Phase and amplitude correlations induced by the switch-on chirp of a detuned laser

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The correlations of the fluctuations of the electric field are studied for a simple model of the buildup from spontaneous-emission noise of the output of a single mode of a laser cavity. Results for when the laser cavity is detuned with respect to the center frequency of the gain medium are compared with the case in which the cavity is resonantly tuned. Resonantly tuned lasers display transient phase and amplitude correlations with a peak at the time that the evolving amplitude departs from the neighborhood of the origin in the complex plane of the amplitude where phase diffusion dominates. Deterministic frequency chirps during the switch-on of a detuned laser delay and significantly strengthen the transient correlation of phase and intensity fluctuations when the detuning is sufficiently large compared with the strength of the noise. In this case the peak in the correlations is related to the anomalous intensity fluctuations characteristic of transient switching.

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I. INTRODUCTION

Lasers and laser amplifiers have been studied for their intensity statistics for many reasons, as examples of Bose-Einstein sources [1], for the transition from thermal to coherent statistics as a laser is brought above threshold [2–4], for the transformation of the statistical fluctuations of input signals [5,6], for the quantum limits to their operation [7–9], and for the effects of gain saturation [10–12]. While measures of the intensity fluctuations or photon statistics have been relatively easy to check experimentally, there have been fewer theoretical treatments or experimental tests of amplitude correlation functions.

In this analysis we explore the evolution of phase and amplitude correlation functions for fluctuations in a single-mode laser during the transient after it is switched on. The intensity statistics of such transient switches were among the earliest of quantum-statistics problems studied after the advent of the laser [2] and they have been studied extensively more recently as indicators of the initial state of the field inside the laser cavity [13–19] or of the strength of injected fields [20–22].

We extend previous work by considering the field correlation functions and by analyzing the additional features that appear in the case of lasers detuned from resonance. For these initial studies we consider the simplest possible model which captures the basic phenomenon of a frequency chirp during the transient. The model involves only a single equation for the complex field (appropriate to what have been called class-A lasers in a classification of lasers by their dynamical properties [23]) and we take only the lowest-order approximation of the gain saturation, which is often called third-order Lamb theory. (Some limits of this approximation as an accurate representation of more exact models have been discussed recently by Christian and Mandel [24].) In this model the detuning causes no modification of the evolution of the intensity statistics, but there can be a significant change in the correlation of the phase and amplitude evolution during the transient. We attribute this to a deterministic phase evolution that dominates the phase of a detuned laser during the initial transient. The deterministic phase evolution arises from the shift in the optical frequency of the laser as the intensity turns on, and it effectively suppresses the importance of phase diffusion once the initial transient has grown to an intensity larger than that of the spontaneous-emission noise.

To understand the origin of the physics incorporated in our simple model of a detuned single-mode laser, consider a generic two-level medium described by the Maxwell-Bloch equations [25]

$$\frac{dE}{dt}=-\kappa E(t)-\kappa AP(t)+i\delta_{AC}E(t)\ , \tag{1a}$$

$$\frac{dP}{dt}=-\gamma_{\perp}[P(t)+E(t)D(t)]+s(t)\ , \tag{1b}$$

$$\frac{dD}{dt}=-\gamma_{\parallel}[D(t)-1]+(\gamma_{\parallel}/2)[P^*(t)E(t)+E^*(t)P(t)]\ . \tag{1c}$$

E, P, and s are complex quantities and all other variables and parameters are real. E and P are slowly varying amplitudes of the electromagnetic field and the atomic polarization, respectively, written in the rotating frame of reference of the atomic resonance frequency, and D is the population difference. The detuning of the laser cavity from the atomic resonance is given by $-\delta_{AC}$.

The intrinsic noise in a macroscopic system of atoms and photons can be well described by semiclassical equations with the fluctuations due to spontaneous emission appearing as a δ-correlated Langevin noise term in the polarization $\langle s(t)s^*(t')\rangle = \Gamma\delta(t-t')$. A recent careful review and elegant demonstration of this is given by Carmichael [26].

