Theory of dressed-state lasers. I. Effective Hamiltonians and stability properties

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(Received 26 December 1990)

We present a detailed theory of dressed-state lasers, i.e., the lasers that operate due to the gain on an inverted transition between dressed states of a coupled atom-field system. We derive effective Hamiltonians that describe such lasers for the case of one- and two-photon resonances. We argue that, due to the presence of nearly resonant intermediate states, the use of gain media consisting of strongly driven two-level atoms provides an unorthodox yet promising means for achieving cw optical two-photon laser oscillation. We disscuss the stability properties of radiation generated by such media.

PACS number(s): 42.50.Hz, 42.55.−f, 42.50.Kb

I. INTRODUCTION

In recent years there has been much interest in the optical instabilities that lead to the generation of various sorts of radiation. The present paper and two others [1,2] that follow relate to two topics within this general area. They are the generation of squeezed light [3–5] and the realization and properties of the two-photon laser [6]. In this series of papers, we aim to show that gain on inverted transitions between dressed-state sublevels may be utilized to produce lasers [7] (dressed-state lasers) that operate via one-, two-, or more-photon stimulated emission and may generate a squeezed output field.

The idea of a dressed-state laser stems from Mollow [8], who predicted that strongly driven homogeneous ensembles of two-level atoms can exhibit optical amplification as well as absorption. Mollow's prediction has been experimentally demonstrated in several works [9,10]. Considerations familiar from the analysis of standard laser systems clearly suggest that the effects of driven-atom gain can be substantially enhanced through the use of a high-finesse optical cavity. In fact, in a cavity containing a sufficiently large number of atoms, gain can be expected to exceed losses, and steady-state lasing with two-level atoms as the amplifying medium is expected to occur.

A two-level atom driven by a strong laser field of the frequency ω_L undergoes dressing [11]. If the driving frequency ω_L is detuned from the atomic transition frequency ω_a by $\Delta_1=\omega_a-\omega_L$, while the Rabi frequency of the resonant driving field is Ω, the atom-field states form a ladder of doublets separated by ω_L and split by the generalized Rabi frequency $\Omega'=(\Omega^2+\Delta^2)^{1/2}$. In Fig. 1 we show a portion of the ladder of dressed states. For nonzero detuning Δ_1, the stationary inversion of the dressed-state doublets is usually different from zero [12]. In particular, in the case of $\Delta_1<0$, the population of the $|+\rangle$-type states is larger than that of the $|-\rangle$-type states (see Fig. 1). If, in this situation, one locates the ensemble of dressed atoms in a high-finesse cavity that is resonant with transitions between $|+\rangle$- and $|-\rangle$-type states of adjacent dressed-state doublets, i.e., with $\omega_c=\omega_L+\Omega'$, lasing will occur for sufficiently large atomic density. The effect of optical gain on such transitions has been studied theoretically by Holm and co-workers [13] and experimentally with confocal optical cavities by Zhu, Lezama, and Mossberg [14]. Dressed-state lasing has now been observed both in atomic-vapor-cell [15(b)] and atomic-beam-type [15(a)] experiments.

By appropriate tuning of the cavity, one can generalize the idea of dressed-state lasing to the case of two- or even

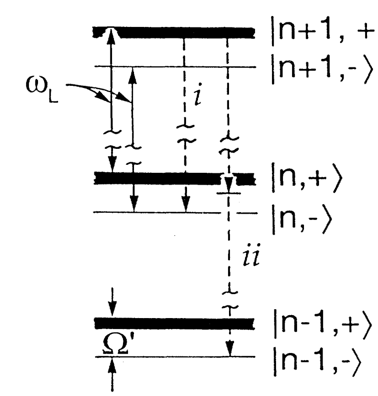

FIG. 1. Standard dressed states describing a two-level atom driven at the frequency ω_L. The lines representing the dressed levels have thicknesses in rough properties to their population. Transition *i* (*ii*) is an inverted one- (two-) photon transition.

multiphoton resonances. One should stress that the gain in dressed-state lasers should be contrasted with gain observed in wave-mixing processes in that it is not subject to any phase-matching condition involving the pump field. Rather, it can be seen as a result of population inversion on multiphoton dressed-state transitions or as a form of multiphoton Raman scattering.

Two-photon lasers are based on stimulated emission of photon pairs and were proposed early on in the laser era [16,17]. Such lasers have been investigated in numerous theoretical works [18–25], and several interesting predictions have been made. For instance, two-photon lasers are expected to have special noise properties [18,21,25], display bistable behavior [22], and require an injected signal to switch them on [17]. Recently, there has been a growing interesting in the study of such lasers [6].

Quite amazingly, however, progress in the experimental realization of two-photon lasers has been quite slow. To realize a two-photon laser, one must have available special media in which two-photon gain is large and competing processes can be suppressed. Ensembles of Rydberg atoms have been shown to be nearly ideal candidates for two-photon amplification in the microwave region. Their nearly equidistant energy levels provide a situation in which near intermediate-state resonance assures large two-photon gain, and competing processes can be suppressed through the use of microwave cavities. Two-photon masers have been experimentally realized in the beautiful experiments of Haroche and co-workers [26,6]. On the other hand, in the optical regime, owing to a lack of suitable two-photon gain media, experimentalists have only been successful in the observation of transient two-photon gain [27,28].

In a recent paper [7], we argued that an optical two-photon laser can be realized within the dressed-state scheme. Our idea stemmed from calculations of susceptibilities describing the response of two-level atoms to two strong fields [29]. The idea of the two-photon dressed-state laser is presented in Fig. 1, in which the schematic energy-level diagram of a driven two-level atom is shown. If, by the proper choice of Δ_1, the $|n,+\rangle$-type states are more populated than the $|n,-\rangle$-type states, two-photon lasing will occur, provided the atoms are located in a cavity that is tuned to two-photon transitions of the form $|n+1,+\rangle$ to $|n-1,-\rangle$ (see transition *ii* in Fig. 1), which implies that $\omega_c=\omega_L+\Omega'/2$. Such a two-photon resonance is strongly enhanced because of the existence of a pair of quasiresonant dressed states (the states $|n,+\rangle$ and $|n,-\rangle$ in Fig. 1).

It is the main aim of this series of papers to formulate a detailed theory of dressed-state lasers and to discuss quantum-statistical properties of the radiation generated by them. This paper is the first in the series and is organized as follows. In Sec. II we introduce the notation and describe a system of two-level atoms, strongly driven by an external laser field and interacting with a single mode of an optical cavity. In Sec. III we discuss the one-photon dressed-state laser; we formulate the semiclassical theory and derive threshold conditions. We compare our results with experiments of Lezama *et al.* [15(a)]. In Sec. IV we present the results for the one-photon dressed-state laser, including effects analogous to those of the Bloch-Siegert shift [30]. Such effects are, in fact, of the same order as those that lead to two-photon lasing and therefore have to be included if one wants to discuss a competition between the process of two-photon resonant lasing and the process of off-resonant one-photon lasing. The Bloch-Siegert shifts are included into our theory through the construction of an effective Hamiltonian [31,32].

A similar approach is used in Sec. V, where we derive the effective Hamiltonian for the dressed-state two-photon laser. The stability and threshold of the two-photon laser, together with the effects of competition with the off-resonant one-photon dressed-state lasing, are discussed within the framework of the semiclassical theory in Sec. VI.

The present paper is followed by two others, the first of which [1] contains a brief description of our treatment of quantum fluctuations based on quantum Langevin equations. Quantum-statistical properties of one- and two-photon dressed-state lasers are presented there and discussed in detail. We show that under suitable conditions significant squeezing of dressed-state-laser radiation may be achieved. The second paper to follow [2] discusses the problem of pump depletion and quantum fluctuations of the pumping laser radiation that arise due to the depletion effects.

II. ATOM-FIELD INTERACTIONS IN THE DRESSED-STATE PICTURE

We consider a system of N two-level atoms located in a cavity (see Fig. 2) pumped by an external driving field of frequency ω_L. The strength of the pump is characterized by the Rabi frequency Ω. The pumping field has a traveling-wave character and is oriented orthogonal to the cavity modes. The orthogonality of the pump and cavity propagation directions serves to discriminate against nonlinear gain processes of a wave-mixing type, which require a phase-matching condition. The cavity may be of various types, but as we will see below and in the subsequent papers of this series, the most interesting cavities are those in which the atom-cavity coupling con-

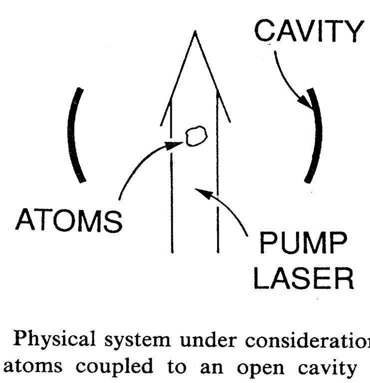

FIG. 2. Physical system under consideration consisting of N two-level atoms coupled to an open cavity and transversely driven by a pump laser of frequency ω_L. The cavity, shown only schematically, may be of various types, e.g., traveling wave or standing wave, and provide for a relatively strong or weak atom-cavity coupling.

stant is large compared to parameters such as the atomic spontaneous emission rate.

