma[z+s-v_0^+(z+s)]\Gamma[z+s-v_0^-(z+s)]}{\Gamma[z-v_0^+(z+s)]\Gamma[z-v_0^-(z+s)]}H(z,s)$$
$$\times\Gamma(-s)\,{}_2F_1(z+s-v_0^+(z+s),z+s-v_0^-(z+s);2z+2s+2\gamma+1;-e^{x_0}/\alpha) . \tag{47}$$

The contour C_1 encircles clockwise the poles of $\Gamma(-s)$ without enclosing other singularities of the integrand. The function $H(z,s)$ is as yet undetermined, but should satisfy $H(z,k)=1$, $k=0,1,2,\ldots$, in order to preserve equivalence with (45). This suggests that $H(z,s)$ can be taken as either a constant or a suitably defined periodic function of s.

For convenience, the further analysis will be related to the *q plane,* which is defined by

$$q=s+z+\gamma . \tag{48}$$

In this plane the locus R of *real non-negative* eigenvalues is found from [see (36)]

$$p_0(z+s)=p_0(q-\gamma)=\alpha(q^2-\gamma^2)=-\lambda ,$$
$$\mathrm{Im}(\lambda)=0, \quad \lambda\geq 0 , \tag{49}$$

and consists of the entire imaginary axis (V),

$$\mathrm{Re}(q)=0 , \tag{50}$$

and a segment S of the real axis (H),

$$\mathrm{Im}(q)=0, \quad |\mathrm{Re}(q)|\leq\gamma . \tag{51}$$

A Fourier-transformed *eigenfunction* representation can now be obtained from (47) by deforming the original contour C_1^* [around the poles of $\Gamma(-q+z+\gamma)$ in the q plane] so as to coincide with R. This is possible if the integrand of (47) has no singularities between R and C_1^*. The singularities to consider are the following.

A. Poles of $\Gamma(s+2z+2\gamma)=\Gamma(q+z+\gamma)$

This is a descending series of simples poles at the points

$$q=q_{1,k}=-k-z-\gamma, \quad k=0,1,2,\ldots , \tag{52}$$

which are all to the left of R and do not obstruct contour deformation.

B. Singularities of $\Gamma[z+s-v_0^{\pm}(z+s)]$

Using (42) and (48), the arguments of these Γ functions may be written as

$$z+s-v_0^{\pm}(z+s)=a^{\pm}(q)$$
$$=(q+\beta-\gamma)\mp(\alpha^2q^2+\beta^2-\alpha^2\gamma^2)^{1/2} . \tag{53}$$

The conditions

$$a^{\pm}(q)=-k, \quad k=0,1,2,\ldots , \tag{54}$$

yield the set of simple poles,

$$q=q_k^{\pm}=\frac{1}{1-\alpha^2}[-(k+\beta-\gamma)\pm D^{1/2}], \quad \mathrm{Re}(D^{1/2})\geq 0 \tag{55}$$

with

$$D=D(k)=\alpha^2(k+\beta-\gamma)^2+(1-\alpha^2)(\beta^2-\alpha^2\gamma^2) . \tag{56}$$

Under the working hypotheses $0\leq\alpha\leq 1$, $\beta<0$, $\gamma>0$ [see (10) or (15) and (12)], two alternatives have to be considered

1. ϵ case

Let

$$\beta^2-\alpha^2\gamma^2=\alpha^2\epsilon^2\geq 0 . \tag{57}$$

The argument functions $a^{\pm}(q)$ [Eq. (53)] have two *imaginary* branch points at

$$q_B^{\pm}=\pm i\epsilon=\pm i\left[\frac{\beta^2}{\alpha^2}-\gamma^2\right]^{1/2} , \tag{58}$$

and $D(k)$ is *semidefinite positive.* All poles $q_k^{\pm}$ are *real,* and they are distributed between the two branches $a^{\pm}$ according to

$$a^+(q_k^+)=-k, \quad a^-(q_k^-)=-k, \quad k=0,1,2,\ldots . \tag{59}$$

The series q_k^- starts at the zero eigenvalue point,

$$q_0^-=\frac{1}{1-\alpha^2}[-\beta+\gamma-D^{1/2}(0)]=\gamma , \tag{60}$$

and is descending. The q_k^- values are *positive* for

$$0\leq k\leq(\gamma-\beta-\alpha\epsilon) . \tag{61}$$

The poles q_k^+ start at

$$q_0^+=\frac{\gamma(1+\alpha^2)-2\beta}{1-\alpha^2}>\gamma \tag{62}$$

and thus inevitably obstruct the intended contour deformation.

