Theory of dressed-state lasers. III. Pump-depletion effects

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We present further developments in the theory of dressed-state lasers, i.e., lasers that operate on inverted transitions between the dressed states of a coupled atom-field system. We take into account the quantum character of the driving field, and consider the effect of its depletion. We derive effective Hamiltonians appropriate for the analysis of both one- and two-photon dressed-state lasers, and then discuss the stability and statistics of such lasers. We show that pump-depletion effects tend to reduce the degree of squeezing found in dressed-state-laser output while simultaneously introducing small squeezing effects into the depleted pump field.

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I. INTRODUCTION

In the two preceding recent papers [1,2] we have presented a theory of lasers that operate on inverted n-photon ($n=1,2$) transitions between the dressed states of a coupled atom-field system. We refer to such lasers as *dressed-state* lasers. Mollow [3] pointed out some time ago that strongly driven two-level atoms may serve as a one-photon gain medium. The predicted one-photon gain [4,5], and indeed one-photon lasing based on the predicted gain, has been observed experimentally [6,7]. Recently, there has been an experimental demonstration of steady-state two-photon gain in the dressed-atom system [8]. As the gain is proportional to the number of atoms involved and may be enhanced through the use of a high-finesse optical cavity, it appears likely that steady-state two-photon lasing may be realizable in the dressed-atom system [9,10].

The principle on which the dressed-state lasers are based can be explained as follows. A two-level atom driven by a strong laser beam of frequency ω_L and resonant Rabi frequency Ω undergoes dressing [11]. If the driving-field frequency ω_L is detuned from the atomic transition frequency ω_a by $\Delta_1=\omega_a-\omega_L$, the dressed-state doublets are split by an amount equal to the effective Rabi frequency $\Omega'=(\Omega^2+\Delta_1^2)^{1/2}$. For nonzero detuning Δ_1, the steady-state population within the dressed-state doublets is polarized [12].

Suppose one locates an ensemble of atoms in a high-finesse cavity of resonance frequency ω_c. We assume that these atoms are dressed-state polarized so as to create an inversion on $|+\rangle\to|-\rangle$ dressed-state transitions. If the cavity is resonant with a one-photon $|+\rangle\to|-\rangle$ dressed-state transition (i.e., $\omega_c=\omega_L+\Omega'$), one-photon lasing will occur for sufficiently large atomic density. If, on the other hand, the cavity is resonant with a two-photon $|+\rangle\to|-\rangle$ transition (i.e., $\omega_c=\omega_L+\Omega'/2$), it is two-photon lasing that may occur.

In Ref. [1] a detailed semiclassical theory of dressed-state lasers has been given. The effective Hamiltonian approach applied allowed us to estimate the range of parameters (detunings, atomic densities, driving-field Rabi frequencies, etc.) over which lasing of the various orders is possible. Importantly, we have found, by stability analysis of the steady-state solutions, that the regions of stability of one- and two-photon lasers are well separated in parameter space; i.e., there should not be severe competition between the two processes.

The quantum-statistical properties of dressed-state-laser radiation have been studied in Ref. [2]. It has been found that emitted laser radiation may be significantly squeezed. This result is true not only for the two-photon laser, but also, astonishingly, for the one-photon dressed-state laser. In both, the squeezing is related to level shifts analogous to Bloch-Siegert [13] and Stark shifts. These shifts appear in our theory as a result of antiresonant transitions between dressed states that are present in the dressed-states Hamiltonian.

In both Refs. [1] and [2], the driving field was treated semiclassically and the effects of pump depletion were neglected. It is the purpose of this paper to generalize the theory developed so as to include pump-depletion effects.

The paper is organized as follows. In Sec. II a general description of atom-field interaction in the presence of a quantized pump is given. We describe here our method of the self-consistent dressing of atoms. In Sec. III the effective Hamiltonians that describe the one- and two-photon dressed-state lasers are derived. In both cases pump depletion is taken into account. In Sec. IV the influence of pump depletion on the squeezing spectra of dressed-state lasers and the pump field itself are studied.

As before [1,2], the technique used is that of quantum Langevin equations [2,14]. In the concluding Sec. V, we consider possible generalizations of our approach.

II. SELF-CONSISTENT DRESSING OF ATOMS BY A QUANTIZED DRIVING FIELD

We consider a system of N two-level atoms of resonance frequency ω_a located in a "laser" cavity and pumped by a driving field of frequency ω_L. The driving (pump) field is assumed to have a traveling-wave character and to be oriented orthogonal to the axis of the laser cavity. This orthogonality will serve, as in Ref. [1], to discriminate against nonlinear gain processes of a wave-mixing type, which require a phase-matching condition.

It will be assumed that the pump field occupies a single traveling-wave mode of a unidirectional ring "pump" cavity. The pump mode will be characterized by the frequency ω_P and the width Γ_b. The pump mode is excited through injection of an external driving field of frequency ω_L. The possibility that $\omega_L \neq \omega_P$ is not excluded.

Let $\sigma_{3\mu}$, $\sigma_\mu^\dagger$, and σ_μ denote the standard Pauli matrices that describe two-level atoms. Individual atoms are indicated by the subscript μ. The atoms interact with the pump mode and a single laser-cavity mode of frequency ω_c. Let the pump-cavity mode (laser-cavity mode) annihilation and creation operators be denoted by b and $b^\dagger$ (a and $a^\dagger$), respectively.

We assume that the atoms may undergo spontaneous emission into electromagnetic-field modes that are neither associated with the pump- nor laser-cavity mode. The density matrix of the system then obeys a Liouville–von Neumann (master) equation of the form [12]

$$\dot{\rho} = -i[\mathcal{H},\rho] + \mathcal{L}_A\rho + \mathcal{L}_F\rho + \mathcal{L}_P\rho \ . \tag{2.1}$$

The Hamiltonian in Eq. (2.1) is given by the expression

$$\begin{aligned}\mathcal{H} = {} & \tfrac{1}{2}\sum_\mu^N (\Delta_1\sigma_{3\mu} + g_\mu\sigma_\mu^\dagger a + G\sigma_\mu^\dagger b + g_\mu^* a^\dagger\sigma_\mu + Gb^\dagger\sigma_\mu) \\ & + i(\Gamma_b + i\Delta_3)\frac{\tilde{\Omega}^*}{G}b^\dagger - i(\Gamma_b - i\Delta_3)\frac{\tilde{\Omega}}{G}b \\ & + \Delta_2 a^\dagger a + \Delta_3 b^\dagger b \ .\end{aligned} \tag{2.2}$$

The Hamiltonian is written in the rotating-wave approximation and does not contain any explicit time dependence, since we have already written it in a frame rotating at the frequency ω_L. As a result of the rotating-frame transformation, the first and last two terms in Eq. (2.2) are proportional to the detunings $\Delta_1 = \omega_a - \omega_L$, $\Delta_2 = \omega_c - \omega_L$, and $\Delta_3 = \omega_P - \omega_L$. The term proportional to the Rabi frequency $\tilde{\Omega}$ describes driving of the pump mode, while the μ-dependent coefficients g_μ and μ-independent coefficients G denote the coupling of atom μ to the laser- and pump-cavity modes, respectively.

The interaction of individual atoms with the pump should, in principle, contain phase factors that characterize the traveling-wave character of the pump field. By an appropriate choice of phases of individual atomic dipoles, we have absorbed these phase factors into the definition of atomic raising and lowering operators $\sigma_\mu^\dagger$ and σ_μ. The phases of individual atomic dipoles are therefore fixed relative to the pump field in the expression (2.2), and thereby a spatially varying phase ϕ_μ is introduced into the g_μ's. We assume, on the other hand, that the moduli of atom-cavity coupling factors are identical, i.e., $|g_\mu| = g$. The latter assumption can be easily justified in the case of a ring laser cavity. Alternatively, in the case of a confocal cavity of the sort used in Refs. [5] and [7], the cavity resonance is in fact highly degenerated and the cavity mode corresponds to a combination of different degenerated modes of the frequency ω_c. For such cavities, $|g_\mu|$ may be assumed to be constant in an average sense.

Note that from the above considerations it follows that our model does not assume any correlation between the phases of the pump and cavity fields. For that reason the atom-cavity couplings factors have spatially varying phases, i.e., $g_\mu = g\exp(i\phi_\mu)$. The fact that these phase factors are practically random influences our results significantly (cf. Ref. [1]). The last three terms in Eq. (2.1) describe laser-cavity-mode and pump-cavity-mode damping, as well as spontaneous emission, i.e.,

$$\mathcal{L}_F\rho = 2\Gamma(a\rho a^\dagger - \tfrac{1}{2}a^\dagger a\rho - \tfrac{1}{2}\rho a^\dagger a) \ , \tag{2.3}$$

$$\mathcal{L}_P\rho = 2\Gamma_b(b\rho b^\dagger - \tfrac{1}{2}b^\dagger b\rho - \tfrac{1}{2}\rho b^\dagger b) \ , \tag{2.4}$$

$$\mathcal{L}_A\rho = 2\gamma\sum_\mu(\sigma_\mu\rho\sigma_\mu^\dagger - \tfrac{1}{2}\sigma_\mu^\dagger\sigma_\mu\rho - \tfrac{1}{2}\rho\sigma_\mu^\dagger\sigma_\mu) \ , \tag{2.5}$$

where $\Gamma$and Γ_b are the laser-cavity- and the pump-cavity-mode half widths at half maximum, respectively. 2γ denotes an atomic spontaneous emission rate (equal to the Einstein A coefficient). As in Refs. [1] and [2], γ will be chosen to be the frequency unit in this paper.