The noise-free steady-state solutions for a detuned laser have several anomalous results. For instance, for the deterministic (noise-free) equations, while the frequency of the lasing solution for fixed cavity detuning is constant and independent of the degree of excitation of the laser, the frequency of the nonlasing state is not defined. While it is convenient in the case of a resonantly tuned laser to say that the trivial and nontrivial solutions both share a common frequency (that of resonant tuning), there is no clear argument to make for the detuned case. Imagine turning the laser on by incrementally increasing the gain [27]; does the laser power first emerge at the detuned frequency of the steady-state lasing solution only after the threshold for laser action has been crossed? From steady-state analysis of the noise-free system, the answer is that there is no frequency (because there is no field) until the nonzero-intensity state begins, and the nonzero-intensity state always has a detuned frequency. But if one abruptly switches on a laser from an initial off state, is there something approximating an instantaneous frequency of the laser output which changes during the transient?

Clearly the only resolution of this problem lies in the inclusion of either nonzero initial conditions for the laser or a steady Langevin noise source. Such additions are also necessary if the real intrinsic noise of the laser is to be modeled and if the system is ever to leave the trivial deterministic solution when the laser is above threshold. One approach is that of studying the spectral properties of stochastic solutions for a sequence of steady-state operating conditions. Recent analyses by Chyba and Abraham [28] (and experiments by Kikuchi [29] on semiconductor lasers) have shown that the spectrum of a laser has a well defined peak(s) even for excitation levels below the threshold for laser action. As the excitation is increased, the peak in the electric-field spectrum moves from the value(s) for the "off" state toward the value for the steady-state detuned laser solution.

This leads to the inference that the instantaneous frequency of a laser would make a similar shift during a transient when the laser is abruptly switched on. Deterministic frequency evolution can be followed in noise free equations and such a frequency-dependent evolution is called a "chirp" when there is a linear (or at least monotonic) sweep in the frequency during the pulse. The transient switch in the frequency may also have consequences for the intensity, or it may be that the frequency simply follows the intensity. In either case it is reasonable to expect that there will be transient correlations in the phase and intensity fluctuations. However, in the presence of noise there is the further complication: namely, that an instantaneous frequency cannot be well defined [30]. Determining what quantities are measurable that might reflect some of the physical evolution of the frequency is a primary motivation of this work.

In particular, we wish to inquire whether results of recent work on switching transients for resonantly tuned lasers [14–16] would change if the laser is detuned from resonance. This should also be of interest for similar switch-on transients in semiconductor lasers [17] where the linewidth enhancement factor (Henry's α factor) provides an intrinsic detuning and thereby a well-known coupling between amplitude and phase evolution and fluctuations. Moreover, the transient evolution of the phase of a laser in individual switch-on events may be measurable for an independent test of our results. Recent heterodyne measurements of Weiss *et al.* for far-infrared lasers [31] demonstrate that something approximating the "instantaneous frequency" (thus the phase evolution) of a laser can be measured during a single transient and single "phase-slip" measurements by optical frequency standards groups [32] indicate that the general technology required for these measurements is available for a wide range of lasers.

II. MODEL

Our model can be constructed from the Maxwell-Bloch equations for a single-mode field interacting with a two-level medium by adiabatic elimination of the amplitudes of the material variables, while keeping careful note of the phase of the material polarization. The resulting differential equation for the complex electric-field amplitude with a third-order approximation of the gain saturation is [33]

$$\frac{dE}{dt} = \alpha_1(1+i\Theta)E(t) - \beta(1+i\Theta)|E(t)|^2E(t) + \xi(t) , \quad \text{(2a)}$$

where

$$\langle \xi(t)\xi^*(t')\rangle = \epsilon\delta(t-t') . \quad \text{(2b)}$$

Here we can consider the electric-field amplitude to be in dimensionless units with the saturation intensity fixed by β^{-1}. The time scale can be conveniently renormalized to the unsaturated gain α_1. The degree of laser excitation is governed by α_1 and the threshold is given by $\alpha_1 = 0$. The equations are written for the slowly varying amplitude of the field in a rotating reference frame which gives a constant amplitude for the steady-state solution with nonzero intensity. The detuning of the laser cavity from the atomic resonance is given by $\alpha_1\Theta$, as indicated by the fact that the trivial (zero-intensity) solution has a frequency of $\alpha_1\Theta$, while the nontrivial solution has a frequency of zero.

We have taken only the third-order approximation of the saturation and in the adiabatic elimination of the polarization we have transferred the noise from the polarization to the amplitude equation. While the quantum mechanics of spontaneous emission dictates that the polarization noise has a white spectrum, if one observes the resulting noise in the field amplitude it has a finite bandwidth given by the relaxation rate of the polarization. The adiabatic elimination process which converts a coherent model for field and atomic variables to an equation only for the field variables transfers the polarization noise to a field noise and filters it. Hence for this equation one must construct a colored-noise process $\xi(t)$, which has a characteristic bandwidth equal to the polarization relaxation rate. However, since the validity of such an adiabatic elimination relies on the polarization decay rate being much larger than the field decay rate, we find that $\xi(t)$ differs very little from white noise in terms of its effect on the field.