2. ν case

Let

$$\beta^2-\alpha^2\gamma^2=-\alpha^2\nu^2<0\ . \tag{63}$$

The arguments $a^{\pm}(q)$ [Eq. (53)] have two *real* branch points *inside* S [Eq. (51)]:

$$q_B^{\pm}=\pm\nu=\pm\left[\gamma^2-\frac{\beta^2}{\alpha^2}\right]^{1/2}\ . \tag{64}$$

Discriminant $D(k)$ [Eq. (56)] is *negative* for k values in the range

$$0<\gamma-\beta-\nu(1-\alpha^2)^{1/2}\le k\le\gamma-\beta+\nu(1-\alpha^2)^{1/2}\ , \tag{65}$$

and the corresponding poles $q_k^{\pm}$ are complex conjugate. They are "obstructing" if their *real parts* are positive; i.e., for k values,

$$\gamma-\beta-\nu(1-\alpha^2)^{1/2}\le k<\gamma-\beta\ . \tag{66}$$

For k outside the range [Eq. (65)], the poles $q_k^{\pm}$ are real again, and (60) and (62) apply.

By back substitution of the $q_k^{\pm}$ in (54) or just by checking the sign of $\mathrm{Re}(q_k^{\pm}+\beta-\gamma+k)$, it can be concluded that all "bad" poles belong to the a^+ branch and that "good" poles are found among the q_k^- belonging to the a^- branch. These last poles are obtained for k values:

$$0\le k\le(\gamma-\beta-\nu)\ . \tag{67}$$

They are real, positive, and satisfy

$$q_B^+<q_k^-\le q_0^-=\gamma\ . \tag{68}$$

For both cases described above, the "bad" poles in a^+ can be "annihilated" by the *simple zeros* of a properly chosen $H(z,s)$ function in (47):

$$H(z,s)=\frac{\sin\pi[z+s-v_0^+(z+s)]}{\sin\pi[z-v_0^+(z+s)]}e^{i\pi s}\ . \tag{69}$$

This $H(z,s)$ merely "flips over" [1] the Γ functions with v_0^+ (i.e., the a^+ branch):

$$\frac{\Gamma(z+s-v_0^+)}{\Gamma(z-v_0^+)}\frac{\sin\pi(z+s-v_0^+)}{\sin\pi(z-v_0^+)}e^{i\pi s}=\frac{\Gamma(1-z+v_0^+)}{\Gamma(1-z-s+v_0^+)}e^{i\pi s}\ , \tag{70}$$

and the integral representation (47) in the q plane becomes

$$\eta(z,t)=\frac{\alpha^z}{2\pi i}\int_{c_1^*}dq\,e^{p_0t}\left[\frac{e^{x_0}}{\alpha}\right]^{q-\gamma}\frac{\Gamma(q+z+\gamma)\Gamma(1-z+v_0^+)\Gamma(q-\gamma-v_0^-)}{\Gamma(2q)\Gamma(1+\gamma-q+v_0^+)\Gamma(z-v_0^-)}\times\Gamma(z+\gamma-q)\,{}_2F_1(q-\gamma-v_0^+,q-\gamma-v_0^-;2q+1;-e^{x_0}/\alpha)\ , \tag{71}$$

where, as functions of q,

$$p_0(q)=\alpha(q^2-\gamma^2)\ ,$$
$$v_0^{\pm}(q)=-\beta\pm(\alpha^2q^2+\beta^2-\alpha^2\gamma^2)^{1/2}\ ,$$
$$\mathrm{Re}[(\cdots)^{1/2}]\ge0\ . \tag{72}$$

The effect of the "flip over" [Eq. (70)] is more far-reaching than in [1], where the $v_0^{\pm}$ were just q independent. The original symmetry of the integrand in $v_0^{\pm}$ is broken by $H(z,s)$ [Eq. (69)], so that branch points (58) or (64) become effective, and *branch cuts* between them are necessary to preserve single valuedness. It should be noted that no new poles are introduced by $H(z,s)$, as the conditions

$$1-z+v_0^+(q)=-k,\quad k=0,1,2,\ldots\ ,$$
$$\mathrm{Re}(v_0^+)\ge0 \tag{73}$$

cannot be met simultaneously.