Note that for infinitely small G it is reasonable to neglect a back interaction of the atoms on the pumping field . In this case the equation of motion for the pump amplitude has the form

$$\dot{b} = -(\Gamma_b + i\Delta_3)\left[b - \frac{\tilde{\Omega}}{G}\right] \ . \tag{2.6}$$

After a transient time of the order of $1/\Gamma_b$, the state of the field in the pump cavity would then become a coherent state, characterized by a complex amplitude $\tilde{\Omega}/G$. In such a case, we could then substitute b and $b^\dagger$ for $\tilde{\Omega}/G$ in the Hamiltonian (2.2). In another words, in the limit of small G the model described by Eq. (2.1) reduces to the one discussed in Ref. [1], i.e., the one in which atoms are located in an *external field* of the strength $|\tilde{\Omega}|$. In such situations the atoms undergo dressing corresponding to a fixed pump-field strength and detuning Δ_1 (compare Sec. II, Ref. [1]).

In the present paper, however, we will study the case of finite atom-pump coupling constant G. Therefore, the pump amplitude in our model is a dynamical variable, and it will depend nontrivially on the atomic response and amplitude of field in the laser cavity. In particular, the stationary amplitude of the pump-cavity field, which we denote as

$$\lim_{t\to\infty} \langle b(t)\rangle = \frac{\Omega}{G} , \tag{2.7}$$

will, in general, be complex and different from $\tilde{\Omega}/G$.

The stationary state of dressed-state lasers will have a phase invariance that leads to phase diffusion [14]. The phase of Ω/G, however, will be uniquely determined (i.e., locked) by the phase of $\tilde{\Omega}/G$. We may adjust the latter phase so that the parameter Ω/G will become real. Such an adjustment of the phase does not change any of the physical properties of the system in question. The absolute value of Ω/G, on the other hand, is not equal to $\tilde{\Omega}/G$, even for $\Delta_3=0$, and cannot be adjusted in any way. The value of Ω/G results from dynamics. It is this stationary value of the pump-mode amplitude that effectively dresses the atoms.

Equations (2.1)–(2.5) are written in the basis of the atomic Hilbert space and consist of bare excited states $|1\rangle_\mu$ and bare ground states $|0\rangle_\mu$. In order to obtain a dressed-state picture, we have to change basis, introducing the dressed states

$$|+\rangle_\mu = \cos\alpha|1\rangle_\mu + \sin\alpha|0\rangle_\mu \tag{2.8a}$$

and

$$|-\rangle_\mu = -\sin\alpha|1\rangle_\mu + \cos\alpha|0\rangle_\mu . \tag{2.8b}$$

In the above expressions the "rotation" angle α, which belongs to the interval $[0,\pi/2]$, is defined through the relations $\Omega=\Omega'\sin2\alpha$ and $\Delta_1=\Omega'\cos2\alpha$, where Ω' denotes an effective Rabi frequency, equal to the dressed-state doublet energy splitting $\Omega'=(\Omega^2+\Delta_1^2)^{1/2}$. The major problem here is that the Rabi frequency Ω that dresses the atoms is a dynamical quantity and has to be determined from the stationary pump-mode amplitude b.

We propose here a novel method of transforming to the dressed-state basis which employs a self-consistency argument. In this method we simultaneously perform two transformations: the first one changes the atomic basis in accordance to Eqs. (2.8) with some arbitrary Ω, and the second one shifts the values of the pump amplitude by Ω/G, i.e.,

$$b \to b + \frac{\Omega}{G} .$$

After performing such a transformation, we also require that the stationary value of the transformed pump-mode amplitude (evaluated within the semiclassical approximation) should be zero. This self-consistency condition determines the value of Ω uniquely. It also fixes the phase of $\tilde{\Omega}$, provided we require that Ω is real.

The explicit form of the Hamiltonian (2.2) in the dressed-state basis may be obtained using the unitary transformation

$$\mathcal{H}' = \mathcal{U}_1\mathcal{U}_2\mathcal{H}\mathcal{U}_2^{-1}\mathcal{U}_2^{-1} , \tag{2.9}$$

where the unitary operator

$$\mathcal{U} = \prod_\mu \exp(i\alpha\sigma_{2\mu}) \tag{2.10}$$

diagonalizes the atomic Hamiltonian in the presence of the dressing field,

$$\mathcal{H}_A = \tfrac{1}{2}\sum_\mu^N [\Delta_1\sigma_{3\mu} + \Omega(\sigma_\mu + \sigma_\mu^\dagger)] . \tag{2.11}$$

The unitary operator $\mathcal{U}$,

$$\mathcal{U}_2 = \exp\left[-\frac{\Omega}{G}(b^\dagger - b)\right] \tag{2.12}$$

shifts the expectation value of b by Ω/G, i.e., provides the dressing of atoms in an external field Ω [see Eq. (2.11)] and redefines b in such a way that the stationary value of b will be zero. A similar transformation has to be applied to the damping terms [Eqs. (2.3)–(2.5)].

After elementary calculations we obtain an expression for the Hamiltonian transformed to the dressed basis that includes nondissipative terms derived from transformation of the damping term (2.4):

$$\begin{aligned}\mathcal{H}' = {} & \sum_\mu \frac{\Omega'}{2}\sigma_{3\mu} + \Delta_2 a^\dagger a + \Delta_3 b^\dagger b - i(\Gamma_b - i\Delta_3)\left[\frac{\tilde{\Omega}-\Omega}{G}\right]b + i(\Gamma_b + i\Delta_3)\left[\frac{\tilde{\Omega}-\Omega}{G}\right]b^\dagger \\ & + \sum_\mu \frac{g_\mu^* a^\dagger + Gb^\dagger}{4}[(1+\cos2\alpha)\sigma_\mu - (1-\cos2\alpha)\sigma_\mu^\dagger + \sin(2\alpha)\sigma_{3\mu}] \\ & + \sum_\mu \tfrac{1}{4}[(1+\cos2\alpha)\sigma_\mu^\dagger - (1-\cos2\alpha)\sigma_\mu + \sin(2\alpha)\sigma_{3\mu}](g_\mu a + Gb) .\end{aligned} \tag{2.13}$$

The same transformation applied to the spontaneous emission term (2.5) yields

$$\begin{aligned}\mathcal{L}_A\rho = \frac{\gamma}{2}\sum_\mu \{ & [\sin(2\alpha)\sigma_{3\mu} + (1+\cos2\alpha)\sigma_\mu - (1-\cos2\alpha)\sigma_\mu^\dagger]\rho[\sin(2\alpha)\sigma_{3\mu} + (1+\cos2\alpha)\sigma_\mu^\dagger - (1-\cos2\alpha)\sigma_\mu] \\ & - [1+\sigma_{3\mu}\cos2\alpha - (\sigma_\mu + \sigma_\mu^\dagger)\sin2\alpha]\rho - \rho[1+\sigma_{3\mu}\cos2\alpha - (\sigma_\mu+\sigma_\mu^\dagger)\sin2\alpha]\} .\end{aligned} \tag{2.14}$$

Obviously, the laser-cavity damping term (2.3) remains unchanged under the action of the transformation (2.9). Also, the dissipative part of the term (2.4) retains its form after the transformation (2.9). We should note, however, that in expressions (2.13)–(2.14) the symbols $\sigma_{3\mu}$, $\sigma_\mu^\dagger$, and σ_μ refer now to the atomic operators in the dressed-state basis and correspond to the dressed-state inversion, raising and lowering operators, respectively.

The master equation in the dressed-state basis can be written using the explicit formulas (2.13)–(2.14) and (2.1) and constitutes a generalization of the result found in Ref. [1] (see Ref. [1] for a discussion). We stress that expressions (2.8)–(2.14) are still undetermined, since we have not yet determined the value of Ω that enters these equations self-consistently. This will be done in Sec. IV, where the semiclassical laser equations are discussed.

III. EFFECTIVE HAMILTONIANS FOR THE ONE- AND TWO-PHOTON DRESSED-STATE LASERS

Just as in Ref. [1], if the one-photon resonance condition $\Omega' \simeq \Delta_2$, $\Delta_3 \simeq 0$, holds and the parameters g/Ω' and G/Ω' are small, we may substitute for the Hamiltonian (2.13) an effective Hamiltonian. Such an effective Hamiltonian includes the lowest-order contributions from nonresonant two-photon processes. We expect that such contributions will have, as in Ref. [1], the form of Stark and Bloch-Siegert shifts [13].