If we consider switching the parameter α_1 from $\alpha_0<0$ to α, we will need to take for initial conditions values which are solutions of the steady-state process given by

$$\frac{dE}{dt}=(\alpha_0+i\alpha\Theta)E+\xi(t)\ , \tag{3}$$

where the frequency shift results from choosing the same rotating reference frame for the complex field before switch-on as after the switch-on.

For the evolution after the switch-on process, we can separate Eq. (2) for the complex field amplitude into equations for the intensity and phase by defining

$$E=\mathrm{R}e^{i\phi}\ ,\quad I=R^2\ ,$$

with R and ϕ real functions of time. Then the process given by Eq. (2) becomes (using Ito calculus [34])

$$\frac{dR}{dt}=\alpha R-\beta R^3+\epsilon/2R+\epsilon^{1/2}\xi_R(t)\ , \tag{4a}$$

$$\frac{d\phi}{dt}=\alpha\Theta-\beta\Theta I+I^{-1/2}\epsilon^{1/2}\xi_\phi(t)\ , \tag{4b}$$

where the noise processes ξ have zero means and correlation functions given by

$$\langle\xi_R(t)\xi_R(t')\rangle=\delta(t-t')\ , \tag{5a}$$

$$\langle\xi_\phi(t)\xi_\phi(t')\rangle=\delta(t-t')\ , \tag{5b}$$

$$\langle\xi_\phi(t)\xi_R(t')\rangle=0\ . \tag{5c}$$

In the noise-free case ($\epsilon=0$), we recover, for reference, the two steady-state solutions

$$I=\alpha/\beta\ , \tag{6a}$$

$$\frac{d\phi}{dt}=\alpha\Theta-\beta\Theta I=0 \tag{6b}$$

and

$$I=0\ , \tag{7a}$$

$$\frac{d\phi}{dt}=\alpha\Theta\ . \tag{7b}$$

The first solution exists and is the only stable one for $\alpha>0$, while the second is stable and is the only solution for $\alpha<0$. In the presence of noise, the bifurcation of the nontrivial branch of solutions is no longer sharply defined as occurring at $\alpha=0$ [35].

For example, for $R\ll1$ and $\alpha\ll0$, we have for steady-state solutions that

$$\left\langle\frac{dI}{dt}\right\rangle=2\alpha I+\epsilon=0 \tag{8}$$

with the result that the intensity is given by

$$\langle I\rangle=\epsilon/2|\alpha|\ . \tag{9}$$

For α negative yet close to zero we must already include the nonlinear correction which gives the result that

$$2\alpha\langle I\rangle-2\beta\langle I^2\rangle+\epsilon=0\ . \tag{10}$$

Then, more exactly, since for such nearly Gaussian amplitude processes

$$\langle I^2\rangle\simeq2\langle I\rangle^2\ ,$$

we have that

$$\langle I\rangle\simeq[\alpha-(\alpha^2+4\beta\epsilon)^{1/2}]/4\beta\ , \tag{11a}$$

in which case

$$\left\langle\frac{d\phi}{dt}\right\rangle=\alpha\Theta-\beta\Theta\langle I\rangle\ . \tag{11b}$$

Both of these revised Eqs. (11) are smoothly varying functions of the parameter α and provide reasonable approximations of the average values as α is increased (though still negative) until the average intensity in Eq. (11a) comes within an order of magnitude of $(\epsilon/\beta)^{1/2}$ (the value at $\alpha=0$).

We return then to the question of whether transient evolution of the frequency and of the intensity and phase statistics follows a simple relation similar to the steady-state relation between the average amplitude and the derivative of the average phase, for example, with the average "instantaneous frequency" [as defined by Eq. (11b)] during a transient simply tracking the average intensity as given by Eq. (7a). Clearly an alternative is that the instantaneous frequency tracks the instantaneous intensity, while the averages significantly differ from the individual transients. The correlations that result from whatever relationships are meaningful are thus enhanced in this region of transient evolution and it is particularly likely that the fluctuations in the switching times of the intensity transients will have a significant impact on the statistical properties of the phase and on its correlation with the intensity.