Formally denoting (71) by

$$\eta(z,t)=\frac{1}{2\pi i}\int_{c_1^*}dq\,M(q,z,x_0,t)\ , \tag{74}$$

the contour can now be deformed, so as to cover the locus R [Eqs. (50) and (51)]. During this process the finite number of "good" q_k^- poles located on the rightmost part of S [Eq. (51)] cross the contour (see Figs. 1 and 2). It is found that

$$\eta(z,t)=\sum_{k=0}^{N}\mathrm{Res}_{q=q_k^-}(M)+\frac{1}{2\pi i}\int_B dq\,M+\frac{1}{2\pi i}\int_V dq\,M\ , \tag{75}$$

where the sum contains the residues at the abovementioned poles, with N given by

$$N=N_\epsilon=\mathrm{int}(\gamma-\beta-\alpha\epsilon) \tag{76}$$

or

$$N=N_\nu=\mathrm{int}(\gamma-\beta-\nu)\ , \tag{77}$$

for the ϵ and ν cases, respectively. The B path is adjacent to the vertical branch cut in the ϵ case (Fig. 1) or turns around the right half of the horizontal cut in the ν case (Fig. 2). Accordingly, the V path consists of the remaining parts of (ϵ case) or the entire (ν case) imaginary axis. Suitable half-circle indentations around the branch points may be shown to have a zero contribution.

The resulting three-part eigenfunction solution is reminiscent of the quantum-mechanical result for a *bounded* Schrödinger potential, where the three components of (75) would correspond, respectively, to bound, reflecting, and free states [7]. It should be noted, however, that there is no simple equivalent Schrödinger potential for the FPE (10) in the *general* case (see Sec. IV F).

Residue calculation proceeds as follows. With q_k^- from (55), $D(k)$ from (56), and $v_0^{\pm}(q)$ from (72), one has

$$h=h(k)=v_0^-(q_k^-)=\frac{1}{1-\alpha^2}[-\beta-\alpha^2(k-\gamma)-D^{1/2}] , \quad (78)$$

$$v_0^+(q_k^-)=-2\beta-h , \quad (79)$$

$$q_k^- -\gamma-v_0^-(q_k^-)=-k, \quad q_k^- -\gamma-v_0^+=-k+2\beta+2h , \quad (80)$$

$$\mathrm{Res}_{q=q_k^-}\Gamma(q-\gamma-v_0^-(q))=\frac{(-1)^k}{k!}\frac{(\beta+h)}{D^{1/2}} , \quad (81)$$

$$p_0(q_k^-)=\alpha[(q_k^-)^2-\gamma^2]=\alpha(h-k)(h-k+2\gamma) , \quad (82)$$

so that the finite sum in (75) can be written as

$$\sum_{k=0}^{N} e^{\alpha t(h-k)(h-k+2\gamma)}\alpha^z\left[\frac{e^{x_0}}{\alpha}\right]^{h-k}\frac{\Gamma(z+h-k+2\gamma)\Gamma(1-z-2\beta-h)}{\Gamma(2\gamma+2h-2k)\Gamma(1-2\beta-2h+k)}\frac{(-1)^k}{k!}\frac{(\beta+h)}{D^{1/2}}\frac{\Gamma(z-h+k)}{\Gamma(z-h)}$$
$$\times{}_2F_1(-k,-k+2h+2\beta;-2k+2h+2\gamma+1;-e^{x_0}/\alpha) . \quad (83)$$