Note, however, that Bloch-Siegert corrections are not with respect to optical frequency ω_L, but rather with respect to Ω'. The latter are indeed of the order of, say, g^2/Ω' and are quite significant.

We shall now derive an effective Hamiltonian using the method of Stenholm [16]. As we have already mentioned, the Hamiltonian (2.13) does not conserve the total number of excitations,

$$\hat{N} = a^\dagger a + \sum_\mu \sigma_\mu^\dagger \sigma_\mu \ . \tag{3.1}$$

On the other hand, an effective Hamiltonian that describes the processes of one-photon lasing should conserve this quantity. Such an effective Hamiltonian will be derived therefore within second-order perturbation theory with respect to the interactions that do not conserve the excitation number (3.1). We shall not attempt to calculate an effective form of the damping terms (2.14); instead, we shall perform a secular approximation and keep only the resonant contribution of this part of the Liouville–von Neumann operator.

On the other hand, since $\Delta_3 \simeq 0$, we expect that transitions between the dressed states that do not change $\sum_\mu \sigma_{3\mu}$ may occur. Such transitions may occur as a result of absorption or emission of pump photons, without dissipating much energy. For this reason, $\mathcal{H}_{\rm eff}$ is not expected to conserve the total number of the pump photons, $b^\dagger b$.

We shall assume that the wave function corresponding to an eigenvector of the Hamiltonian (2.13) has the form

$$\begin{aligned}|\psi\rangle = & \sum_{m=0}^{M} \psi_m |M-m\rangle + \sum_{m=0}^{M} \varphi_m^+ |M-m+1\rangle \\ & + \sum_{m=0}^{M-1} \varphi_m^- |M-m-1\rangle + \sum_{m=0}^{M} \eta_m^+ |M-m+2\rangle \\ & + \sum_{m=0}^{M-2} \eta_m^- |M-m-2\rangle \ .\end{aligned} \tag{3.2}$$

The coefficients ψ_m, φ_m, and η_m are vectors in the atomic Hilbert space and in the space of the pump-mode photons. The lower index m indicates that these vectors correspond to m excited atoms. The vectors $|M-m\rangle$, $|M-m\pm 1\rangle$, and $|M-m\pm 2\rangle$ are the elements of the Hilbert space that describes laser-cavity photons. They correspond to the indicated, definite number of laser-cavity photons.

The first term in the above expression thus corresponds to a definite total number of excitations, $\hat{N}=M$. The remaining four terms describe lower-order corrections that correspond to $\hat{N}=M\pm 1$ and $\hat{N}=M\pm 2$, respectively.

Let us denote the atomic operators that enter Eq. (2.13) as

$$G_0 = \sum_\mu \frac{g_\mu^*}{4}(1+\cos 2\alpha)\sigma_\mu \ , \tag{3.3a}$$

$$G_0^\dagger = \sum_\mu \frac{g_\mu}{4}(1+\cos 2\alpha)\sigma_\mu^\dagger \ , \tag{3.3b}$$

$$G_1 = \sum_\mu \frac{g_\mu^*}{4}\sin(2\alpha)\sigma_{3\mu} \ , \tag{3.3c}$$

$$G_1^\dagger = \sum_\mu \frac{g_\mu}{4}\sin(2\alpha)\sigma_{3\mu} \ , \tag{3.3d}$$

$$G_2^\dagger = -\sum_\mu \frac{g_\mu^*}{4}(1-\cos 2\alpha)\sigma_\mu^\dagger \ , \tag{3.3e}$$

$$G_2 = -\sum_\mu \frac{g_\mu}{4}(1-\cos 2\alpha)\sigma_\mu \ , \tag{3.3f}$$

and the operators that combine atomic and pump variables as

$$F_0 = \sum_\mu \frac{G}{4}(1+\cos 2\alpha)\sigma_\mu \ , \tag{3.4a}$$

$$F_0^\dagger = \sum_\mu \frac{G}{4}(1+\cos 2\alpha)\sigma_\mu^\dagger \ , \tag{3.4b}$$

$$F_1 = \sum_\mu \frac{G}{4}\sin(2\alpha)\sigma_{3\mu} \ , \tag{3.4c}$$

$$F_2^\dagger = -\sum_\mu \frac{G}{4}(1-\cos 2\alpha)\sigma_\mu^\dagger \ , \tag{3.4d}$$

$$F_2 = -\sum_\mu \frac{G}{4}(1-\cos 2\alpha)\sigma_\mu \ . \tag{3.4e}$$

If we introduce for brevity the notation

$$E_1^* = i(\Gamma_b + i\Delta_3)\left[\frac{\tilde{\Omega}^* - \Omega}{G}\right], \tag{3.5a}$$

$$E_1 = -i(\Gamma_b - i\Delta_3)\left[\frac{\tilde{\Omega} - \Omega}{G}\right], \tag{3.5b}$$

$$D_1 = \frac{G}{4}(1+\cos 2\alpha), \tag{3.5c}$$

$$D_2 = \frac{G}{4}(1-\cos 2\alpha), \tag{3.5d}$$

$$D_3 = \frac{G}{4}\sin 2\alpha, \tag{3.5e}$$

we may write down the Schrödinger equations for the coefficients ψ_m, φ_m, and η_m in the compact form

$$\begin{aligned} E\psi_m = {} & [\Delta_2(M-m) + \Omega' m + \Delta_3 b^\dagger b + E_1^* b^\dagger + E_1 b + F_1(b+b^\dagger)]\psi_m + (F_0 b^\dagger + F_2 b)\varphi_{m+1}^+ + (F_0^\dagger b + F_2^\dagger b^\dagger)\varphi_{m-1}^- \\ & + G_0\sqrt{(M-m)}\psi_{m+1} + G_0^\dagger\sqrt{(M-m+1)}\psi_{m-1} + G_1\sqrt{(M-m)}\varphi_m^- + G_1^\dagger\sqrt{(M-m+1)}\varphi_m^+ \\ & + G_2\sqrt{(M-m+1)}\eta_{m+1}^+ + G_2^+\sqrt{(M-m)}\eta_{m-1}^- . \end{aligned} \tag{3.6}$$

The above equation is exact in the sense that it contains all couplings of the vectors with $\hat{N}=M$ to those with $\hat{N}=M\pm1$, $M\pm2$. We stress here that we are effectively constructing the lowest-order expansion with respect to g/Δ_2 or g/Ω', which is formally the same as an expansion in those of the operators (3.3) and (3.4) that do not conserve $\hat{N}$, i.e., $G_1, G_1^\dagger, G_2, G_2^\dagger$ or $F, F^\dagger, F_2, F_2^\dagger$. Therefore, in the equations for the coefficients φ_m and η_m, we may keep only the terms that describe back coupling of these vectors to the ψ_m's:

$$E\varphi_m^+ = [\Omega' m + \Delta_2(M-m+1) + \Delta_3 b^\dagger b + E_1^* b^\dagger + E_1 b + F_1(b+b^\dagger)]\varphi_m^+ + G_1\sqrt{(M-m+1)}\psi_m + (F_0^\dagger b + b^\dagger F_2^\dagger)\psi_{m-1}, \tag{3.7a}$$

$$E\varphi_m^- = [\Omega' m + \Delta_2(M-m-1) + \Delta_3 b^\dagger b + E_1^* b^\dagger + E_1 b + F_1(b+b^\dagger)]\varphi_m^- + G_1^\dagger\sqrt{(M-m)}\psi_m + (F_0 b^\dagger + b F_2)\psi_{m+1}, \tag{3.7b}$$

$$E\eta_{m+1}^+ = [\Omega'(m+1) + \Delta_2(M-m+1) + \Delta_3 b^\dagger b + E_1^* b^\dagger + E_1 b + F_1(b+b^\dagger)]\eta_{m+1}^+ + G_2^\dagger\sqrt{(M-m+1)}\psi_m, \tag{3.7c}$$

$$E\eta_{m-1}^- = [\Omega'(m-1) + \Delta_2(M-m-1) + \Delta_3 b^\dagger b + E_1^* b^\dagger + E_1 b + F_1(b+b^\dagger)]\eta_{m-1}^- + G_2\sqrt{(M-m)}\psi_m . \tag{3.7d}$$

In the expressions (3.6) and (3.7), the energy is shifted by a constant $E = E' + N\Omega'/2$, where E' is an eigenvalue of the Hamiltonian (2.13). The vectors φ_m and η_m may be eliminated from Eqs. (3.6) by solving Eqs. (3.7). In the lowest order, the solutions of Eqs. (3.7) are obtained by substituting for E its approximate zeroth-order value $E = \Delta_2(M-m) + \Omega' m$ or by any other equivalent (with respect to the resonance condition $\Omega' \simeq \Delta_2$) combination of Δ_2 and Ω'. In the process of eliminating the vectors $\varphi_m^\pm, \eta_m^\pm$, we may also neglect the operator

$$\delta\hat{E} = \Delta_3 b^\dagger b + E_1^* b^\dagger + E_1 b + F_1(b+b^\dagger), \tag{3.8}$$

which enters the right-hand side of Eqs. (3.7). Self-consistency of our resonance approximation requires that $\Delta_3 \simeq 0$ and that the stationary values of the pump amplitude b and its complex conjugate b^* be zero. Neglect of the terms (3.8) turns out to lead to errors having magnitudes on the order of G/Ω'.