Let $\sigma_{3\mu}$, $\sigma_\mu^\dagger$, and σ_μ denote the standard Pauli matrices that describe two-level atoms, each of which is enumerated by an index μ. The atoms interact with the pumping field and with a single, nearly resonant mode of the cavity. Let us denote the cavity-photon creation and annihilation operators by $a^\dagger$ and a, respectively. Additionally, the atoms may undergo spontaneous emission into the modes of the electromagnetic field that are not associated with the cavity resonance. The density matrix of the system obeys then the Liouville–von Neumann (master) equation of the form [33].

$$\dot{\rho}=-i[\mathcal{H},\rho]+\mathcal{L}_A\rho+\mathcal{L}_F\rho \ . \tag{2.1}$$

The Hamiltonian in Eq. (2.1) is given by the expression

$$\mathcal{H}=\tfrac{1}{2}\sum_{\mu}^{N}[\Delta_1\sigma_{3\mu}+\Omega(\sigma_\mu+\sigma_\mu^\dagger)+g_\mu\sigma_\mu^\dagger a+g_\mu^* a^\dagger\sigma_\mu] + \Delta_2 a^\dagger a \ , \tag{2.2}$$

where the rotating wave has been employed to eliminate the explicit time dependence of the pump field at the frequency ω_L. It follows that the free parts of the Hamiltonian, corresponding to the first and last terms in Eq. (2.2), are proportional to the appropriately defined detunings $\Delta_1=\omega_a-\omega_L$ and $\Delta_2=\omega_c-\omega_L$, where ω_a and ω_c denote the atomic and cavity resonance frequencies, respectively. The term proportional to the Rabi frequency Ω describes the driving of the atoms, while the coefficient g_μ denotes the coupling of atom μ to the cavity mode. In principle, the interaction of individual atoms with the pump contains phase factors determined by the local phase of the traveling-wave pump field. But by appropriately choosing the phases of the individual atomic dipoles, we have absorbed these phase factors into the definition of atomic raising and lowering operators $\sigma_\mu^\dagger$ and σ_μ. The phases of individual atomic dipoles are therefore fixed relative to the pump field in expression (2.2), and thereby a spatially varying phase ϕ_μ is introduced into the g_μ's. The magnitude of the atom-cavity coupling constant, on the other hand, is assumed to be a constant, i.e., $|g_\mu|=g$. The latter assumption can be easily justified in the case of traveling-wave single-mode cavities. Alternatively, in the case of standing-wave mode-degenerate cavities of the sort used in Refs. [14] and [15], the cavity resonance corresponds to a combination of different degenerated modes of the frequency ω_c, all of which exhibit atom-cavity coupling variations on the scale of the optical wavelength. For such cavities, $|g_\mu|$ may be assumed to be constant in an average sense.

The last two terms in Eq. (2.1) describe cavity damping and spontaneous emission, i.e.,

$$\mathcal{L}_F\rho=2\Gamma\left[a\rho a^\dagger-\tfrac{1}{2}a^\dagger a\rho-\tfrac{1}{2}\rho a^\dagger a\right] \tag{2.3}$$

and

$$\mathcal{L}_A\rho=2\gamma\sum_\mu\left[\sigma_\mu\rho\sigma_\mu^\dagger-\tfrac{1}{2}\sigma_\mu^\dagger\sigma_\mu\rho-\tfrac{1}{2}\rho\sigma_\mu^\dagger\sigma_\mu\right] , \tag{2.4}$$

where Γ is the cavity half width at half maximum and 2γ denotes the free-space atomic spontaneous emission rate (equal to the Einstein coefficient A).

The system described by Eq. (2.1) is quite similar to the one describing optical bistability [34]. The only difference is that in Eq. (2.1) the pump field does not occupy a cavity mode. Quite recently, de Oliveira and Knight [35] observed a unitary equivalence between transverse and cavity-mode pumping provided that one assumes a correlation between the phases of the pump and cavity fields. Such correlation does not exist in our model, and for that reason the atom-cavity couplings contain spatially varying phase factors $g_\mu=g\exp(i\phi_\mu)$. Assuming that the atoms in our sample are spread throughout a volume whose characteristic size is large compared to the wavelength, these phase factors will be spread essentially uniformly between -1 and 1. This is the main difference between our results and those of Ref. [35].

Equations (2.1)–(2.4) are written in the basis of the atomic Hilbert space, consisting of bare excited states $|1\rangle_\mu$ and bare ground states $|0\rangle_\mu$. In order to obtain a dressed-state picture, we have to change the basis, introducing the dressed states

$$|+\rangle_\mu=(\cos\alpha)|1\rangle_\mu+(\sin\alpha)|0\rangle_\mu \tag{2.5a}$$

and

$$|-\rangle_\mu=-(\sin\alpha)|1\rangle_\mu+(\cos\alpha)|0\rangle_\mu \ . \tag{2.5b}$$

In the above expressions the "rotation" angle α, which belongs to the interval $[0,\pi/2]$, is defined through the relations $\Omega=\Omega'\sin2\alpha$ and $\Delta_1=\Omega'\cos2\alpha$, where Ω' denotes an effective Rabi frequency, equal to the dressed-state energy splitting $\Omega'=(\Omega^2+\Delta_1^2)^{1/2}$. The form of the Hamiltonian [Eq. (2.2)] in the dressed-state basis may be obtained using the unitary transformation

$$\mathcal{H}'=\mathcal{U}\mathcal{H}\mathcal{U}^{-1} , \tag{2.6}$$

where the unitary operator

$$\mathcal{U}=\prod_\mu\exp(i\alpha\sigma_{2\mu}) \tag{2.7}$$

diagonalizes the sum of the first two terms in Eq. (2.2)—the term that corresponds to the free atomic Hamiltonian and the term that describes atom-pump interactions.

After elementary calculations we obtain the explicit form of the Hamiltonian transformed to the dressed basis:

$$\mathcal{H}'=\sum_\mu \frac{\Omega'}{2}\sigma_{3\mu}+\Delta_2 a^\dagger a+\sum_\mu \frac{g_\mu^*}{4}a^\dagger[(1+\cos2\alpha)\sigma_\mu-(1-\cos2\alpha)\sigma_\mu^\dagger+\sin(2\alpha)\sigma_{3\mu}]$$
$$+\sum_\mu \frac{g_\mu}{4}[(1+\cos2\alpha)\sigma_\mu^\dagger-(1-\cos2\alpha)\sigma_\mu+\sin(2\alpha)\sigma_{3\mu}]a \ . \qquad (2.8)$$

The same transformation applied to the spontaneous emission term (2.4) yields

$$\mathcal{L}_A\rho=\frac{\gamma}{2}\sum_\mu \{[\sin(2\alpha)\sigma_{3\mu}+(1+\cos2\alpha)\sigma_\mu-(1-\cos2\alpha)\sigma_\mu^\dagger]\rho[\sin(2\alpha)\sigma_{3\mu}+(1+\cos2\alpha)\sigma_\mu^\dagger-(1-\cos2\alpha)\sigma_\mu]$$
$$-[1+\sigma_{3\mu}\cos2\alpha-(\sigma_\mu+\sigma_\mu^\dagger)\sin2\alpha]\rho-\rho[1+\sigma_{3\mu}\cos2\alpha-(\sigma_\mu+\sigma_\mu^\dagger)\sin2\alpha]\} \ . \qquad (2.9)$$

Obviously, the cavity damping term [Eq. (2.3)] remains unchanged under the action of the transformation of Eq. (2.7). We should note, however, that in Eqs. (2.8) and (2.9) the symbols $\sigma_{3\mu}$, $\sigma_\mu^\dagger$, and σ_μ refer now to the atomic operators in the dressed-state basis and correspond to the dressed-state inversion raising and lowering operators, respectively.

The master equation in the dressed-state basis can be written using the explicit formulas in Eqs. (2.8), (2.9), and (2.3). This form of the master equation will serve for the derivation of approximate, effective equations that govern the evolution of dressed-state lasers. That issue will be discussed in the next section. Here we would only like to point out a couple of properties of the dressed-state description. First of all, the Hamiltonian (2.8), when written in the dressed-state basis, is similar to that of the Jaynes-Cummings model without the rotating-wave approximation [36]. It contains, however, several terms that do not conserve the total number of atomic- and cavity-mode excitations. Some of these terms change the number of excitations by ±1, some of them by ±2. Thus, in contrast to the case of the Jaynes-Cummings model, this Hamiltonian does not conserve parity.

Inspection of the spontaneous emission term (2.9) indicates that apart from damping it now describes incoherent and coherent processes. This fact has an obvious physical interpretation. Namely, spontaneous emission may cause transitions from the upper to the lower dressed states and from the lower to the upper dressed state, as well as transitions between the same dressed states (see Fig. 1). The possibility of multiphoton lasing in the presence of an appropriately tuned cavity arises from the two general properties of the dressed-state description just mentioned.

We finish this section with a short discussion of the parameters characterizing systems of driven atoms coupled to cavities and the ranges of these parameters experimentally accessible. The relevant parameters all have the dimension of frequency and can be expressed in units of the spontaneous emission width γ. In the case of many commonly used atomic species (e.g., Ba and Na), γ is of the order of 10 MHz. Pump lasers used in experiments are usually tunable, providing for arbitrary detunings Δ_1. In particular, the range $\Delta_1\le\pm100\gamma$ is easily accessible. Available pump lasers are sufficiently powerful to generate Rabi frequencies up to 100γ. Cavity resonance frequencies can be easily controlled using, for instance, piezoelectric transducers so that the cavity-pump detuning Δ_2 also spans the range up to $\pm100\gamma$ as well.

The crucial question is how large the atom-cavity coupling can be made. In experiments with confocal cavities, modern techniques allow for achieving values of g on the order of γ, while at the same time the cavity resonance width is of the same order.

The number of atoms, N, in the active volume in such cavities can reach as high as 10^5 or even 10^6. As we shall see below, the parameter that is crucial for the laser action is

$$t=\frac{g^2N}{\Gamma\gamma} \ . \qquad (2.10)$$

Lasing is possible usually for $t\gg1$, a condition that can indeed be fulfilled in confocal cavities.