From this expression it appears that the *discrete eigenvalue spectrum* is given by

$$\lambda_k=-\alpha(h-k)(h-k+2\gamma)=\alpha\gamma^2-\alpha(h-k+\gamma)^2, \quad k=0,1,2,\ldots,N \quad (84)$$

with N from (76) or (77), and that the corresponding *eigenfunctions* $\varphi_k(x)$ are *proportional* to

$$\chi_k(x)=(\alpha e^{-x})^{k-h}{}_2F_1(-k,-k+2h+2\beta;-2k+2h+2\gamma+1;-e^x/\alpha) , \quad (85)$$

where the *polynomial part* ${}_2F_1$ is expressible by *Jacobi polynomials* [8]

$${}_2F_1(-k,-k+2h+2\beta;-2k+2h+2\gamma+1;-e^x/\alpha)$$
$$=\frac{\Gamma(k-2h-2\gamma)}{\Gamma(2k-2h-2\gamma)}k!(\alpha e^{-x})^{-k}P_k^{(\pi_1,\pi_2)}(1+2\alpha e^{-x}) , \quad (86)$$

with parameters π_1 and π_2:

$$\pi_1=-2h-2\beta ,$$
$$\pi_2=2\beta-2\gamma-1 . \quad (87)$$

Via h [Eq. (78)], the parameter π_1 is *nonlinearly* dependent upon the degree k.

The line integrals in (75) will not be elaborated further

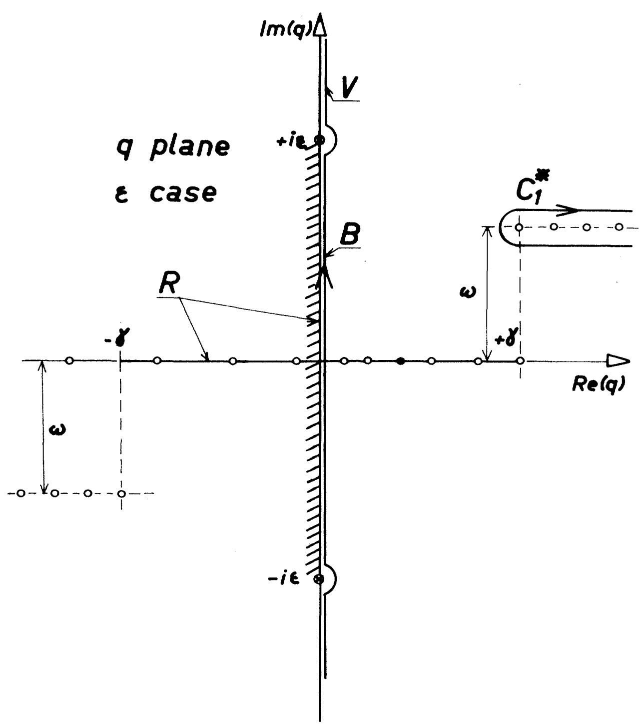

FIG. 1. Final q-plane configuration for the ϵ case.

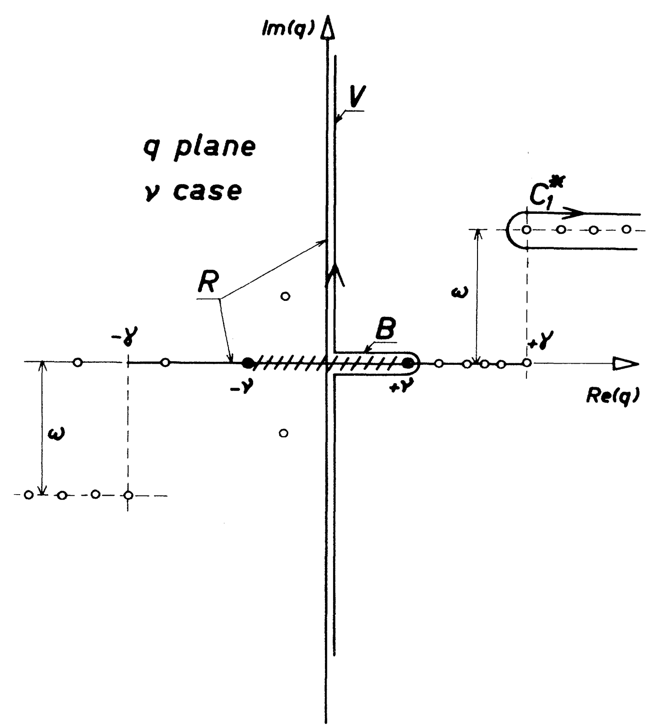

FIG. 2. Final q-plane configuration for the ν case.