After employing the approximations discussed, we obtain the effective Hamiltonian that conserves $\hat{N}$:

$$\begin{aligned} \mathcal{H}_{\text{eff}} = {} & \Delta_2 a^\dagger a + \sum_\mu \frac{\Omega'}{2}\sigma_{3\mu} + G_0^\dagger a + a^\dagger G_0 + \Delta_3 b^\dagger b + E_1^* b^\dagger + E_1 b + F_1(b+b^\dagger) - \frac{2G_1^\dagger G_1}{\Delta_2+\Omega'} + \frac{2a^\dagger a}{3\Delta_2+\Omega'}[G_2^\dagger, G_2] \\ & - \frac{2G_2 G_2^\dagger}{3\Delta_2+\Omega'} - \frac{2}{\Delta_2+\Omega'}[F_0 b^\dagger + F_2 b, F_0^\dagger b + F_2^\dagger b^\dagger] + \frac{2}{\Delta_2+\Omega'}[G_1 a^\dagger, F_0 b^\dagger + F_2 b] + \frac{2}{\Delta_2+\Omega'}[F_0^\dagger b + F_2^\dagger b^\dagger, G_1^\dagger a] . \end{aligned} \tag{3.9}$$

Finally, we may make use of the fact that the phases ϕ_μ of the g_μ's that enter formula (3.9) through the definitions (3.3) and (3.4) are random. Summing over μ may therefore be replaced by averaging over φ_μ. Then, except for a nonessential constant, we obtain the final expression

$$\begin{aligned} \mathcal{H}_{\text{eff}} = {} & \frac{\Omega'}{2}S_3 + \Delta_2 a^\dagger a + \Delta_3 b^\dagger b + E_1 b + E_1^* b^\dagger + D_3 S_3(b+b^\dagger) + g_2 S_3(a^\dagger a + \tfrac{1}{2}) + g_1(S^\dagger a + a^\dagger S) \\ & + \lambda_2 S_3(b^\dagger b + \tfrac{1}{2}) + \lambda_1 S_3[b^2 + (b^\dagger)^2] + \lambda_3(b^\dagger S^\dagger a + a^\dagger b S)\lambda_4(S^\dagger b a + a^\dagger b^\dagger S) , \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
g_1 &= \frac{1+\cos 2\alpha}{2} g \ , \\
g_2 &= \frac{(1-\cos 2\alpha)^2}{8(3\Delta_2+\Omega')} g \ , \\
\lambda_1 &= -\frac{(1-\cos^2 2\alpha)}{8(\Delta_2+\Omega')} G^2 \ , \\
\lambda_2 &= \frac{(1+\cos^2 2\alpha)}{4(\Delta_2+\Omega')} G^2 \ , \\
\lambda_3 &= \frac{(1-\cos 2\alpha)\sin 2\alpha}{4(\Delta_2+\Omega')} gG \ , \\
\lambda_4 &= -\frac{(1+\cos 2\alpha)\sin 2\alpha}{4(\Delta_2+\Omega')} gG \ .
\end{aligned} \tag{3.11}
$$

In the above formula, we have introduced the macroscopic polarization

$$S=\sum_\mu \sigma_\mu \exp(-i\phi_\mu) \ , \tag{3.12a}$$

its conjugate

$$S^\dagger=\sum_\mu \sigma_\mu^\dagger \exp(+i\phi_\mu) \ , \tag{3.12b}$$

and the macroscopic inversion

$$S_3=\sum_\mu \sigma_{3\mu} \ . \tag{3.12c}$$

The influence of pump dynamics on dressed-state lasers is manifested through the appearance of several new terms in the Hamiltonian (3.10). Those are (a) the terms proportional to $S_3(b^\dagger b+\frac{1}{2})$ that describe dynamical Stark shifts of the atomic frequency, due to virtual emission and absorption of the pump-mode photons; (b) the terms of the form $b+b^\dagger$, $S_3(b^\dagger+b)$, and $S_3[(b^\dagger)^2+b^2]$ that describe the creation and annihilation of single pump photons and pairs of pump photons, such processes being associated with atomic transitions between dressed states of the same type; and (c) quasiresonant and off-resonant Raman processes described by the terms $b^\dagger S^\dagger a$, $a^\dagger Sb$, etc.

A discussion of the one-photon dressed-state-laser dynamics consequent to the Hamiltonian (3.10) will be presented in the next section.

The same method can be applied to the case of two-photon resonance, when $\Omega'\simeq 2\Delta_2$ and Δ_3 remains small. The effective Hamiltonian will then conserve (for details, see Ref. [1], Sec. V) a generalized number of excitations given by

$$\hat{N}_{\rm eff}=a^\dagger a+2\sum_\mu \sigma_\mu^\dagger\sigma_\mu \ . \tag{3.13}$$

Using steps similar to those employed in the one-photon case, we obtain the effective Hamiltonian for the two-photon dressed-state laser:

$$
\begin{aligned}
\mathcal{H}_{\rm eff}=&\frac{\Omega'}{2}S_3+\Delta_2 a^\dagger a+\Delta_3 b^\dagger b+E_1 b+E_1^* b^\dagger \\
&+D_3S_3(b+b^\dagger)+f_2S_3(a^\dagger a+\tfrac{1}{2}) \\
&+f_1[S^\dagger a^2+(a^\dagger)^2S]+\Lambda_2S_3(b^\dagger b+\tfrac{1}{2}) \\
&+\Lambda_1S_3[b^2+(b^\dagger)^2] \ ,
\end{aligned} \tag{3.14}
$$

where

$$
\begin{aligned}
f_1&=(1+\cos 2\alpha)\sin 2\alpha\frac{g^2}{4\Omega'} \ , \\
f_2&=\left[\frac{(1+\cos 2\alpha)^2}{\Omega'}+\frac{(1-\cos 2\alpha)^2}{4\Delta_2+\Omega'}\right]\frac{g^2}{8} \ ,
\end{aligned} \tag{3.15a}
$$

$$
\begin{aligned}
\Lambda_1&=-\frac{\sin^2 2\alpha}{2\Delta_2+\Omega'}\frac{G^2}{8} \ , \\
\Lambda_2&=\frac{1+\cos^2 2\alpha}{2\Delta_2+\Omega'}\frac{G^2}{4} \ .
\end{aligned} \tag{3.15b}
$$

In the above formulas, we have introduced the two-photon polarization

$$S=\sum_\mu \sigma_\mu \exp(-2i\phi_\mu) \ , \tag{3.16}$$

and its conjugate.

As we see, the pump dynamics introduce new terms into the two-photon effective Hamiltonian (3.14) that are similar to the first (a) and second (b) type found in the one-photon case. No terms of the Raman type [(c) above] are present in Eq. (3.14).

IV. SEMICLASSICAL LASER EQUATIONS AND QUANTUM FLUCTUATIONS

From the Hamiltonian (3.10), one can easily derive the semiclassical equations that describe the one-photon dressed-state laser.

It is convenient to assume that, in the stationary limit, the atomic polarization and laser-cavity field amplitude will oscillate as

$$
\begin{aligned}
S(t)&=e^{-i\Delta_L t}S \ , \\
S^*(t)&=e^{+i\Delta_L t}S^*
\end{aligned}
$$

and

$$
\begin{aligned}
a(t)&=e^{-i\Delta_L t}a \ , \\
a^\dagger(t)&=e^{+i\Delta_L t}a^\dagger \ ,
\end{aligned}
$$

where Δ_L will be determined from the stationary solution through the "frequency-pulling formula" (see Eq. (4.12) in Ref. [1]). After introducing this ansatz, we obtain

$$
\begin{aligned}
\dot{a}=&-[\Gamma+i(\Delta_2-\Delta_L)]a-ig_2S_3a-ig_1S \\
&-i\lambda_3bS-i\lambda_4b^*S \ ,
\end{aligned} \tag{4.1a}
$$

$$
\begin{aligned}
\dot{a}^*=&-[\Gamma-i(\Delta_2-\Delta_L)]a^*+ig_2S_3a^*+ig_1S^* \\
&+i\lambda_3b^*S^*+i\lambda_4bS^* \ ,
\end{aligned} \tag{4.1b}
$$

$$
\begin{aligned}
\dot{S}=&-[\gamma_1+i(\Omega'-\Delta_L)]S-2iD_3(b+b^*)S \\
&-2ig_2S(a^*a+\tfrac{1}{2})+ig_1S_3a-2i\lambda_1S[b^2+(b^*)^2] \\
&-2i\lambda_2S(b^*b+\tfrac{1}{2})+i\lambda_3b^*S_3a+i\lambda_4S_3ba \ ,
\end{aligned} \tag{4.1c}
$$