On the other hand, in the case of standard single-mode cavities, the typical values of g are of the order of $10^{-2}\gamma$ or even smaller. Such a dramatic decrease of the coupling constant, however, can be compensated by an increase of the cavity finesse, which in turn leads to a decrease of the cavity width Γ. As a result, the relevant parameter t may still achieve values that exceed the laser threshold.

III. ONE-PHOTON DRESSED-STATE LASERS

The explicit expression for the Hamiltonian [Eq. (2.8)] and damping term [Eq. (2.9)] may now be used to derive different approximate expressions that describe one or multiphoton dressed-state lasers. We shall start our discussion with the simplest case of one-photon resonance.

One-photon dressed-state lasing may be experimentally selected by tuning the cavity into resonance with inverted one-photon transitions between the dressed-state levels. Lasing will occur at the cavity frequencies $\omega_c\simeq\omega_L\pm\Omega'$ (corresponding to $\Delta_2\simeq\pm\Omega'$). Here the plus sign corresponds to $\Delta_1<0$, as shown in Fig. 1. If we assume that the driving field is strong, i.e., that both Ω' and $|\Delta_2|$ are much larger than g, we may invoke the standard rotating-wave approximation [37] and drop the antiresonant terms [38] from Eq. (2.8). Similar arguments may be used in order to neglect the antiresonant terms in expression (2.9), provided that Ω' is also much larger than γ. Self-consistency of our analysis demands that the cavity width Γ be much smaller than Ω', guaranteeing that the cavity resonance is sufficiently well defined to make the resonance condition specified above meaningful. As we shall see below, the rotating-wave approximation, when applied to Eqs. (2.8) and (2.9), enables us to define collective order parameters that effectively describe the behavior of the atoms. For instance, the macroscopic dressed-state polarization is defined as

$$\mathcal{S}=\sum_{\mu}\sigma_{\mu}\exp(-i\phi_{\mu}) \ . \tag{3.1}$$

The definitions of macroscopic order parameters contain atom-dependent phase factors and imply specific correlations between the phases of individual atomic dipoles. The appearance of ordering in the system, i.e., the emergence of nonzero values of the quantities of the form (3.1), means automatically that phase correlations between different atoms must emerge. Such phase correlations imply, on the other hand, that the antiresonant terms in the Hamiltonian [Eq. (2.8)] and in the spontaneous emission term [Eq. (2.9)] will contain atom-specific phase factors, which tend to zero when summed over μ. This fact justifies the neglect of the antiresonant terms in a manner that does not depend on the largeness of Ω' or Δ_2 as compared to other relevant frequencies.

Neglecting the antiresonant terms in Eqs. (2.8) leads to the following effective Hamiltonian that describes the one-photon dressed-state laser for $\Delta_1<0$:

$$\mathcal{H}_{\text{eff}}=\sum_{\mu}\frac{\Omega'}{2}\sigma_{3\mu}+\Delta_2 a^{\dagger}a + \frac{g}{4}(1+\cos 2\alpha)\sum_{\mu}(\sigma_{\mu}^{\dagger}e^{i\phi_{\mu}}a+a^{\dagger}e^{-i\phi_{\mu}}\sigma_{\mu}) \ . \tag{3.2}$$

Similarly, the spontaneous emission term in the rotating-wave approximation takes the form

$$\mathcal{L}_A=\frac{\gamma}{2}\sum_{\mu}\{[\sin^2(2\alpha)\sigma_{3\mu}\rho\sigma_{3\mu}+(1+\cos 2\alpha)^2\sigma_{\mu}\rho\sigma_{\mu}^{\dagger} +(1-\cos 2\alpha)^2\sigma_{\mu}^{\dagger}\rho\sigma_{\mu}]-[1+\cos(2\alpha)\sigma_{3\mu}]\rho -\rho[1+\cos(2\alpha)\sigma_{3\mu}]\} \ . \tag{3.3}$$

The above equations describe a standard one-photon-laser model [39], with the only difference that the parameters entering the definitions of the effective Hamiltonian and damping term are determined by a specific kind of pumping in the system and may be dynamically controlled. For instance, the parameter Ω' is here an analog of the atomic transition frequency in standard laser theory and depends explicitly on the parameters that describe the dynamics of pumping, i.e., Ω and Δ_1. The atom-cavity coupling constant is given through the expression

$$\lambda_1=\frac{g}{4}(1+\cos 2\alpha) \ . \tag{3.4}$$

The atomic polarization damping constant, which is sometimes called γ transverse in the standard theory, is defined here as

$$\gamma_1=\gamma(2+\sin^2 2\alpha)/2 \ . \tag{3.5}$$

The atomic inversion damping (corresponding to γ parallel in the standard theory) takes the form

$$\gamma_2=\gamma(1+\cos^2 2\alpha) \ . \tag{3.6}$$

Not only are the parameters defined above dynamically controlled, they are also correlated one with another (for instance, an increase of γ_1 implies a decrease of γ_2, etc.). The complicated relations (3.4)–(3.6) that define the dependence of the effective parameters on those used in Eqs. (2.8) and (2.9) constitute the only difference between our theory and that of the standard laser. For every choice of parameters, a one-photon dressed-state laser may be investigated using the standard methods.

The semiclassical laser equations associated with our model thus take a well-known form

$$\begin{aligned}\dot{S}&=-(\gamma_1+i\Omega')S+i\lambda_1 S_3 a \ ,\\ \dot{S}_3&=-\gamma_2(S_3-\bar{S}_3)+2i\lambda_1(a^*S-S^*a) \ ,\\ \dot{a}&=-(\Gamma+i\Delta_2)a-i\lambda_1 S \ ,\end{aligned} \tag{3.7}$$

where S denotes the macroscopic dressed-state polarization defined in Eq. (3.1), S^* its complex conjugate, and

$$S_3=\sum_{\mu}\sigma_{3\mu}$$

is the macroscopic dressed-state inversion. a and a^* denote the complex amplitudes of the cavity field, which is equal to the quantum-averaged values of the cavity-photon annihilation and creation operators, respectively. The stationary value of the dressed-state inversion $\bar{S}_3$ is given by

$$\bar{S}_3=\frac{-2N\cos 2\alpha}{1+\cos^2 2\alpha} \ . \tag{3.8}$$

Note that for any $\Delta_1\neq 0$ some dressed-state transitions are inverted, i.e., $\bar{S}_3\neq 0$. In particular, for negative Δ_1, α is larger than $\pi/4$ and the upper dressed state is more populated than the lower one; consequently, the inversion $\bar{S}_3$ is positive, corresponding to the situation shown in Fig. 1.

The stationary solutions of Eqs. (3.7) can be found with the assumption that in the long time limit the polarization and cavity-field amplitude behave as

$$a(t)=e^{-i\Delta_L t}a \tag{3.9}$$

and

$$S(t)=e^{-i\Delta_t t}S \ . \tag{3.10}$$

The resulting equation for the stationary value of the cavity-field amplitude then takes the form

$$a=\frac{\lambda_1^2 S_3 a}{[\Gamma+i(\Delta_2-\Delta_L)][\gamma_1+i(\Omega'-\Delta_L)]} \ . \tag{3.11}$$

As usual, Eq. (3.10) admits two kinds of solutions. The first one is trivial and corresponds to $a=a^*=0$. This solution is stable below the laser threshold. A nontrivial solution of Eq. (3.11) can be determined from

$$1=\frac{1}{[\Gamma+i(\Delta_2-\Delta_L)][\gamma_1+i(\Omega'-\Delta_L)]}\times\frac{\gamma_2\lambda_1^2\bar{S}_3}{\gamma_2+4\lambda_1^2\gamma_1|a|^2/[\gamma_1^2+(\Omega'-\Delta_L)^2]} \, . \quad (3.12)$$

From Eq. (3.12) we obtain immediately the analog of the "frequency-pulling formula" for the one-photon dressed-state-laser frequency $\omega_L^{1\gamma}$. We will write $\omega_L^{1\gamma}=\omega_L+\Delta_L$, where

$$\Delta_L=\frac{\gamma_1\Delta_2+\Gamma\Omega'}{\gamma_1+\Gamma} \, . \quad (3.13)$$

The averaged photon number becomes then

$$|a|^2=\frac{\gamma_2\bar{S}_3}{4\Gamma}-\frac{\gamma_2[\gamma_1^2+(\Omega'-\Delta_L)^2]}{4\gamma_1\lambda_1^2} \, . \quad (3.14)$$

This solution is stable above the laser threshold, i.e., when

$$1\geq-\frac{Ng^2\cos(2\alpha)(1+\cos2\alpha)^2}{4\gamma\Gamma(2+\sin^2 2\alpha)(1+\cos^2 2\alpha)}\times\left[1+\frac{(\Delta_2-\Omega')^2}{(\gamma_1+\Gamma)^2}\right]^{-1} \, . \quad (3.15)$$

It is worth noting that relation (3.15), correctly predicts the one-photon dressed-state lasing observed in the experiment of Ref. [15(a)]. In this experiment the detuning Δ_1 was chosen to be of the order of the half of the Rabi frequency, i.e., $\cos2\alpha\simeq-\frac{1}{2}$. The cavity resonance was at $\Delta_2\simeq\Omega'\simeq20\gamma$ and the cavity width was $\Gamma\simeq2\gamma$. Using these values, the threshold condition can be approximately written as

$$\frac{Ng^2}{\gamma^2}\geq200 \, .$$

With $g=0.04\gamma$, as in Ref. [15(a)], one expects one-photon lasing to occur for $N\simeq10^5$, in agreement with observations.

One can now apply the method of quantum Langevin equations [39] or the quasiprobability approach [39,40] to study quantum fluctuations of dressed-state-laser radiation. However, since our model is fully equivalent to the standard one, dressed-state-laser radiation will have the same statistical properties as the radiation from conventional one-photon lasers. A detailed description of these properties can be found in the literature [39], and we shall not discuss it here.