here. It is clear that the corresponding continuous spectra are found from

$$q=\nu+\rho e^{\pm i\pi} , \tag{88}$$

$$\lambda(\rho)=\alpha[\gamma^2-(\nu-\rho)^2], \quad \rho\in[0,\nu] \tag{89}$$

for the B integral around the horizontal branch cut in the ν case (reflecting states), and from

$$q=i\mu , \tag{90}$$

$$\lambda(\mu)=\alpha(\gamma^2+\mu^2), \quad \mu\in[-\infty,+\infty] \tag{91}$$

for the integrals along the imaginary axis in *both* cases (free or reflecting plus free states).

Finally, Fourier inversion of the coefficient of the kth time exponential in (83) yields

$$\begin{aligned}\frac{w_s}{\alpha e^x+1}\varphi_k(x)\varphi_k(x_0) &= \frac{(-1)^k}{k!}\frac{(\beta+h)}{D^{1/2}}\\ &\times\frac{\Gamma(2k-2h-2\gamma)}{\Gamma(k-2h-2\gamma)B(2\gamma+2h-2k,1-2\beta-2h+k)}\\ &\times(\alpha e^{-x})^{1-2\beta}(1+\alpha e^{-x})^{2\beta-2\gamma-1}\chi_k(x)\chi_k(x_0) ,\end{aligned} \tag{92}$$

where B is the β function [8] and χ_k is given by (85). Setting $k=0$ gives, for the *normalized steady-state* PDF,

$$\begin{aligned}w_s(x)&=\left[\left[1-\frac{\alpha^2\gamma}{\beta}\right]B(2\gamma,1-2\beta)\right]^{-1}\\ &\times(1+\alpha e^x)(\alpha e^{-x})^{1-2\beta}\\ &\times(1+\alpha e^{-x})^{2\beta-2\gamma-1}, \quad \beta<0, \quad \gamma>0\end{aligned} \tag{93}$$

and factorizing this out from (92) yields the product of the *normalized* eigenfunctions,

$$\begin{aligned}\varphi_k(x)\varphi_k(x_0)&=\frac{(-1)^k}{k!}\frac{(\beta+h)}{D^{1/2}}\frac{\Gamma(2k-2h-2\gamma)}{\Gamma(k-2h-2\gamma)}\\ &\times\frac{B(2\gamma,1-2\beta)(1-\alpha^2\gamma/\beta)}{B(2\gamma+2h-2k,1-2\beta-2h+k)}\\ &\times\chi_k(x)\chi_k(x_0) ,\end{aligned} \tag{94}$$

which clearly displays the normalization constant.

IV. APPLICATIONS

A. Systems with $\alpha>1$

The preceding analysis has been performed under the working hypothesis $0\le\alpha\le1$ [see Eq. (10)]. When $\alpha>1$ the transformations

$$\begin{aligned}&\bar\alpha=1/\alpha<1, \quad \bar\beta=-\gamma<0, \quad \bar\gamma=-\beta>0 ,\\ &\bar x=-x, \quad \bar t=t ,\\ &\bar w(\bar x,\bar t)=w(-x,t)\end{aligned} \tag{95}$$

leave Eq. (10) absolutely invariant, so that the relation

$$w(x,t|x_0;\alpha,\beta,\gamma)=w(-x,t|-x_0;1/\alpha,-\gamma-\beta) \tag{96}$$

provides the continuation of the derived solution for α values larger than 1 and expresses the symmetry of the PDF under reflexion of x.

For calculational purposes the following transforms are useful [see (56) and (78)]:

$$\bar D(k)=\frac{1}{\alpha^4}D(k), \quad \bar h(k)=k-h(k) . \tag{97}$$

Further, it is seen that

$$\bar\beta^2-\bar\alpha^2\bar\gamma^2=-\frac{1}{\alpha^2}(\beta^2-\alpha^2\gamma^2) , \tag{98}$$

which means that for an ϵ-case combination of parameters α, β, and γ, with $\alpha>1$, one has to use the formulas for a $\bar\nu$ case, with

$$\bar\nu^2=\alpha^2\epsilon^2 , \tag{99}$$

and vice versa for a ν case.