$$\dot{S}^* = -[\gamma_1 - i(\Omega' - \Delta_L)]S^* + 2iD_3(b+b^*)S^* + 2ig_2S^*(a^*a+\tfrac{1}{2}) - ig_1S_3a^* + 2i\lambda_1S^*[b^2+(b^*)^2] + 2i\lambda_2S^*(b^*b+\tfrac{1}{2}) - i\lambda_3a^*bS_3 - i\lambda_4a^*b^*S_3 \ , \quad (4.1\text{d})$$

$$\dot{S}_3 = -\gamma_2(S_3 - \bar{S}_3) + 2ig_1(a^*S - S^*a) + 2i\lambda_3(a^*bS - b^*S^*a) + 2i\lambda_4(a^*b^*S - S^*ba) \ , \quad (4.1\text{e})$$

$$\dot{b} = -(\Gamma_b + i\Delta_3)b - iE_1^* - iD_3S_3 - i\lambda_2S_3b - 2i\lambda_1b^*S_3 - i\lambda_3S^*a - i\lambda_4a^*S \ , \quad (4.1\text{f})$$

$$\dot{b}^* = -(\Gamma_b - i\Delta_3)b + iE_1 + iD_3S_3 + i\lambda_2S_3b^* + 2i\lambda_1bS_3 + i\lambda_3Sa^* + i\lambda_4S^*a \ , \quad (4.1\text{g})$$

where the damping rates of the dressed-state polarization and inversion are given by

$$\gamma_1 = \frac{2+\sin^2 2\alpha}{2}\gamma$$

and

$$\gamma_2 = (1+\cos^2 2\alpha)\gamma \ ,$$

respectively, whereas the dressed-state inversion in the absence of lasing is

$$\bar{S}_3 = -\frac{2N\cos 2\alpha}{1+\cos^2 2\alpha} \ .$$

Self-consistency requires that

$$b = b^* = 0 \ .$$

This condition immediately leads to an implicit equation for Ω:

$$\tilde{\Omega} + \frac{G}{\Delta_3 + i\Gamma_b}[D_3S_3 + i\lambda_3S^*a + i\lambda_4a^*S] = \Omega \ . \quad (4.2)$$

From the above expression, it is easily seen that the phase of Ω is controlled by the phase of $\tilde{\Omega}$. Since the physics cannot depend on this phase, as we mentioned already, we can adjust the phase of $\tilde{\Omega}$, so that Ω will be real.

It is worth checking that Eq. (4.1) allows for a below threshold solution in which $a = a^* = 0$ and $S_3 = \bar{S}_3$. In such a case,

$$\Omega = \tilde{\Omega} + \frac{GD_3\bar{S}_3}{\Delta_3 + i\Gamma_b} \ . \quad (4.3)$$

Obviously, for small G and below threshold, $|\tilde{\Omega}| \simeq |\Omega|$, as we expect. This relation will also hold above threshold, provided that the correct branch of the solutions of the Eqs. (4.1) is chosen. Above the threshold, in principle, the equation for Ω may admit bistable solutions. For one of these solutions, the relation $|\tilde{\Omega}| \simeq |\Omega|$ will also hold. The second solution will have a form dissimilar to (4.3). We shall not discuss the problem of bistability here in detail (see also the next section), and we shall consider only the solutions of (4.2) that fulfill $|\tilde{\Omega}| \simeq |\Omega|$ in the limit of small G.

Stationary solutions of the system of equations (4.1) can be easily found. In fact, since $b = 0$, the equations for stationary values of other variables decouple and can be solved analytically for a fixed Ω, as discussed in Appendix A of Ref. [1]. Solutions may then be inserted into Eq. (4.2), thereby reducing the problem to solving a single nonlinear equation for Ω. The stability of the solutions can be investigated by linearizing Eqs. (4.1) around the stationary solution. The solutions are stable when all of the eigenvalues of the stability matrix (see Appendix) have negative real parts.

The quantum properties of the laser radiation as well as the pump radiation may be studied using the quantum Langevin equations [14,15] as in Ref. [2]. In order to do it, one introduces quantum noise terms into Eqs. (4.1) and studies linear fluctuations of the solutions around the stationary state, which are linear functions of the noise terms. The only new element of such a calculation in comparison to the one presented in detail in Ref. [2] is the introduction of the new noise forces $F_b, F_b^\dagger$ that characterize the pump mode and fulfill the standard white-noise commutation relations,

$$[F_b(t), F_b^\dagger(t')] = 2\Gamma_b\delta(t-t') \ . \quad (4.4)$$

The same approach can be used in the case of the two-photon laser. Semiclassical equations are conveniently represented after eliminating the asymptotic temporal behavior:

$$S(t) = e^{-2i\Delta_L t}S \ ,$$

$$S^*(t) = e^{+2i\Delta_L t}S^*$$

and

$$a(t) = e^{-i\Delta_L t}a \ ,$$

$$a^\dagger(t) = e^{+i\Delta_L t}a^\dagger \ ,$$

where Δ_L will be determined from the frequency-pulling formula (see Eq. (6.2), Ref. [1]).

The semiclassical equations have the form

$$\dot{a} = -[\Gamma + i(\Delta_2 - \Delta_L)]a - 2if_1a^*S - if_2S_3a \ , \quad (4.5\text{a})$$

$$\dot{a}^* = -[\Gamma - i(\Delta_2 - \Delta_L)]a^* + 2if_1aS^* + if_2S_3a^* \ , \quad (4.5\text{b})$$

$$\dot{S} = -[\gamma_1 + i(\Omega' - 2\Delta_L)]S - 2iD_3(b+b^*)S - 2if_2S(a^*a+\tfrac{1}{2}) + if_1S_3a^2 - 2i\Lambda_1S[b^2+(b^*)^2] - 2i\Lambda_2S(b^*b+\tfrac{1}{2}) \ , \quad (4.5\text{c})$$

$$\dot{S}^* = -[\gamma_1 - i(\Omega' - 2\Delta_L)]S^* + 2iD_3(b+b^*)S^* + 2if_2S^*(a^*a+\tfrac{1}{2}) - if_1S_3a^* + 2i\Lambda_1S^*[b^2+(b^*)^2] + 2i\Lambda_2S^*(b^*b+\tfrac{1}{2}) \ , \quad (4.5\text{d})$$

$$\dot{S}_3 = -\gamma_2(S_3 - \bar{S}_3) + 2if_1[(a^*)^2S - S^*a^2] \ , \quad (4.5\text{e})$$

$$\dot{b} = -(\Gamma_b + i\Delta_3)b - iE_1^* - iD_3S_3 - i\Lambda_2S_3b - 2i\Lambda_1b^*S_3 \ , \quad (4.5\text{f})$$

$$\dot{b}^* = -(\Gamma_b - i\Delta_3)b + iE_1 + iD_3S_3 + i\Lambda_2S_3b^* + 2i\Lambda_1bS_3 \ . \quad (4.5\text{g})$$

The above equations have to be solved in accordance with the self-consistency condition $b=b^*=0$, i.e.,

$$\Omega=\tilde{\Omega}+\frac{GD_3S_3}{\Delta_3+i\Gamma_b} . \tag{4.6}$$

Below threshold, when $a=S=0$, we obtain the same relation between $\tilde{\Omega}$ and Ω as in Eq. (4.3). Above threshold the phase of $\tilde{\Omega}$ determines the phase of Ω and vice versa. We may therefore choose the first one so that the second one will be zero. Solving Eqs. (4.5) in the stationary limit above threshold may be reduced to solving a single equation for Ω, just as in the one-photon case (see Appendix B, Ref. [1]). Stability of the stationary solutions is governed by the linear stability matrix, given in the Appendix. Again, the linearization of Eqs. (4.5) supplied with the quantum noise terms allows for studies of the quantum properties of the laser and pump radiation.

We have performed a detailed analysis of the stability and quantum-statistical properties of the dressed-state lasers in the presence of the pump depletion. We have concentrated on the regime of small G, i.e., the case when the pump depletion effects are small. Quite generally, the stability properties of the one- and two-photon dressed-state lasers are practically unchanged and are satisfactorily described by the theory presented in Ref. [1]. In this paper we shall concentrate therefore on the discussion of the quantum-statistical properties of the laser and pump radiation.