The simple estimation of the threshold condition given above indicates that our theory can indeed describe the dressed-state laser quite accurately. We hope that the generalization of this theory to the case of multiphoton resonances will give us unequally accurate estimates of threshold conditions [7]. However, before we turn to the discussion of the two-photon laser, we should stress that the process of one-photon lasing is, in principle, possible even if the cavity is strongly detuned from the one-photon resonance. Such an off-resonance process will therefore always compete with other processes, such as two-photon resonant lasing. Of course, the threshold for off-resonant lasing lies much higher and requires higher atomic densities. This effect is already partially taken into account in the formula of Eq. (3.15), which contains a resonant Lorentzian factor that decreases the magnitude of the left-hand side of Eq. (3.15) dramatically when Δ_2 is not equal to Ω'. Our predictions for the stability of two-photon lasing in Ref. [7] were in fact based on this observation. One should stress, however, that the effective Hamiltonian [Eq.(3.2)] was derived on a basis of the rotating-wave approximation and it does not provide a good description of our system in the off-resonant case. For a more accurate discussion of the competition between the resonant two-photon lasing and off-resonant one-photon lasing, one has to include the lowest-order off-resonant correction to the Hamiltonian (3.2). As we shall see in the next section, such corrections have the form of Bloch-Siegert shifts. Although the inclusion of these corrections does not influence the main conclusions of Ref. [7], they do alter stability conditions for lasing in some situations, as well as the quantum-statistical properties of the generated radiation [1].

IV. EFFECTIVE HAMILTONIAN AND BLOCH-SIEGERT SHIFTS

As we shall see in Sec. V, two-photon processes are characterized by an effective coupling constant of the order of g^2/Ω'. When one wants to discuss a competition between the resonant two-photon processes and off-resonant one-photon processes, one has to describe the latter ones with the accuracy up to the order of g^2/Ω' as well. In order to do it, one has to substitute for the Hamiltonian [Eq. (3.2)] an effective one that includes desired corrections. Such corrections, as we shall see, have the form of Bloch-Siegert shifts.

Note that the antiresonant processes in our dressed-state laser correspond to different transitions between the dressed states [38]. As such, making the standard rotating-wave approximation relative to the optical frequency and assuming that the resonant terms correspond to transitions at the frequency $\omega_L+\Omega'$, then the antiresonant terms correspond to transitions at the frequency $\omega_L-\Omega'$. Thus the Bloch-Siegert corrections are not with respect to the optical frequency ω_L, compared to which they are completely negligible, but rather with respect to Ω'. They are of the order of g^2/Ω' and cannot be neglected. We shall now derive such a Hamiltonian using the method of Stenholm [31]. As we have already mentioned, the Hamiltonian of Eq. (2.8) does not conserve the total number of excitations,

$$\hat{N}=a^\dagger a+\sum_\mu\sigma_\mu^\dagger\sigma_\mu \, . \quad (4.1)$$

On the other hand, an effective Hamiltonian that describes the processes of one-photon lasing should conserve this quantity. Such an effective Hamiltonian will be derived using second-order perturbation theory with respect to the interactions that do not conserve the excitation number [Eq. (4.1)]. We shall not attempt to calculate an effective form of the damping terms [Eq. (2.9)], and we shall keep only the resonant contribution of the damping part of the Liouville–von Neumann operator.

We shall assume that the wave function corresponding to an eigenvector of the Hamiltonian [see Eq. (2.8)] has the form

$$|\psi\rangle=\sum_{m=0}^{M}\psi_M|M-m\rangle+\sum_{m=0}^{M}\varphi_m^+|M-m+1\rangle+\sum_{m=0}^{M-1}\varphi_m^-|M-m-1\rangle+\sum_{m=0}^{M}\eta_m^+|M-m+2\rangle+\sum_{m=0}^{M-2}\eta_m^-|M-m-2\rangle\ . \tag{4.2}$$

The coefficients ψ_m, $\varphi_m^{\pm}$, and $\eta_m^{\pm}$ are vectors in the atomic Hilbert space. The lower index m indicates that these vectors correspond to m excited atoms. The vectors $|M-m\rangle$, $|M-m\pm1\rangle$, and $|M-m\pm2\rangle$ are the elements of the Hilbert space that describe cavity photons. They correspond to the indicated definite number of cavity photons.

The first term in the above expression thus corresponds to a definite total number of excitations, $\hat{N}=M$. The remaining four terms describe lowest-order corrections that correspond to $\hat{N}=M+-1$ and $\hat{N}=M\pm2$, respectively.

If we denote the atomic operators that enter Eq. (2.8) as

$$G=\sum_\mu \frac{g_\mu^*}{4}(1+\cos2\alpha)\sigma_\mu\ , \tag{4.3a}$$

$$G^\dagger=\sum_\mu \frac{g_\mu}{4}(1+\cos2\alpha)\sigma_\mu^\dagger\ , \tag{4.3b}$$

$$G_1=\sum_\mu \frac{g_\mu^*}{4}\sin(2\alpha)\sigma_{3\mu}\ , \tag{4.3c}$$

$$G_1^\dagger=\sum_\mu \frac{g_\mu}{4}\sin(2\alpha)\sigma_{3\mu}\ , \tag{4.3d}$$

$$G_2^\dagger=-\sum_\mu \frac{g_\mu^*}{4}(1-\cos2\alpha)\sigma_\mu^\dagger\ , \tag{4.3e}$$

$$G_2=-\sum_\mu \frac{g_\mu}{4}(1-\cos2\alpha)\sigma_\mu\ , \tag{4.3f}$$

we may write down the Schrödinger equations for coefficients ψ_m, φ_m, and η_m in the compact form

$$E\psi_m=[\Delta_2(M-m)+\Omega'm]\psi_m+G\sqrt{(M-m)}\psi_{m+1}+G^\dagger\sqrt{(M-m+1)}\psi_{m-1}+G_1\sqrt{(M-m)}\varphi_m^-+G_1^\dagger\sqrt{(M-m+1)}\varphi_m^++G_2\sqrt{(M-m+1)}\eta_{m+1}^++G_2^\dagger\sqrt{(M-m)}\eta_{m-1}^-\ . \tag{4.4}$$

The above equation is exact in the sense that it contains all couplings of the vectors with $\hat{N}=M$ to those with $\hat{N}=M\pm1$ and $M\pm2$. We stress here that we are effectively constructing the lowest-order expansion with respect to g/Δ_2 or g/Ω', which is formally the same as an expansion in terms of the operators [see Eq. (4.3)] that do not conserve $\hat{N}$, i.e., G_1, $G_1^\dagger$, G_2, and $G_2^\dagger$. Therefore, in the equations for φ_m and η_m, we need only to keep terms describing back coupling of those vectors to the ψ_m's:

$$E\varphi_m^+=[\Omega'm+\Delta_2(M-m+1)]\varphi_m^++G_1\sqrt{(M-m+1)}\psi_m\ , \tag{4.5a}$$

$$E\varphi_m^-=[\Omega'm+\Delta_2(M-m-1)]\varphi_m^-+G_1^\dagger\sqrt{(M-m)}\psi_m\ , \tag{4.5b}$$

$$E\eta_{m+1}^+=[\Omega'(m+1)+\Delta_2(M-m+1)]\eta_{m+1}^++G_2^\dagger\sqrt{(M-m+1)}\psi_m\ , \tag{4.5c}$$

$$E\eta_{m-1}^-=[\Omega'(m-1)+\Delta_2(M-m-1)]\eta_{m-1}^-+G_2\sqrt{(M-m)}\psi_m\ . \tag{4.5d}$$

In Eqs. (4.4) and (4.5), the energy is shifted by a constant $E=E'+N\Omega'/2$, where E' is an eigenvalue of the Hamiltonian [Eq. (2.8)]. The vectors φ_m and η_m may be eliminated from Eq. (4.4) by solving Eq. (4.5). In the lowest order, the solutions of Eqs. (4.5) are obtained by substituting for E by its approximate zeroth-order value $E=\Delta_2(M-m)+\Omega'm$. Such a substitution leads to the relations

$$\varphi_m^+=-G_1\frac{\sqrt{(M-m+1)}}{\Delta_2}\psi_m\ , \tag{4.6a}$$

$$\varphi_m^-=G_1^\dagger\frac{\sqrt{(M-m)}}{\Delta_2}\psi_m\ , \tag{4.6b}$$

$$\eta_{m+1}^+=-G_2^\dagger\frac{\sqrt{(M-m+1)}}{\Delta_2+\Omega'}\psi_m\ , \tag{4.6c}$$

$$\eta_{m-1}^-=G_2\frac{\sqrt{(M-m)}}{\Delta_2+\Omega'}\psi_m\ . \tag{4.6d}$$

Inserting the above formulas into Eq. (4.4), we obtain

$$E\psi_m=[\Delta_2(M-m)+\Omega'm]\psi_m+G\sqrt{(M-m)}\psi_{m+1}+G^\dagger\sqrt{(M-m+1)}\psi_{m-1}+(M-m)\frac{G_1G_1^\dagger}{\Delta_2}\psi_m-(M-m+1)\frac{G_1^\dagger G_1}{\Delta_2}\psi_m+(M-m)\frac{G_2^\dagger G_2}{\Delta_2+\Omega'}\psi_m-(M-m+1)\frac{G_2G_2^\dagger}{\Delta_2+\Omega'}\psi_m\ . \tag{4.7}$$

From Eq. (4.7) we may read off the effective Hamiltonian that conserves $\hat{N}$:

$$\mathcal{H}_{\text{eff}}=\Delta_2 a^\dagger a+\sum_\mu \frac{\Omega'}{2}\sigma_{3\mu}+G^\dagger a+a^\dagger G-\frac{G_1^\dagger G_1}{\Delta_2}+\frac{a^\dagger a}{\Delta_2+\Omega'}[G_2^\dagger,G_2]-\frac{G_2 G_2^\dagger}{\Delta_2+\Omega'} . \tag{4.8}$$