B. Constant diffusion subclass: $\alpha=1$

The physical significance of this subclass was already mentioned in relation to the SDE (14). The derived solution is adapted for this case by substitution of

$$D^{1/2}(k)=\gamma-\beta-k , \tag{100}$$

and a *limit* for $h(k)$,

$$\begin{aligned}h(k)&=\lim_{\alpha\to1}\frac{1}{1-\alpha^2}(\alpha^2\gamma-\beta-\alpha^2k-D^{1/2})\\ &=\frac{\gamma^2-(\gamma-k)^2}{2(\gamma-\beta-k)} .\end{aligned} \tag{101}$$

The steady-state PDF becomes

$$\begin{aligned}w_s(x)&=\left[\left[1-\frac{\gamma}{\beta}\right]B(2\gamma,1-2\beta)\right]^{-1}\\ &\times e^{2\gamma x}(1+e^x)^{2(\beta-\gamma)} .\end{aligned} \tag{102}$$

but otherwise no particular simplifications of the results are possible *in general,* as the expression for h [Eq. (101)] is still nonlinear in k.

Results for the classical tanh model [4] are retrieved by *additionally* specifying

$$\beta=-\gamma , \tag{103}$$

which eliminates the constant drift from the SDE (14):

$$\dot x(t)=-2\gamma\tanh(x/2)+F(t) . \tag{104}$$

One finds

$$w_s(x)=\frac{\Gamma(4\gamma)}{[\Gamma(2\gamma)]^2}[2\cosh(x/2)]^{-4\gamma} , \tag{105}$$

and with

$$\begin{aligned}&D^{1/2}=2\gamma-k ,\\ &h=k/2 ,\end{aligned} \tag{106}$$

one has, for the discrete part of the spectrum,

$$\lambda_k=\tfrac{1}{4}k(4\gamma-k),\quad k=0,1,\ldots,N=\mathrm{int}(2\gamma)\ . \tag{107}$$

From (85) it can be seen that the eigenfunctions are proportional to

$$\chi_k(x)=e^{-kx/2}{}_2F_1(-k,-2\gamma;1-k+2\gamma;-e^x)\ , \tag{108}$$

where ${}_2F_1$ has suitable parameters for a *quadratic transformation* [8] to be applicable, such that

$$\begin{aligned}\chi_k(x)&=(1-e^x)^k(e^{-x/2})^k{}_2F_1\left[-\frac{k}{2},\frac{1-k}{2};1-k+2\gamma;-\frac{4e^x}{(1-e^x)^2}\right]\\&=[-2\sinh(x/2)]^k{}_2F_1\left[-\frac{k}{2},\frac{1-k}{2};1-k+2\gamma;-\sinh^{-2}\left[\frac{x}{2}\right]\right],\end{aligned} \tag{109}$$

which are *polynomials* in $\sinh(x/2)$. This reconstructs the results found by Wong [4], in terms of the variable $x_{(W)}=\sinh(x/2)$ and the parameter $\alpha_{(W)}=2\gamma$, although it should be clear that many alternative representations (e.g., in terms of Legendre functions) are possible.

C. Extended parameter set

While the preceding solution has been obtained in terms of only three parameters α, β, and γ, additional parameters may be introduced so as to obtain a more general (stationary) FPE,

$$\frac{\partial w}{\partial \tau}=\frac{\partial^2}{\partial y^2}\left[\frac{ae^{\mu y}+b}{fe^{\mu y}+g}w\right]-\frac{\partial}{\partial y}\left[\frac{ce^{\mu y}+d}{fe^{\mu y}+g}w\right],$$
$$y\in[-\infty,+\infty],a,b,f,g\geq 0 \tag{110}$$

by applying the following transformations to the standard equation (10):

$$\begin{aligned}x&=\mu y+\frac{1}{2}\ln\left[\frac{af}{bg}\right],\\t&=\mu^2\left[\frac{ab}{fg}\right]^{1/2}\tau,\\\alpha&=\left[\frac{bf}{ag}\right]^{1/2},\quad \beta=\frac{c}{2\mu a},\quad \gamma=\frac{d}{2\mu b}\ .\end{aligned} \tag{111}$$

This proves the equivalence of (110) [or (9)] with (10) and prepares for the subsequent retrieval of interesting limiting cases.