As in Ref. [2], we shall present results concerning squeezing spectra of the laser and pump radiation:

$$S_a(\omega)=\min_{\varphi_q}\lim_{t\to\infty}2\Gamma\int_{-\infty}^{+\infty}d\tau\, e^{-i\omega\tau}\langle \mathcal{T}[\delta\hat{a}(t+\tau)e^{i\varphi_g}+\delta\hat{a}^\dagger(t+\tau)e^{-i\varphi_q}][\delta\hat{a}(t)e^{i\varphi_q}+\delta\hat{a}^\dagger(t)e^{-i\varphi_q}]\rangle \tag{4.7}$$

and

$$S_b(\omega)=\min_{\varphi_q}\lim_{t\to\infty}2\Gamma_b\int_{-\infty}^{+\infty}d\tau\, e^{-i\omega\tau}\langle \mathcal{T}[\delta\hat{b}(t+\tau)e^{i\varphi_q}+\delta\hat{b}^\dagger(t+\tau)e^{-i\varphi_q}][\delta\hat{b}(t)e^{i\varphi_q}+\delta\hat{b}^\dagger(t)e^{-i\varphi_q}]\rangle , \tag{4.8}$$

where the operators $\delta\hat{a}$, $\delta\hat{a}^\dagger$, $\delta\hat{b}$, and $\delta\hat{b}^\dagger$ denote quantum fluctuations around the semiclassical solutions of the cavity and pump fields, correspondingly. The symbol $\mathcal{T}$ indicates the appropriate ordering of the operators in Eqs. (4.7) and (4.8), i.e., normal ordering of creation and annihilation operators and apex time ordering.

Both spectra, defined above, are optimized with respect to the quadrature angle (see Refs. [17] and [18]); i.e., they correspond, for each ω, to the specific quadrature of the field that has possibly the smallest fluctuations. In expression (4.7) phase-diffusion effects of the generated laser field are neglected. That means that the calculated spectrum corresponds to the transient case, when the observation time is larger than all the other characteristic time scales of the system, but much smaller than the time scale of the phase diffusion [19]. Other possibilities, as discussed in detail in Ref. [2], are that the calculated spectrum corresponds to the self-homodyning of the laser radiation or the locking of its phase. It should be stressed that no phase-diffusion effects are present in the dynamics of the pump mode. The stationary amplitude of the pump mode has a locked phase, uniquely determined by the phase of the driving field $\tilde{\Omega}$. The quantum fluctuations of the pump amplitude that enter Eq. (4.8) have correlation functions that decay on a much faster time scale than that of the phase diffusion. Operators that describe such fluctuations commute with the operator that corresponds to the diffusing phase.

Apart from squeezing spectra (4.7) and (4.8), we shall also discuss, as in Ref. [2], results concerning optimized squeezing factors, defined as

$$S_{a,b}=\min_{\omega}S_{a,b}(\omega) . \tag{4.9}$$

V. RESULTS AND DISCUSSION

The stability regions and the squeezing properties of radiation produced by dressed-state lasers have been studied in detail in Refs. [1] and [2], respectively. Therefore, we concentrate here on the statistical properties of the depleted pump and the influence of the pump depletion on the squeezing spectra of dressed-state lasers.

To study the effect of pump depletion, we have introduced in the previous sections three parameters G, Δ_3, and Γ_b. It will be recalled that the coupling constant G describes the strength of the coupling of the pump mode with the active atoms; Γ_b is the half-width of the additional, external pump-mode cavity and gives the decay rate of the pump mode, while Δ_3 is the detuning of this cavity with respect to the driving-field frequency. Together with the quantities describing the laser itself, the number of parameters involved in the problem is rather large. Therefore, to keep the discussion within reasonable bounds, we have to restrict ourselves to selected values of the parameters. In particular, we shall always set $\Delta_3=0$, assuming, in quite a natural way, that the pump cavity is driven by a resonant signal. When the coupling constant G is too large, the active atoms interact mainly with the external pump cavity. This prevents the buildup of the dressed-laser mode. In the limiting case when the pump mode dominates the dynamics, our system smoothly evolves into a model for optical bistability [20]. This is beyond the scope of the present paper. Thus we shall typically choose $G \ll g$, especially discussing dressed-state lasers in a confocal cavity (large g).

Note also that in all the graphs presented below we give the value of the effective Rabi frequency Ω', inside the cavity. This parameter better characterizes the atomic (and dressed-laser-mode) dynamics than the "external"

Rabi frequency $\tilde{\Omega}$, since it measures the dressed-state energy splitting. On the other hand, one has to remember that $\tilde{\Omega}$ is the quantity that is directly controlled in experiment, whereas Ω may be controlled only indirectly.

Let us being our discussion with the one-photon dressed-state laser in the case of relatively small $g=0.01$—compare Refs. [1] and [2]. In Fig. 1 the optimized squeezing factor [as defined by Eq. (4.9)] is plotted as a function of Δ_2 for several different values of G. The solid curve gives the limiting result corresponding to $G=0$, i.e., no pump depletion. In this case one sees two Δ_2 regions in which the optimized squeezing parameter is negative. One should keep in mind, however, that in between these two regions the laser is operating in a stable state, but without any squeezing, i.e., with positive values of the optimized squeezing factor that are not displayed. Small pump depletion leads to a small increase of squeezing (dashed curve), which, however, stays below 1% level. Larger values of G destroy the squeezing of the dressed-laser mode (the remnants of squeezing may be seen as a small bump around $\Delta_2=13.5$). For the relatively large value of G chosen here ($G=g=0.01$), pump depletion strongly affects the laser stability properties. In fact, for the value of $G=g=0.01$, the dressed-state laser is stable only in the Δ_2 regions around 13.5 and 18.7 and no squeezing is observed at all. The inset in Fig. 1 shows squeezing spectra calculated for different values of G and $\Delta_2=13.41$ chosen so as to maximize squeezing.

Figure 2 shows the optimized squeeze factor for the pump mode. Note that the depletion of the pump may

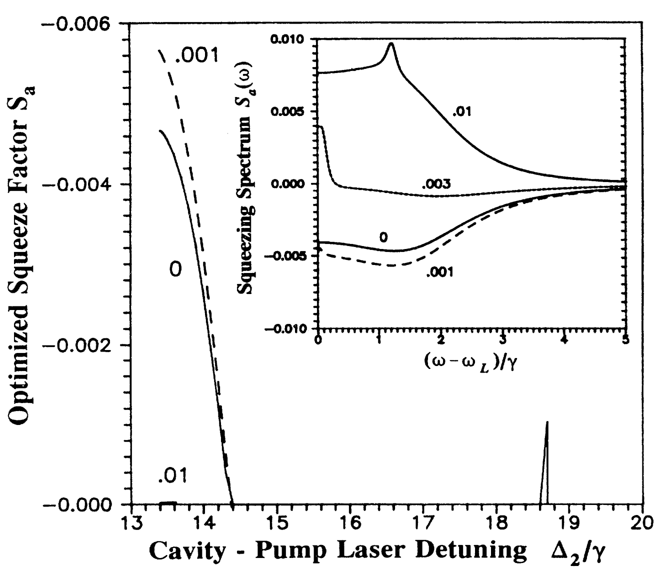

FIG. 1. Optimized squeezing factor S_a vs Δ_2 for the dressed-state one-photon laser for different values of the pump-mode coupling constant G. Lasing is characterized by the parameters $\Omega'=16$, $\Delta_1=-12$, $g=0.01$, $\Gamma=0.1$, and $N=10^6$. The pump-cavity parameters are $\Delta_3=0$ and $\Gamma_b=0.1$. The inset shows the squeezing spectrum of the one-photon laser, $S_a(\omega)$, with Δ_2 fixed at 13.41 and with the same values of other parameters. The frequencies Δ_2 and $(\omega-\omega_L)$ are given in units of γ, i.e., one-half of the free-space atomic spontaneous emission width.

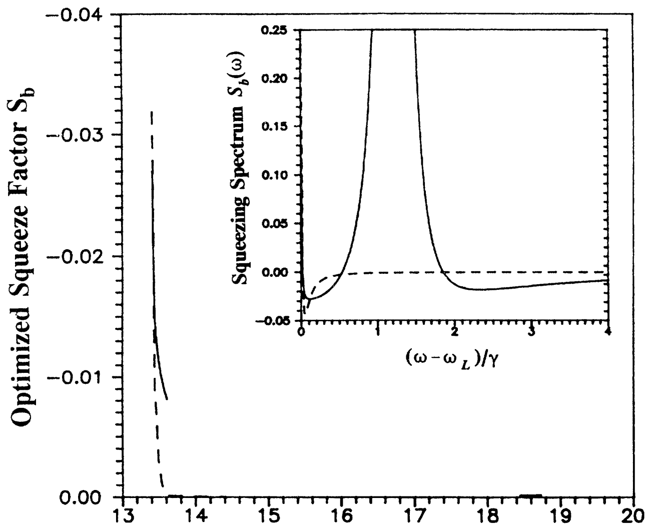

FIG. 2. Optimized squeeze factor S_b for the depleted pump vs Δ_2 and (inset) the pump squeezing spectra $S_b(\omega)$ with Δ_2 fixed at 13.41. The solid and dashed curves are drawn for $G=0.01$ and 0.001, respectively. Other parameters as in Fig. 1. The frequencies Δ_2 and $(\omega-\omega_L)$ are given in units of γ, i.e., one-half of the free-space atomic spontaneous emission width.

produce about 3% squeezing in the pump mode (i.e., squeezing in the pump mode may be one order of magnitude larger than in the dressed-state-laser mode in this case). One should stress, however, that in this parameter regime (small g) squeezing of both laser and pump radiation is typically very small and remains at most of the order of few percent.