Finally, we may make use of the fact that the phases ϕ_μ of g_μ's that enter Eq. (4.3) are random. Summing over μ may be therefore substituted by averaging over ϕ_μ, and we obtain

$$G_1^\dagger G_1=\sum_{\mu,\mu'}\frac{g_\mu g_{\mu'}^*}{16}\sin^2(2\alpha)\sigma_{3\mu}\sigma_{3\mu'}\approx\frac{N}{16}g^2\sin^2 2\alpha \tag{4.9a}$$

and

$$G_2 G_2^\dagger=\sum_{\mu,\mu'}\frac{g_\mu g_\mu'}{16}(1-\cos 2\alpha)^2\sigma_\mu\sigma_{\mu'}^\dagger\approx\sum_\mu\frac{g^2}{32}(1-\cos 2\alpha)^2(1-\sigma_{3\mu}) . \tag{4.9b}$$

Up to the inessential constant in Eqs. (4.9a) and (4.9b), we obtain the final expression

$$\mathcal{H}_{\text{eff}}=\Delta_2 a^\dagger a+\sum_\mu\left[\frac{\Omega'}{2}\sigma_{3\mu}+\frac{g^2(1-\cos 2\alpha)^2}{16(\Delta_2+\Omega')}\sigma_{3\mu}\left[a^\dagger a+\frac{1}{2}\right]\right]+G^\dagger a+a^\dagger G . \tag{4.10}$$

Introducing the macroscopic polarizations S,S^* and inversion S_3, we may easily derive the semiclassical laser equations from Eq. (4.10):

$$\dot{S}=-(\gamma_1+i\Omega')S+i\lambda_1 S_3 a-2i\lambda_2 S(a^*a+\tfrac{1}{2}) , \tag{4.11a}$$

$$\dot{S}_3=-\gamma_2(S_3-\bar{S}_3)+2i\lambda_1(a^*S-S^*a) , \tag{4.11b}$$

$$\dot{a}=-(\Gamma+i\Delta_2)a-i\lambda_1 S-i\lambda_2 S_3 a , \tag{4.11c}$$

where

$$\lambda_2=\frac{g^2}{16(\Delta_2+\Omega')}(1-\cos 2\alpha)^2 .$$

As before, one stationary solution of the above equations corresponds to $a=S=0$ and $S_3=\bar{S}_3$ and describes the situation below threshold. The above-threshold solution is obtained by substituting

$$a(t)=e^{-i\Delta_L t}a ,$$

$$S(t)=e^{-i\Delta_L t}S .$$

The "frequency-pulling formula" now includes the effect of the dynamical Bloch-Siegert shift:

$$\Delta_L=\frac{\gamma_1(\Delta_2+\lambda_2 S_3)+\Gamma(\Omega'+\lambda_2+2\lambda_2|a|^2)}{\gamma_1+\Gamma} . \tag{4.12}$$

The problem of finding the stationary solutions of Eqs. (4.11) can be reduced to solving the quadratic equation for the stationary photon number $|a|^2$ (see Appendix A). Such an equation has two solutions. If both of them are positive, i.e., physically acceptable, the larger solution may be stable, while the smaller one is always unstable. A linear stability analysis of the stationary solutions can be performed using the standard methods. We have performed such a stability analysis and the stability regions of the one-photon laser will be discussed in Sec. VI, where a comparison with the stability regions of the two-photon laser will be presented.

V. EFFECTIVE HAMILTONIAN FOR THE TWO-PHOTON LASER

The effective Hamiltonian for the two-photon laser can be derived using the same method of Stenholm [31] as in Sec. IV, with the assumption that the cavity is tuned close to the two-photon resonance, $2\Delta_2\simeq\Omega'$. Unlike the Hamiltonian of Eq. (4.10), the effective Hamiltonian will conserve the generalized total excitation number

$$\hat{N}_{\text{eff}}=a^\dagger a+2\sum_\mu\sigma_\mu^\dagger\sigma_\mu . \tag{5.1}$$

We shall therefore assume that the eigenfunctions of the Hamiltonian [Eq. (2.8)] have the form

$$|\psi\rangle=\sum_{m=0}^{[M/2]}\psi_m|M-2m\rangle+\sum_{m=0}^{[M/2]}\varphi_m^+|M-2m+1\rangle+\sum_{m=0}^{[M/2]-1}\varphi_m^-|M-2m-1\rangle+\sum_{m=0}^{[M/2]}\eta_m^+|M-2m+3\rangle+\sum_{m=0}^{[M/2]-2}\eta_m^-|M-2m-3\rangle , \tag{5.2}$$

where $[x]$ denotes the integer part of x. The coefficients in Eq. (5.2) are again vectors in the atomic Hilbert space corresponding to m excited atoms. The first term in Eq. (5.2) corresponds to the definite generalized number of excitations, $\hat{N}_{\text{eff}}=M$. The remaining terms are corrections that correspond to $\hat{N}_{\text{eff}}=M\pm1$ and $M\pm3$, respectively.

The Schrödinger equations for the coefficients ψ_m, ϕ_m, and η_m take the form

$$E\psi_m=[\Delta_2(M-m)+\Omega' m]\psi_m+G\sqrt{(M-2m)}\varphi^+_{m+1}+G^\dagger\sqrt{(M-2m+1)}\varphi^-_{m-1}+G_1\sqrt{(M-2m)}\varphi^-_m+G_1^\dagger\sqrt{(M-2m+1)}\varphi^+_m+G_2^\dagger\sqrt{(M-2m)}\eta^-_{m-1}+G_2\sqrt{(M-2m-1)}\eta^+_{m+1} \quad (5.3)$$

and

$$E\varphi^+_m=[\Omega' m+\Delta_2(M-2m+1)]\varphi^-_m+G_1\sqrt{(M-2m+1)}\psi^\dagger_m+G^\dagger\sqrt{(M-2m+2)}\psi_{m-1}\ , \quad (5.4a)$$

$$E\varphi^-_m=[\Omega' m+\Delta_2(M-2m-1)]\varphi^-_m+G_1^\dagger\sqrt{(M-2m)}\psi_m+G\sqrt{(M-2m+1)}\psi_{m+1}\ , \quad (5.4b)$$

$$E\eta^-_{m-1}=[\Omega'(m-1)+\Delta_2(M-2m-1)]\eta^-_{m-1}+G_2\sqrt{(M-2m)}\psi_m\ , \quad (5.4c)$$

$$E\eta^+_{m+1}=[\Omega'(m+1)+\Delta_2(M-2m+1)]\eta^+_{m+1}+G_2^\dagger\sqrt{(M-2m+1)}\psi_m\ . \quad (5.4d)$$

Inserting $E\simeq\Omega' m+\Delta_2(M-m)$ into Eqs. (5.4) and eliminating $\varphi^\pm_m$ and $\eta_m\pm$ from Eq. (5.3), we may read off the effective Hamiltonian, in the same manner as was done in Eqs. (4.7) and (4.8). After simplifying the terms that contain averages with respect to ϕ_μ [just as in the case of Eq. (4.9)], we obtain

$$\mathcal{H}_{\rm eff}=\frac{\Omega'}{2}S_3+\Delta_2 a^\dagger a-\Lambda_1[S^\dagger a^2+(a^\dagger)^2 S]+\Lambda_2 S_3(a^\dagger a+\tfrac{1}{2})\ , \quad (5.5)$$

where

$$\Lambda_1=\frac{g^2}{4\Omega'}\sin 2\alpha(1+\cos 2\alpha) \quad (5.6a)$$

and

$$\Lambda_2=\frac{g^2}{8}\left[\frac{(1+\cos 2\alpha)^2}{\Omega'}+\frac{(1-\cos 2\alpha)^2}{4\Delta_2+\Omega'}+\frac{2\sin^2(2\alpha)(\Omega'-2\Delta_2)}{\Omega'(4\Delta_2-\Omega')}\right]\ . \quad (5.6b)$$

Note that Eq. (5.6b) is valid only for $\Omega'\approx 2\Delta_2$, and for $\Omega'=2\Delta_2$, it reduces to the corresponding expression for the parameter δ in Ref. [7]. In Eq. (5.5) we have introduced the new macroscopic polarization operator that constitutes a proper order parameter for the two-photon lasing process,

$$S=\sum_\mu \sigma_\mu \exp(-2i\phi_\mu)\ , \quad (5.7)$$

and its conjugate $S^\dagger$. Note that the phases of the individual dipoles required for one-photon lasing [Eq. (3.1)] and two-photon lasing are different. Therefore, we expect that the two-photon processes will have a strong anticorrelation with the one-photon processes. As soon as the macroscopic polarization required for the buildup of the two-photon laser is created, it will automatically destroy the coherence needed for one-photon lasing and vice versa. Consequently, when two-photon lasing is turned on by the requisite injection signal, a strong bias is set up against competing process.