We start with an initial condition given by a steady-state solution of Eq. (3) where the excitation parameter is $\alpha_0<0$. Then for $t>0$ we take the system to evolve according to Eq. (2a) with parameter α ($\alpha>0$).

For the theory of the transient switch-on in the presence of noise, the quasideterministic theory (QDT) [13,36] for the field E gives

$$E_{\mathrm{QDT}}(t)=H(t)e^{\alpha(1+i\Theta)t}/\{[1+(\beta/\alpha)|H(t)|^2\times(e^{2\alpha t}-1)]^{(1+i\Theta)/2}\}\ , \tag{12}$$

which is the approximation of deterministic amplification of random initial conditions

$$H(t)=\int_0^t dt' e^{-\alpha(1+i\Theta)t'}\xi(t')+E(0) \ , \tag{13a}$$

$$\langle H(t)H^*(t)\rangle=\epsilon(1-e^{-2\alpha t})/2\alpha+\epsilon/2|\alpha_0| \ , \tag{13b}$$

where

$$\langle |E(0)|^2\rangle=\epsilon/2|\alpha_0| \ . \tag{13c}$$

For $t \gg 1/2\alpha$, $H(t)$ is replaced by a time-independent complex Gaussian variable H, whose variance is

$$\langle |H|^2\rangle=\epsilon/2\alpha+\epsilon/2|\alpha_0| \ . \tag{14}$$

The QDT approximation in Eq. (12) neglects fluctuations in the final state. Therefore it neglects final phase diffusion. We proceed with this approximation for the intensity (assuming no final intensity fluctuations) while keeping the exact Eq. (4b) for the phase in order to see the phase diffusion. Then for the intensity

$$\begin{aligned}I(t)\simeq I_{\rm QDT}(t)&=|E_{\rm QDT}(t)|^2\\&=|H(t)|^2e^{2\alpha t}/[1+(\beta/\alpha)|H(t)|^2\\&\qquad\times(e^{2\alpha t}-1)] \ ,\end{aligned} \tag{15}$$

which is independent of detuning, while for the phase

$$\begin{aligned}\phi(t)=\phi(0)+\alpha\Theta t-\beta\Theta\int_0^t dt' I_{\rm QDT}(t')\\+\sqrt{\epsilon}\int_0^t dt'[I_{\rm QDT}(t')]^{-1/2}\xi_\phi(t') \ .\end{aligned} \tag{16}$$

Equations (15) and (16) form the basic equations of our approximation. We see that the intensity evolution under these assumptions is independent of the phase evolution, while the phase evolution depends on deterministic evolution, on the stochastic quantity depending on the evolution of the intensity, and on a phase diffusion process.

III. PHASE FLUCTUATIONS AND PHASE-AMPLITUDE CORRELATIONS

There are two averages that might be taken in Eq. (16): one with respect to the phase noise ξ_ϕ, $\langle\ \rangle_{\xi_\phi}$, and the other with respect to the ensemble of the $I_{\rm QDT}$ realizations.

A. Averages over the phase noise

We can compute an "average instantaneous frequency" $\Omega(t)$ defined as follows:

$$\Omega(t)\equiv\left\langle\frac{d\phi}{dt}\right\rangle_{\xi_\phi}=\alpha\Theta-\beta\Theta I_{\rm QDT}(t) \ . \tag{17}$$

Thus the first three terms of Eq. (16) give a frequency $(d\phi/dt)$ which changes from $\alpha\Theta$ (if the initial intensity is negligible) to zero as the intensity rises to its final value, since $I_{\rm QDT}(\infty)=\alpha/\beta$. Subtracting this deterministic frequency evolution for a given intensity trajectory, we define

$$\Delta\phi=\phi(t)-\phi(0)-\int_0^t\Omega(t')dt' \ . \tag{18}$$

Then the phase fluctuations along the trajectory are given by

$$\langle(\Delta\phi)^2\rangle_{\xi_\phi}=\epsilon\int_0^t ds[I_{\rm QDT}(s)]^{-1} \tag{19}$$

and are independent of the detuning. Note that $\Omega(t)$ in Eq. (17) is still a stochastic quantity during the transient.

B. Ensemble average

If we perform an ensemble average we obtain divergences which indicate that the phase itself is meaningless in the ensemble average:

$$\begin{aligned}\langle(\Delta\phi)^2\rangle_{\xi_\phi}&>I_{\rm QDT}\\&=\epsilon\int_0^t ds\langle I_{\rm QDT}(s)\rangle^{-1}\\&=\epsilon\int_0^t ds\int\int dH\,dH^*P(H)P(H^*)\\&\qquad\times[I_{\rm QDT}(s,H)]^{-1} \ ,\end{aligned} \tag{20}$$

where $\langle I_{\rm QDT}(s)\rangle$ is the average obtained from Eq. (16) after averaging over the Gaussian distribution of H, $P(H)$, with its standard deviation specified in Eq. (14).