In Fig. 3 representative squeezing spectra for the single-photon dressed-state laser in a high-g cavity ($g=0.5$) are shown. One characteristic feature is that, for relatively small values of G, pump depletion affects the squeezing spectra of the dressed-laser mode only in the vicinity of $\omega-\omega_L=0$. Since maximal squeezing generally appears in relatively high-frequency regions (cf. Ref. [2]), pump depletion only affects the optimized squeeze factors slightly. Note that all curves presented in Fig. 3 practically coincide for $\omega>5$, i.e., in the region of maximal squeezing. Thus the corresponding optimized squeeze factors will also coincide and have a form such as that shown in Fig. 5 of Ref. [2].

In Figs. 4–7 squeezing properties of the high-g-cavity-based two-photon dressed-state lasers are discussed. In Fig. 4 the influence of pump depletion on the optimized squeeze factor of the two-photon-laser mode is shown. The letter C in the figure indicates the consistency limit of the effective Hamiltonian approach (see Eqs. (6.11) and (6.12) in Ref. [1] and Sec. V of Ref. [2]), while S denotes the stability border. We present here results that lie inside the region bounded by these conditions. As we see from Fig. 4, a very small but nonzero value of G does not affect the optimized squeeze factor dramatically. However, larger values of G lead to a dramatic decrease in the degree of squeezing. Interestingly, the same degree

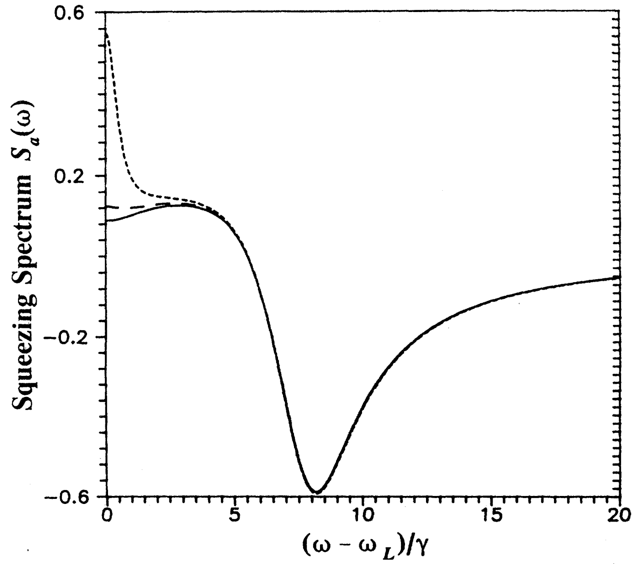

FIG. 3. Squeezing spectrum for the one-photon dressed-state laser, $S_a(\omega)$, in the high-g case. Here $\Omega'=110$, $\Delta_1=-100$, $\Delta_2=110$, $\Gamma=2$, $g=0.5$, and $N=10^5$. The solid curve gives the spectrum for $G=0$ (no pump depletion). In the case of the dashed curves, $G=0.01$, $\Delta_3=0$. For the long-dashed (short-dashed) curve, $\Gamma_b=1$ ($\Gamma_b=0.03$). The frequency $(\omega-\omega_L)$ is given in units of γ, i.e., one-half of the free-space atomic spontaneous emission width.

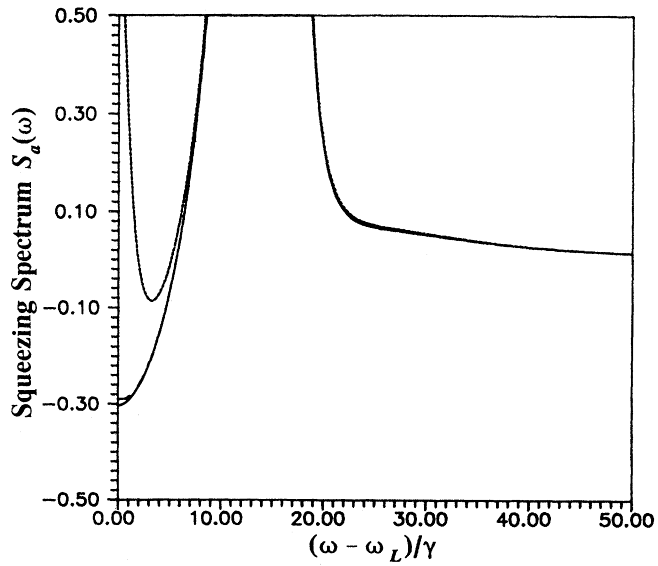

FIG. 5. Squeezing spectrum of the two-photon dressed-state laser, $S_a(\omega)$, corresponding to conditions of Fig. 4 and $\Delta_2=8.5$. Solid curve, $G=0$; (long-dashed curve, $G=0.001$ (noticeable only around $\omega-\omega_L=0$); short-dashed curve, $G=0.01$. The frequency $(\omega-\omega_L)$ is given in units of γ, i.e., one-half of the free-space atomic spontaneous emission width.

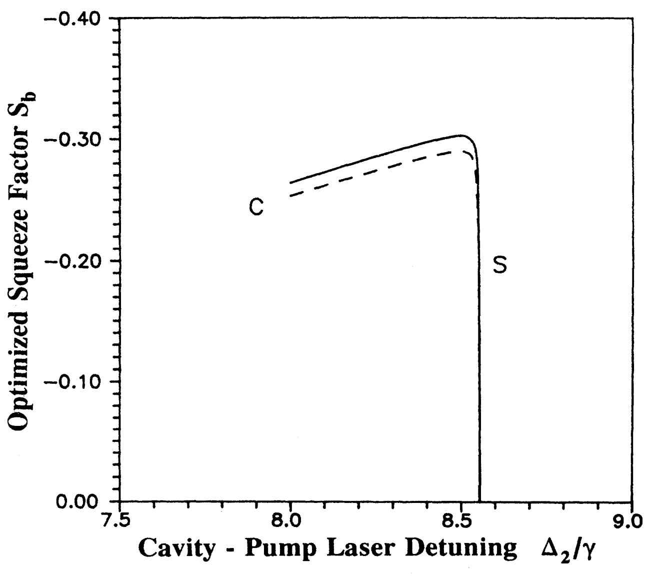

FIG. 4. Optimized squeeze parameter S_a for the two-photon dressed-state laser in the high-g case. C denotes the consistency limit of the theory; S denotes the stability border. The parameters have the values $\Omega'=20$, $\Delta_1=-12$, $g=0.4$, $\Gamma=10$, and $N=10^5$. The pump parameters for the dashed curve are $\Delta_3=0$, $\Gamma_b=1$, and $G=0.001$. In the case of the solid curve, the parameters are the same except that $G=0$ (no pump depletion). The frequency Δ_2 is given in units of γ, i.e., one-half of the free-space atomic spontaneous emission width.

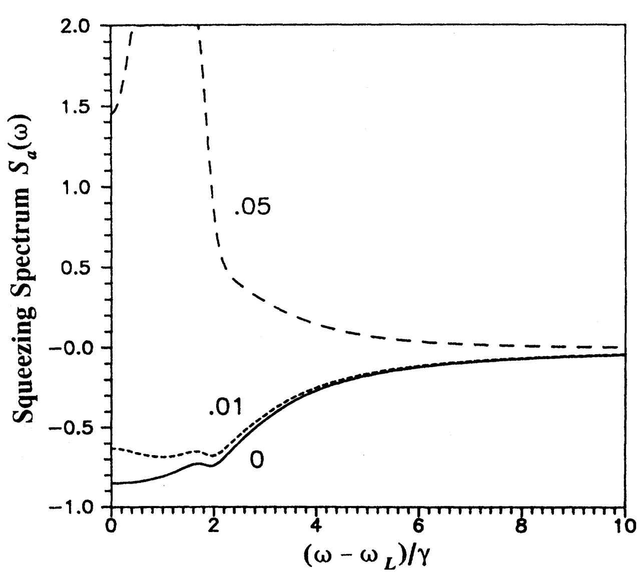

FIG. 6. Squeezing spectrum of the two-photon dressed-state laser, $S_a(\omega)$, for the different values of G indicated in the figure. Other parameters are $\Omega'=140$, $\Delta_1=-100$, $\Delta_2=61$, $\Gamma=2$, $g=0.5$, $N=50000$, $\Gamma_b=1$, and $\Delta_3=0$. As before, pump depletion tends to destroy squeezing around $\omega-\omega_L=0$. The frequency $(\omega-\omega_L)$ is given in units of γ, i.e., one-half of the free-space atomic spontaneous emission width.