Note also that $\mathcal{H}_{\rm eff}$ does not have the same form as the standard two-photon-laser Hamiltonian in that it contains a dynamical Stark-shift term [39], which causes the laser frequency to be intensity dependent. The importance of the Stark-shift term in the process of two-photon lasing has been extensively discussed by Boone and Swain [32]. In our case, as we shall show below, the Stark-shift term may profoundly affect the stability properties of the two-photon dressed-state laser. Semiclassical two-photon-laser equations can be easily derived from the Hamiltonian [Eq. (5.5)], after dropping the antiresonant terms in $\mathcal{L}_A$:

$$\dot S=-(\gamma_1+i\Omega')S-i\Lambda_1 S_3 a^2-2i\Lambda_2 S(|a|^2+\tfrac{1}{2})\ ,$$
$$\dot S_3=-\gamma_2(S_3-\bar S_3)-2i\Lambda_1[(a^*)^2 S-S^* a^2]\ , \quad (5.8)$$
$$\dot a=-(\Gamma+i\Delta_2)a+2i\Lambda_1 S a^*-i\Lambda_2 S_3 a\ .$$

Equations (5.8) admit the stationary solution $a=S=0$, $S_3=\bar S_3$, which is always locally stable. Thus, in order to turn a two-photon laser on, it is generally necessary to inject an external signal [41]. Apart from the trivial solution, we may look for a stationary solution of the form

$$a(t)=e^{-i\Delta_L t}a\ , \quad (5.9a)$$

$$S(t)=e^{-2i\Delta_L t}S\ . \quad (5.9b)$$

Analytical expressions for these solutions are derived in Appendix B. The existence and stability region of such solutions that describe the process of two-photon lasing will be discussed in Sec. VI.

VI. STABILITY OF DRESSED-STATE LASERS

The possibility of two-photon lasing depends on two facts: the existence of nontrivial solutions to Eqs. (5.8) and their local (linear) stability. As discussed in Appendix B, if there exists one nonzero solution of the form (5.9), there exists also a second one. The smaller of the two solutions is always unstable [39].

In the vicinity of the laser threshold, our stability analysis shows (see below) that the upper-branch solution is stable in a large part of its existence domain. Therefore, simple estimates may be given for the two-photon-lasing threshold [7], which are based entirely on the existence of the nonzero solution to Eq. (5.8).

Now we substitute the proposed solutions of Eq. (5.9) into Eq. (5.8) and obtain the expression

$$a=\frac{2\Lambda_1^2 S_3|a|^2 a}{[\Gamma+i(\Delta_2-\Delta_L+\Lambda_2 S_3)][\gamma_1+i(\Omega'-2\Delta_L+2\Lambda_2|a|^2+\Lambda_2)]} . \tag{6.1}$$

The two-photon-laser frequency, which may be deduced from the above expression, is found to depend on the laser intensity due to the dynamical Stark-shift effect:

$$\Delta_L=\frac{\gamma_1(\Delta_2+\Lambda_2 S_3)+\Gamma(\Omega'+\Lambda_2+2\Lambda_2|a|^2)}{\gamma_1+2\Gamma} . \tag{6.2}$$

In particular, far above the threshold and for a large photon number, the presence of Stark shifts may considerably narrow the region over which the upper-branch solution is stable relative to its existence region.

Close to the lasing threshold, however, we may assume for simplicity that the cavity is tuned to compensate for the dynamical Stark shift. This requires that the following condition hold:

$$\Delta_2-\Delta_L+\Lambda_2 S_3=0 . \tag{6.3}$$

In such a case, the quantity Y, introduced in Appendix B, and the laser intensity may be determine from Eq. (B4):

$$\frac{2\Lambda_1^2\gamma_2\bar{S}_3|a|^2}{\Gamma(\gamma_1\gamma_2+4\Lambda_1^2|a|^4)}=1 . \tag{6.4}$$

Equation (6.4) admits solutions, provided $\bar{S}_3>0$ and

$$\frac{g^2\bar{S}_3\sin 2\alpha(1+\cos 2\alpha)}{8\Omega T}\left[\frac{\gamma_2}{\gamma_1}\right]^{1/2}\geq 1 . \tag{6.5}$$

Note that a factor of 2 was missing in the corresponding estimates (14) and (15) in Ref. [7]. This fact, obviously, does not influence the conclusions of Ref. [7]. Slightly above threshold the laser intensity is simply given by

$$|a|^2=\frac{\gamma_2\bar{S}_3}{4\Gamma} . \tag{6.6}$$

Our estimates must be consistent with the assumption that the two lasing processes (one and two photon) shown in Fig. 1 can be spectrally distinguished. That means that the Stark shift must remain small, i.e, $\Delta_L\simeq\Delta_2$. In other words, we have to impose an additional condition that

$$\begin{aligned}\frac{2\Delta_L-\Omega'}{\Omega'}&\approx\frac{2\Lambda_2|a|^2}{\Omega'}\\&\approx\frac{g^2\gamma_2\bar{S}_3}{2\Delta_2\Omega'\Gamma}[(1+\cos 2\alpha)^2+\tfrac{1}{3}(1-\cos 2\alpha)^2]\\&\ll 1 .\end{aligned} \tag{6.7}$$

It is easy to check that for $\Omega'\simeq 50$ and other parameters chosen as in the one-photon dressed-state experiment discussed above [15(a)], condition (6.5) implies

$$\left[\frac{g}{\gamma}\right]^2 N\geq 2\times 10^3 , \tag{6.8}$$

whereas condition (6.7) implies

$$\left[\frac{g}{\gamma}\right]^2 N\leq 8\times 10^4 , \tag{6.9}$$

so that one has roughly 1.5 orders of magnitude in parameter range over which the present analysis is valid and two-photon-laser action is expected to occur. Note also that under the experimental conditions of Ref. [15(a)] the two-photon-lasing threshold is only about 5 times higher than the one-photon-lasing threshold.

One should stress that two-photon lasing will compete with the strongly detuned one-photon lasing. For the same choice of parameters as above, Eq. (3.15) indicates that detuned one-photon lasing is possible for

$$\left[\frac{g}{\gamma}\right]^2 N\geq 10^4 . \tag{6.10}$$

We therefore conclude, as previously pointed out in Ref. [7], that an optical two-photon laser may be realized using a gain medium comprised of strongly driven two-level atoms. This seemingly unlikely system has a relatively large two-photon gain and lacks channels that strongly compete with the desired two-photon lasing process. In fact, generalization of the analysis presented here indicates that three- or more-photon amplification and lasing may be possible in driven two-level atom systems.

Let us stress again, however, that the condition embodied in Eq. (6.5) is based only on the existence of the nonzero solution and that it does not necessarily imply lasing. The latter requires the solution to be linearly stable. Fortunately, exact stability analysis shows that not too far from the threshold the stability region fills practically all of the existence domain of the nontrivial solution [Eq. (5.9)]. For this reason the simple condition of Eq. (6.5) gives an accurate estimate of the lasing threshold. In the remainder of this section, we present several results concerning stability that support the above statements. Since we want to explore the competition of one- and two-photon processes, we present phase diagrams for both kinds of instabilities simultaneously. In Figs. 3–6 we have displayed the stability diagrams for two- and one-photon-laser action in the Ω'-Δ_2 plane. In each figure we present two curves representing the stability boundaries of the one-photon laser. Dashed curves correspond to the stability region for lasing based on the simple model of Sec. III, which does not include the Bloch-Siegert shifts. Solid curves represent the stability of the one-photon dressed-state laser, including the Bloch-Siegert shifts described in Sec. IV. The laser is stable inside the area surrounded by the curves. In presenting the stability curve, we have taken into account that the derivation of the Hamiltonian [Eq. (2.8) or (4.10)] requires the resonant condition $\Omega'\simeq\Delta_2$, although both models may formally be studied in the whole region of Δ_2 and Ω'. We have therefore presented only the parts of

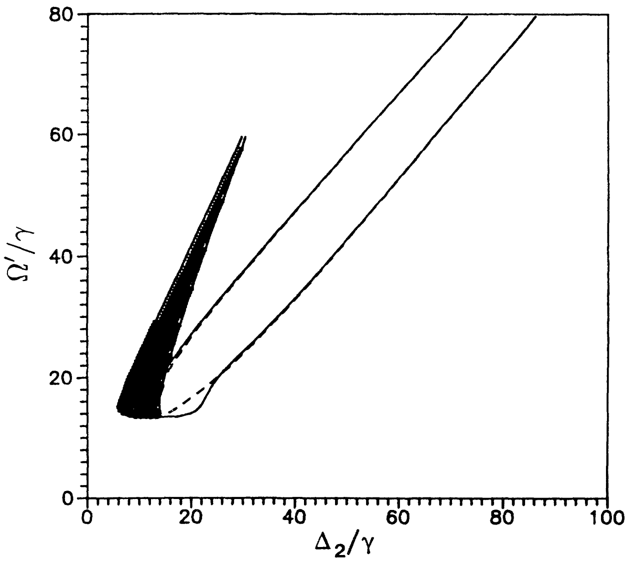

FIG. 3. Stability and existence domain of one- and two-photon dressed-state lasers in the effective-Rabi frequency–cavity-laser-detuning (Ω'-Δ_2) plane. The parameters are $\Delta_1 = -12$, $\Gamma = 0.1$, $g = 0.01$, and $N = 2\times 10^6$. The dashed line surrounds the stability region of the one-photon laser in the absence of Bloch-Siegert shifts. The solid line on the right surrounds the stability domain of the one-photon laser with Bloch-Siegert terms included. The solid line on the left surrounds the existence region of the two-photon dressed-state laser. The blackened area in this region is the stability domain of that laser. Note that for both axes the quantities are plotted in units of γ, i.e., one-half the free-space atomic spontaneous emission rate.