D. Retrieval of the subclass of Ref. [1]

This subclass, which was shown to contain the Verhulst-Landau and generalized Rayleigh processes, is obtained by the following choice of parameters in (110) [see also (8)]:

$$\mu=1,\quad g=1,\quad f=0\ . \tag{112}$$

The singularities in x and t [Eq. (111)] for $f\to 0$ are compensated by $\alpha\to 0$ in a suitable limiting process. As an example, one has [see (56) and (78)]

$$\begin{aligned}&\lim_{f\to 0}D^{1/2}=-\beta=-\frac{c}{2a}\ ,\\&\lim_{f\to 0}h(k)=0\ ,\\&\lim_{f\to 0}(\lambda_k t)=\tau k\left[k-\frac{d}{b}\right]\lim_{f\to 0}\alpha\left[\frac{ab}{f}\right]^{1/2}\\&\qquad\qquad=b\tau k\left[k-\frac{d}{b}\right],\\&\lim_{f\to 0}\left[\frac{e^x}{\alpha}\right]=e^y\lim_{f\to 0}\frac{(af/b)^{1/2}}{\alpha}=\frac{a}{b}e^y\ ,\end{aligned} \tag{113}$$

which clearly allows one to reproduce the results of [1] (with y and τ standing for x and t, respectively).

E. Ornstein-Uhlenbeck process

One possible way to reduce the unifying stochastic process to the Ornstein-Uhlenbeck process is first to set

$$a=f=b=g\ , \tag{114}$$

in (110), so that a constant diffusion ($\alpha=1$) FPE results. Choosing further

$$d=-c=\frac{2a\sigma^2}{\mu}\ , \tag{115}$$

and taking the limit $\mu\to 0$, the FPE (110) becomes

$$\frac{\partial w}{\partial \tau}=\frac{\partial^2 w}{\partial y^2}+\sigma^2\frac{\partial}{\partial y}(yw),\quad y\in[-\infty,+\infty] \tag{116}$$

which clearly is the FPE for the Ornstein-Uhlenbeck process. The limiting processes necessary to produce the entirely discrete spectrum and the well-known Hermite-polynomial eigenfunctions of (116) out of the general solution will not be given here as they are rather tedious and of academic interest only.

The example illustrates the possibility of other FPE's, with *non exponentially* state-dependent coefficients, belonging to the same class. The *one-sided Ornstein-Uhlenbeck* process, i.e., (116) with $y\in[0,+\infty]$ and reflecting boundary at $y=0$, has been considered by Wong [4] and Stratonovich [9]. It is clear that this process can be seen as a special case of the generalized Rayleigh process, with Laguerre polynomials [1,9] turning into *even* Hermite polynomials. So it turns out that part of the class [4] (Pearson's equation class) is also a subclass of this unifying stochastic process and thus has *a common representation* for the transition PDF's too.

F. Associated Schrödinger equations

It is well known [2] that a *constant diffusion* FPE with drift $f(x)$,

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}-\frac{\partial}{\partial x}(fw)\ , \tag{117}$$

can be transformed to an equivalent (imaginary time) Schrödinger equation which has the potential

$$V(x)=\frac{f^2}{4}+\frac{f'}{2}\ . \tag{118}$$

A constant diffusion subclass of the unifying stochastic process is obtained *parametrically* by choosing $\alpha=1$ [see Sec. IV B and Eq. (14)]. The drift in this case becomes

$$f(x)=\frac{2\beta e^x+2\gamma}{e^x+1}=(\beta+\gamma)+(\beta-\gamma)\tanh(x/2)\ , \tag{119}$$

and the associated solvable Schrödinger case has the two-parametric potential

$$V(x)=\frac{(\beta e^x+\gamma)^2+(\beta-\gamma)e^x}{(e^x+1)^2},\quad x\in[-\infty,+\infty] \tag{120}$$

which, for $\gamma>0$ and $\beta<0$, is a *bounded* potential well.