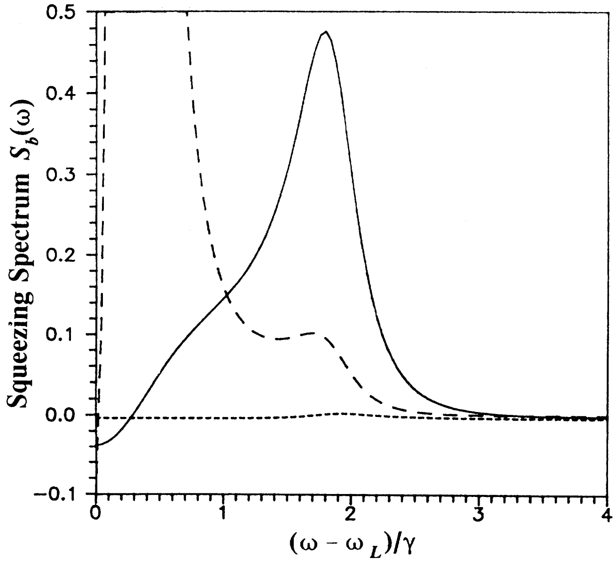

FIG. 7. Squeezing spectrum of the depleted pump, $S_b(\omega)$, for different widths of the external cavity. The solid curve corresponds to $\Gamma_b=1$, the long-dashed curve to $\Gamma_b=0.1$, and the short-dashed curve to $\Gamma_b=10$. Here $G=0.5$ and the values of other parameters are the same as in Fig. 6. The frequency $(\omega-\omega_L)$ is given in units of γ, i.e., one-half of the free-space atomic spontaneous emission width.

of pump depletion tends to decrease the squeezing of the dressed-laser mode more in the case of two-photon lasing than in the case of one-photon lasing. This is illustrated clearly in Fig. 5, which shows the corresponding squeezing spectra for the two-photon laser. Two-photon lasers evidently display maximal squeezing close to the $\omega-\omega_L=0$ frequency region. This is at the same time the frequency region is most strongly affected by pump depletion, as we already learned from the analysis of Figs. 2 and 3. That is why squeezing of two-photon lasers is so strongly degraded by the pump depletion.

In Fig. 6 squeezing spectra for a different regime of dressed-state lasing (strongly off-resonant pumping, $\Delta_1=-100$) is shown. As previously, squeezing at frequencies located around $\omega-\omega_L=0$ is affected most strongly by pump depletion, whereas squeezing at higher frequencies is only weakly affected. For strong enough coupling with the external cavity $G=g$, squeezing is destroyed completely. The pump depletion leads to a small squeezing of the pump mode as shown in Fig. 7. Again, as in the one-photon lasing case, the squeezing is only of the order of few percent. In Fig. 7 we also study the influence of cavity width on squeezing. The largest squeezing is obtained for narrow cavities (long-dashed line). The squeezing then, however, is narrow band and limited to a very narrow region of frequencies around zero. Moderate values of cavity width ($\Gamma_b=\gamma$) decrease the degree of squeezing, but broaden the frequency region over which it occurs. Finally, very broad cavities reduce and broaden both squeezing and other features in the pump squeezing spectrum.

In conclusion, the results presented show how pump depletion may affect the stability and statistical properties of dressed-state lasers. We have formulated a novel theory of self-consistently transforming to a dressed-state basis. The dressing pump field in such a transformation is chosen in such a way, so that its mean value is determined from semiclassical laser equations (4.1) and (4.5). These equations include, of course, the dynamics of the pump. Our method allows for a self-consistent construction of the effective Hamiltonian of the dressed-state lasers.

For small values of atom-pump coupling G, pump-depletion effects practically do not influence the stability properties of dressed-state lasers. The influence of pump depletion on squeezing is typically destructive and slightly more pronounced. It turns out that in the case of resonant pumping ($\Delta_3=0$) squeezing in the lasing mode is most strongly affected at frequencies close to the lasing frequency (i.e., around $\omega-\omega_L=0$ in Figs. 3, 5, and 6). On the other hand, spectral components well separated from the pump-light frequency are only weakly affected by pump depletion (as shown in Fig. 3 for the single-photon dressed-state laser). Similar effects are observed for the two-photon laser. It is worth stressing, however, that in the limit of high-g cavities and not too large G, pump depletion does not destroy large squeezing effects as predicted in Ref. [2]. We have also found that pump depletion leads to a small (several percent) squeezing in the depleted pump radiation.

The present paper completes our presentation of the theory of dressed-state lasers based on the effective Hamiltonian approach. We restricted ourselves to one- and two-photon lasing as these effects have either been observed experimentally [6,7] or, we hope, as in the two-photon lasing case, should be accessible to experiments in the immediate future. The theory presented is easily generalized to N-photon (e.g., three-photon) dressed-state lasing. The effective cavity enhanced coupling between dressed states involved in the lasing transition scales, however, such as g^N, while nonresonant effects (Bloch-Siegert terms or Stark shifts) such as g^2 in the lowest order. Therefore, the effective Hamiltonian approach developed in Refs. [1], [2], and [8] and in this paper has to be used very carefully in the case of higher N. In particular, the competition between different lasing processes (e.g., N-photon and $N-1$ off-resonant photon) may play a crucial role. Ideally, a theory should be developed that will allow one to treat different competing processes on the same footing. Such a theory must go beyond the effective Hamiltonian approach utilized here. Work in this direction is in progress.

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APPENDIX

In this appendix we present explicit forms of stability matrices that result from the linearization of Eqs. (4.1) and (4.5) around the stationary-state solutions and that govern the stability of those solutions. The stability matrix for the one-photon dressed-state laser in the presence of the quantized pump [Eq. (4.1)] is

$$\tilde{\mathcal{M}}=\begin{bmatrix} -\Gamma_a & 0 & -ig_1 & 0 & -ig_2a & -i\lambda_3S & -i\lambda_4S \\ 0 & -\Gamma_a^* & 0 & ig_1 & ig_2a^* & +i\lambda_4S^* & +i\lambda_3S^* \\ ig_1S_3-2ig_2Sa^* & -2ig_2Sa & -\Gamma_1 & 0 & ig_1a & -2iD_3S+i\lambda_4S_3a & -2iD_3S+i\lambda_3S_3a \\ 2ig_2S^*a^* & -ig_1S_3+2ig_2S^*a & 0 & \Gamma_1^* & -ig_1a & +2iD_3S-i\lambda_3S_3a & +2iD_3-i\lambda_4a^*S_3 \\ -2ig_1S^* & 2ig_1S & 2ig_1a^* & -2ig_1a & -\gamma_2 & 2i\lambda_3a^*S-2i\lambda_4S^*a & 2i\lambda_4a^*S-2i\lambda_3S^*a \\ -i\lambda_3S^* & -i\lambda_4S & -i\lambda_4a^* & -i\lambda_3a & -iD_3 & -(\Gamma_b+i\Delta_3) & 0 \\ +i\lambda_4S^* & +i\lambda_3S & +i\lambda_3a^* & +i\lambda_4a & iD_3 & 0 & -(\Gamma_b-i\Delta_3) \end{bmatrix}, \tag{A1}$$

where

$$\Gamma_a=\Gamma+i(\Delta_2-\Delta_L)+ig_2S_3,\quad \Gamma_a^*=\Gamma-i(\Delta_2-\Delta_L)-ig_2S_3,\quad \Gamma_1=\gamma_1+i(\Omega'-\Delta_L)+2ig_2(|a|^2+0.5)\ ,$$

and

$$\Gamma_1^*=\gamma_1-i(\Omega'-\Delta_L)-2ig_2(|a|^2+0.5)\ .$$

Analogously, the stability matrix for the two-photon dressed-state laser in the presence of the quantized pump [Eq. (4.5)] is

$$\tilde{\mathcal{M}}=\begin{bmatrix} -\Gamma_a & -2if_1S & -2if_1a^* & 0 & -if_2a & 0 & 0 \\ 2if_1S^* & -\Gamma_a^* & 0 & if_1a & if_2a^* & 0 & 0 \\ 2if_1S_3a-2if_2Sa^* & -2if_2Sa & -\Gamma_1 & 0 & if_1a^2 & -2iD_3S & -2iD_3S \\ 2if_2S^*a^* & -2if_1S_3a^*+2if_2S^*a & 0 & -\Gamma_1^* & -if_1(a^*)^2 & +2iD_3S^* & +2iD_3S^* \\ -4if_1S^*a & 4if_1a^*S & 2if_1(a^*)^2 & -2if_1a^2 & -\gamma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -iD_3 & -(\Gamma_b+i\Delta_3+i\Lambda_2S_3) & -2i\Lambda_1S_3 \\ 0 & 0 & 0 & 0 & iD_3 & 2i\Lambda_1S_3 & -(\Gamma_b-i\Delta_3-i\Lambda_2S_3) \end{bmatrix}, \tag{A2}$$

where this time we have denoted

$$\Gamma_a=\Gamma+i(\Delta_2-\Delta_L)+if_2S_3,\quad \Gamma_a^*=\Gamma-i(\Delta_2-\Delta_L)-if_2S_3\ ,$$

$$\Gamma_1=\gamma_1+i(\Omega'-2\Delta_L)+2if_2(|a|^2+0.5),\quad \Gamma_1^*=\gamma_1-i(\Omega'-2\Delta_L)-2if_2(|a|^2+0.5)\ .$$

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