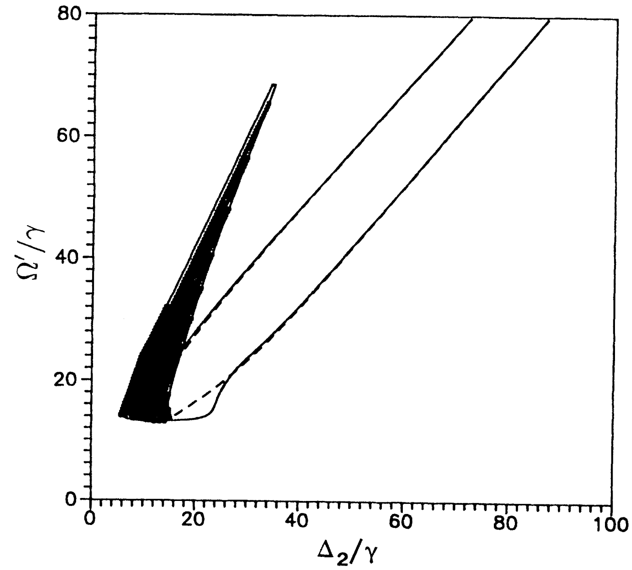

FIG. 5. Same as Fig. 3, but for the parameters $\Delta_1 = -12$, $\Gamma = 0.1$, $g = 0.05$, and $N = 10^5$.

the stability region, which additionally fulfill the condition

$$\Omega' \leq 1.5\Delta_2 \ . \tag{6.11}$$

That additional restriction (consistency condition) can indeed be imposed, since, as we shall see below, all of our results confirm the more or less simple conditions for lasing discussed above and in Ref. [7]. The analysis presented here indicates that, for reasonable choices of the parameters characterizing our system, no severe competition of the one- and two-photon-lasing processes takes

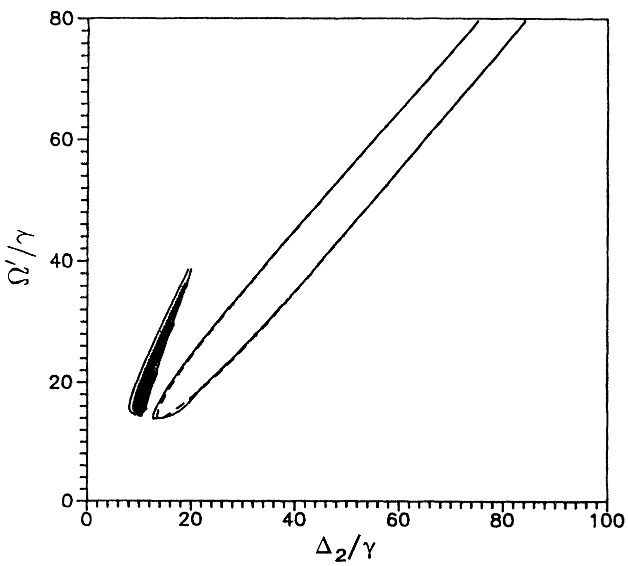

FIG. 4. Same as Fig. 3, but for $N = 10^6$.

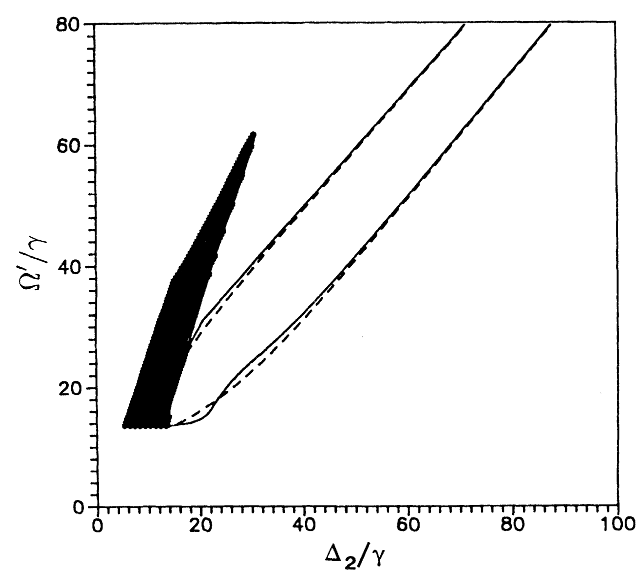

FIG. 6. Same as Fig. 3, but for the parameters $\Delta_1 = -12$, $\Gamma = 0.5$, $g = 0.1$, and $N = 10^5$.

place. Stability regions of the two processes are typically disjoint.

The other set of solid curves represent existence regions of the nonzero solution of the two-photon-laser equations. Solid dots inside the region surrounded by those curves represent stability regions. Again, in order to retain self-consistency, we have limited the stability regions to the vicinity of the two-photon resonance, $2\Delta_2 \simeq \Omega'$. In fact, we used the additional condition

$$\Omega' \leq 2.5\Delta_2 \, , \tag{6.12}$$

which indeed limits the domain of stability to physically acceptable regions.

We have also checked that our results are consistent with the condition relating to the maximum size of the ac Stark shift [see Eq. (6.7)]. In fact, throughout all of the two-photon laser stability domains shown in Figs. 3–6 Stark shifts do not exceed 15%. Figures 3 and 4 present the results that correspond to typical low-g cavities. Here $\Gamma \simeq 0.2\gamma$ and g is of the order of $10^{-2}\gamma$. The results show that Bloch-Siegert effects are rather small and that they lead only to a small modification of the stability region for the one-photon laser. It can already be clearly seen from Fig. 3 that large effective Rabi frequencies Ω' correspond to weaker Bloch-Siegert effects. This fact has a simple explanation; i.e., Bloch-Siegert shifts are caused by the correction terms in the effective Hamiltonian [Eq. (4.10)], which are of the order of g^2/Ω'. For the two-photon laser, it is interesting to note that the stability region extends over nearly the entire existence domain. This region is concentrated around the line $\Omega' = 2\Delta_2$, but it is bounded from above; i.e., no two-photon lasing is possible for Ω' or Δ_2 too large. This effect is due to the fact that the effective coupling constant Λ_1 decreases with increasing Ω' or Δ_2. It also exhibits the destructive influence of Stark shifts on the laser stability. Figure 4 is calculated for a smaller value of N, i.e., closer to the threshold. Figure 4 reveals that the one- and two-photon-laser stability domains are separated in a more pronounced fashion when the lasers are operated close to threshold than when the lasers are operated far above it. This result appears to be true rather generally. Figures 5 and 6 present results for the intermediate situation of a moderately large g and a moderately small Γ. The results here show how the small increase of the cavity coupling g tends to stabilize two-photon action for smaller values of N.

In Figs. 7 and 8 the parameter values employed correspond to conditions achievable in confocal-type cavities [15(a)]. Such cavities are notable in that even when they attain centimeter-scale dimensions they may exhibit high relative g values. For $g = 0.2$ the regions of stability of the two-photon laser are rather small for experimentally accessible values of $N \simeq 10^5$. One should also note that the stability domain of two-photon lasing does not cover the whole existence domain. Nevertheless, the estimate of Eq. (6.5) still gives quite an accurate description of this case, provided we limit ourselves to the vicinity of the resonance $\Omega' = 2\Delta_2$ and to relatively small values of Ω'.

In Figs. 9 and 10 the same situation is considered, except that a large negative detuning $\Delta_1 = -100$ is used. The results in this figure confirm once more the accuracy of the simple estimates discussed in the beginning of this section. It is worth noting that the stability regions of the one- and two-photon laser becomes more distinct as Δ_1 increases. This effect can be observed even for the case of very small Ω.

In conclusion, we stress once more that all of the presented figures show clearly two important effects.

(i) In most cases the regions of stability of two- and one-photon lasers are well separated; i.e., there will be no

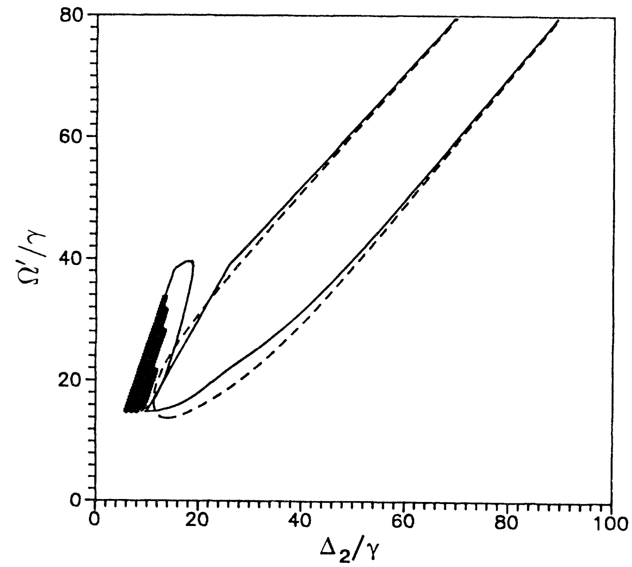

FIG. 7. Same as Fig. 3, but for the parameters $\Delta_1 = -12$, $\Gamma = 0.2$, $g = 0.2$, and $N = 5 \times 10^5$.

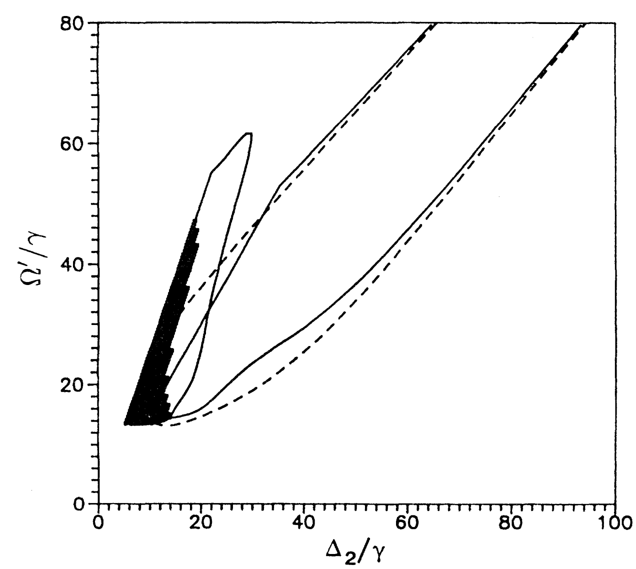

FIG. 8. Same as Fig. 3, but for the parameters $\Delta_1 = -12$, $\Gamma = 2$, $g = 0.2$, and $N = 10^5$.