The other way to arrive at a constant diffusion FPE such as (117) is a transformation of variables. Considering the generalized form [Eq. (110)] of the unifying stochastic process FPE, the necessary transformation is given by

$$z(y)=\int dy\left[\frac{fe^{\mu y}+g}{ae^{\mu y}+b}\right]^{1/2}\ . \tag{121}$$

Although this integral can be worked out, the result is hardly useful. The transformation is not readily invertible in general so that the drift of the resulting constant diffusion FPE (and the associated "unifying" Schrödinger potential) cannot be expressed in terms of known simple functions of z.

For special parameter combinations and limiting cases, however, (121) is invertible, and as a nontrivial example, the Schrödinger potential for the subclass of Ref. [1] (see Sec. IV D) can be obtained as follows:

Let, without loss of generality,

$$\begin{aligned}&\mu=a=b=g=1\ ,\\&f=0\ .\end{aligned} \tag{122}$$

Then, from (121),

$$z(y)=\int dy(e^y+1)^{-1/2}=-2\,\mathrm{arcsinh}(e^{-y/2})\ , \tag{123}$$

or, after removing the minus sign by $z\to -z$,

$$e^{-y/2}=\sinh(z/2),\quad z\in[0,+\infty]\ . \tag{124}$$

The FPE drift becomes [see also Eq. (16) of [1]]

$$\begin{aligned}f(z)&=\frac{1-2c}{\sinh(z)}-d\tanh(z/2)\\&=\frac{\frac{1}{2}-c}{\tanh(z/2)}-(d-c+\tfrac{1}{2})\tanh(z/2)\\&=\frac{(\frac{1}{2}-c)e^z-d(e^z-1)^2}{(e^{2z}-1)},\quad z\in[0,+\infty]\end{aligned} \tag{125}$$

and the associated *one-sided* Schrödinger potential follows from (118),

$$V(z)=\frac{c_1}{\tanh^2(z/2)}+c_2\tanh^2(z/2)+c_3\ , \tag{126}$$

with

$$\begin{aligned}&c_1=\tfrac{1}{4}(c^2-\tfrac{1}{4})\ ,\\&c_2=\tfrac{1}{4}[(d-c+1)^2-\tfrac{1}{4}]\ ,\\&c_3=\tfrac{1}{4}[2(c-d)(1-c)-\tfrac{1}{2}]\ .\end{aligned} \tag{127}$$

V. CONCLUSIONS

Expressing the non-negative character of a fluctuating physical quantity $q(t)$ in an obvious way by setting $q=e^x$ gives rise to FPE's with *exponentially* state-dependent coefficients, which eventually suggests special solution methods [10,11]. The most general FPE allowing for the maximum complexity of a *first-order* functional recurrence relation, such as (23), has been identified, and the underlying stochastic process turns out to have particularly rich unifying characteristics. The process not only synthesizes many well-known and separately studied cases into one single solvable (three-parametric) class, but is also interesting in itself because of the boundedness of its drift and diffusion coefficients. Although *globally* stochastically stable in probability (for $\beta<0$, $\gamma>0$), the process behaves for $x\to\pm\infty$ as two different and essentially unstable Wiener processes with constant drift, as can be seen from the coefficients of the FPE (10). The sign of the drift at $\pm\infty$ is such that the process is restored toward the origin, resulting in global stability. Further, it is clear that by suitable transformations [e.g., $y=e^x$, $y=\tanh(x/2)$, etc.], the SDE (11) can be given many different appearances. It should be mentioned, however, that an additive noise version of (11) is not evident, as the transformation $y(x)$ leading to a unitary diffusion [i.e., the coefficient of $F(t)$] is *non invertible* in general (see also Sec. IV F).

The connection between very different stochastic models and between their PDF's may be interpreted *a posteriori* as the physical counterpart of the mathematical interrelations between orthogonal polynomials, as they have been synthesized in "Askey's scheme for hypergeometric orthogonal polynomials" [12]. It is believed that the results in this paper give a convincing example of this parallelism.

Finally, the solution method developed in [1] and slightly generalized in this paper proves to be a powerful tool for "extended operational calculus." The usual guesswork for the identification or solution of a new eigenvalue problem is completely absent here. [Consider, e.g., the reduction of (10) to a hypergeometric differential equation, which *a posteriori* appears to be possible.] The use of the Fourier transform, which in this particular case is linked to the choice of the state variable x and the type of equation, is not a prerequisite for the constructive solution method to be applicable.

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