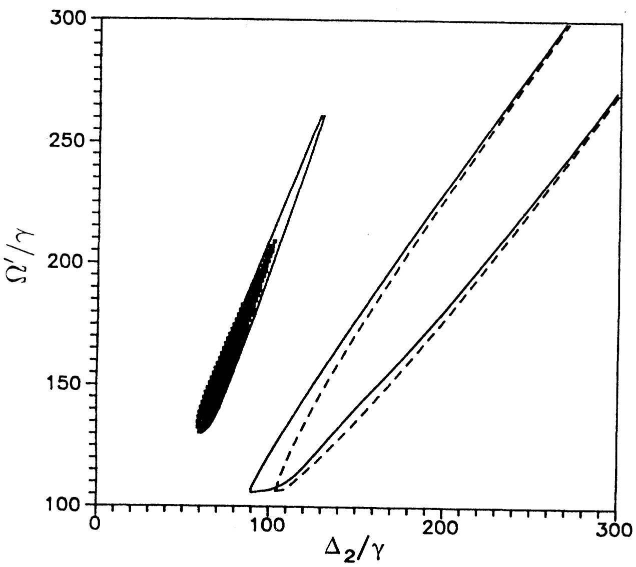

FIG. 9. Same as Fig. 3, but for the parameters $\Delta_1=-100$, $\Gamma=2$, $g=0.5$, and $N=5\times10^4$.

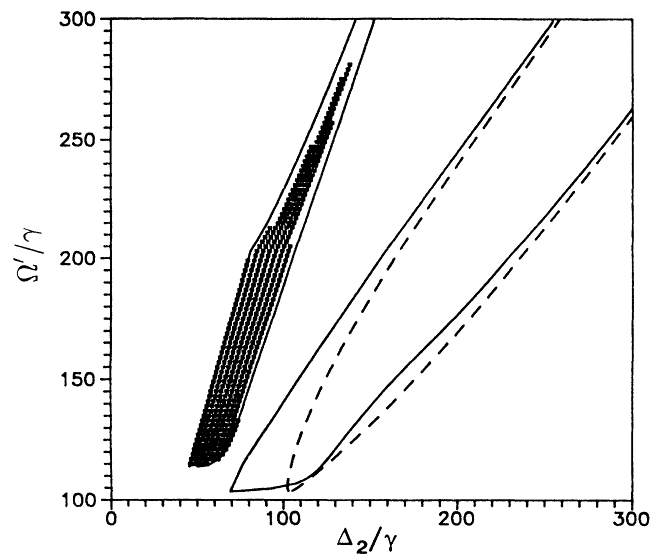

FIG. 10. Same as Fig. 3, but for the parameters $\Delta_1=-100$, $\Gamma=2$, $g=0.5$, and $N=10^5$.

severe competition between the two processes.

(ii) The simple estimates of Ref. [7] give quite an accurate description of the threshold conditions for one- and two-photon dressed-state lasers.

After completion of this series of papers, we have learned of the works of Agarwal [42,43] devoted to the same problem. He considers, however, one-photon lasing only in the simplest rotating-wave approximation.

Finally, we should mention that two-photon gain and preliminary evidence of two-photon lasing have recently been observed by us [44] in the driven two-level atom system.

ACKNOWLEDGMENTS

We acknowledge the fruitful collaboration and discussions with Dr. J. R. Kuliński, who took part in the initial phase of this project. M.L. and J.Z. acknowledge the support by the Polish Government, Grant No. CPBP 0.107, and T.W.M. acknowledges the financial support of the U.S. Army Research Office under Contract No. DAAL03-91-G-0313.

APPENDIX A

Here we present our method of finding analytic solutions for the stationary state of a one-photon laser. From the laser equations (4.11), we obtain

$$S_3=\frac{\gamma_2\bar{S}_3}{\gamma_2+4\lambda_1^2\gamma_1|a|^2/[\gamma_1^2+(\Omega'-\Delta_L+2\lambda_2|a|^2+\lambda_2)^2]} \tag{A1}$$

and

$$S=\frac{i\lambda_1 S_3 a}{\gamma_1+i(\Omega'-\Delta_L+2\lambda_2|a|^2+\lambda_2)} . \tag{A2}$$

Inserting Eq. (A2) into the equation for the cavity-mode amplitude, we get

$$a=\frac{\lambda_1^2 S_3 a}{[\Gamma+i(\Delta_2-\Delta_L+i\lambda_2 S_3)][\gamma_1+i(\Omega'-\Delta_L+2\lambda_2|a|^2+\lambda_2)]} . \tag{A3}$$

Assuming that $a\neq0$, we obtain immediately from Eq. (A3) a "frequency-pulling formula." Defining $Y=\Omega'-\Delta_L+2\lambda_2(|a|^2+\frac{1}{2})$ and using Eq. (4.12), we obtain, from Eq. (A3),

$$1=\frac{\gamma_1\lambda_1^2 S_3}{\Gamma(Y^2+\gamma_1^2)} , \tag{A4}$$

and using Eq. (A2),

$$1=\frac{\gamma_1\gamma_2\lambda_1^2\bar{S}_3}{\Gamma[\gamma_2(Y^2+\gamma_1^2)+4\lambda_1^2\gamma_1|a|^2]} . \tag{A5}$$

Combining Eqs. (A4) and (A5) we obtain

$$S_3=\bar{S}_3-\frac{4\Gamma}{\gamma_2}|a|^2 . \tag{A6}$$

Inserting Eq. (A6) into the definition of Δ_L [Eq. (4.12)], we obtain an explicit representation of Δ_L as a linear function of $|a|^2$. Y becomes then a linear function of $|a|^2$ also. From Eq. (A6) we obtain an explicit quadratic equation for $|a|^2$,

$$\tilde{\alpha}|a|^4+\tilde{\beta}|a|^2+\tilde{\gamma}=0 , \tag{A7}$$

where

$$\tilde{\alpha}=\frac{4\lambda_2^2\gamma_1^2}{(\gamma_1+\Gamma)^2\gamma_2^2}(\gamma_2+2\Gamma)^2 . \tag{A8a}$$

$$\tilde{\beta}=\frac{4\lambda_1^2\gamma_1}{\gamma_2}+\frac{4\lambda_2\gamma_1^2}{\gamma_2(\gamma_1+\Gamma)^2}(\gamma_2+2\Gamma)[\Omega-\Delta_2-\lambda_2(\bar{S}_3-1)] , \tag{A8b}$$

and

$$\tilde{\gamma}=\frac{\gamma_1^2[\Omega'-\Delta_2-\lambda_2(\bar{S}_3-1)]^2}{(\gamma_1+\Gamma)^2}+\gamma_1-\frac{\gamma_1\lambda_1^2\bar{S}_3}{\Gamma} . \tag{A8c}$$

APPENDIX B

Here we derive analytic expressions for stationary solutions [see Eq. (5.9)] describing two-photon-laser action. From Eq. (5.8c) we obtain

$$a=\frac{2\Lambda_1^2S_3|a|^2a}{[\Gamma+i(\Delta_2-\Delta_L+\Gamma_2S_3)][\gamma_1+i(\Omega'-2\Delta_L+2\Lambda_2|a|^2+\Lambda_2)]} . \tag{B1}$$

Assuming $a\neq0$, the "frequency-pulling formula" follows immediately from Eq. (B1). Denoting $Y=\Omega'-2\Delta_L+2\Lambda_2(|a|^2+\frac{1}{2})$, we obtain, from Eq. (B1),

$$1=\frac{2\Lambda_1^2\gamma_1S_3|a|^2}{\Gamma(\gamma_1^2+Y^2)} . \tag{B2}$$

From Eq. (5.8b) we get

$$S_3=\frac{\gamma_3\bar{S}_3}{\gamma_2+4\Lambda_1^2\gamma_1|a|^4/(\gamma_1^2+Y^2)} , \tag{B3}$$

and combining Eqs. (B2) and (B3),

$$1=\frac{2\Lambda_1^2\gamma_1\gamma_2\bar{S}_3|a|^2}{\Gamma[\gamma_2(\gamma_1^2+Y^2)+4\Lambda_1^2\gamma_1|a|^4]} . \tag{B4}$$

From Eqs. (B4) and (B3) we obtain

$$S_3=\bar{S}_3-\frac{2\Gamma}{\gamma_2}|a|^2 . \tag{B5}$$

Equation (B5) allows us to express Δ_L and thereby Y as a linear function of $|a|^2$. We obtain then a quadratic equation for $|a|^2$ from Eq. (B2),

$$\tilde{\alpha}|a|^4+\tilde{\beta}|a|^2+\tilde{\gamma}=0 , \tag{B6}$$

where

$$\tilde{\alpha}=\left[\frac{2\Lambda_2\gamma_1(\gamma_2+2\Gamma)}{\gamma_2(\gamma_1+2\Gamma)}\right]^2+4\Lambda_1^2\frac{\gamma_1}{\gamma_2} , \tag{B7a}$$

$$\tilde{\beta}=2\frac{2\Lambda_2\gamma_1^2(\gamma_2+2\Gamma)(\Omega'+\Lambda_2-2\Delta_2-2\Lambda_2\bar{S}_3)}{\gamma_2(\gamma_1+2\Gamma)^2}-\frac{2\Lambda_1^2\gamma_1}{\Gamma}\bar{S}_3, \tag{B7b}$$

$$\tilde{\gamma}=\frac{\gamma_1^2(\Omega'+\Lambda_2-2\Delta_2-2\Lambda_2\bar{S}_3)^2}{(\gamma_1+2\Gamma)^2}+\gamma_1^2 . \tag{B7c}$$

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