C. Correlation functions

A different way of looking at transient phase fluctuations is to consider correlation functions which depend on the coupling of the amplitude and phase. Correlation functions for transients are functions of two times rather than one and averages are only over the ensemble since time averages cannot be meaningfully taken during the transient. For example, the electric-field correlation function is given by

$$\begin{aligned}\langle E(t)E^*(t')\rangle_{\xi_\phi}=[I_{\rm QDT}(t)I_{\rm QDT}(t')]^{1/2}\\\times\langle e^{i[\phi(t)-\phi(t')]}\rangle_{\xi_\phi} \ .\end{aligned} \tag{21}$$

From Eq. (16) we obtain

$$\langle e^{i[\phi(t)-\phi(t')]}\rangle_{\xi_\phi}=e^{i\theta\Psi(t,t')}e^{-(\epsilon/2)\chi(t,t')} \ , \tag{22}$$

with

$$\Psi(t,t')=\alpha(t-t')-\beta\int_{t'}^t ds\, I_{\rm QDT}(s) \tag{23}$$

and

$$\chi(t,t')=\int_{t'}^t ds[I_{\rm QDT}(s)]^{-1} \ , \tag{24}$$

where $\Psi(t,t')$ is a function of $I_{\rm QDT}$, which gives the frequency shift, and $\chi(t,t')$ is a function of $I_{\rm QDT}$, which for long time gives the phase diffusion in the final state. $\chi(t,t')$ gives a linewidth which displays both narrowing with amplification and further changes when the amplification saturates.

As t and $t'\to\infty$,

$$\chi(t,t')\to(\beta/\alpha)(t-t') \ , \tag{25}$$

while

$$\Psi(t,t')\to 0 \ . \tag{26}$$

We introduce a notation for the arguments of the transient correlation function $\langle E(t_1)E^*(t_2)\rangle$ as follows:

$$T=(t_1+t_2)/2 \ , \quad t=t_1-t_2 \ .$$

Before discussing the ensemble averages, several observations are in order.

From Eqs. (23) and (15) and when $\alpha T \gg 1$ while $\alpha t \ll 1$, we get

$$\Psi(T,t) \simeq \alpha t[1-(\beta/\alpha)I_{\rm QDT}(T)] \tag{27}$$

and

$$\chi(T,t) \simeq t/[I_{\rm QDT}(T)] \ , \tag{28}$$

up to orders $(t/T)^2$. Equations (27) and (28) for each realization of the intensity give an effective stochastic instantaneous frequency $\alpha\Theta[1-(\beta/\alpha)I_{\rm QDT}(T)]$ [see Eq. (18)] and an effective decay time (or inverse bandwidth) $\tau=(2/\epsilon)I_{\rm QDT}(T)$. The effective frequency and decay time depend on T through $I_{\rm QDT}$. There is then a natural scaling of the experimental results with the intensity in the case in which intensity and phase are measured simultaneously.

In the same limit $\alpha T \gg 1$ and $\alpha t \ll 1$, from Eq. (16) we obtain

$$[I_{\rm QDT}(t_1)I_{\rm QDT}(t_2)]^{1/2}=I_{\rm QDT}(T)+O((t/T)^2) \ , \tag{29}$$

in agreement with the numerical results discussed and displayed in the figures below. In this limit and for a given intensity realization

$$\langle E(t_1)E^*(t_2)\rangle_{\xi_\phi}=I_{\rm QDT}(T)e^{i[\alpha\Theta-\beta\Theta I_{\rm QDT}(T)]t} \times e^{\epsilon t/2I_{\rm QDT}(T)} \ . \tag{30}$$

The function

$$C(t_1,t_2)\equiv[\langle E(t_1)E^*(t_2)\rangle - \langle R(t_1)R(t_2)\rangle\langle e^{i[\phi(t_1)-\phi(t_2)]}\rangle] \ , \tag{31}$$

provides a measure of the degree to which the amplitude and phase fluctuations are correlated. In the steady state this function goes to zero in our approximation. In principle, the exact solutions of our model and those of even more exact laser models such as that of the Maxwell-Bloch equations or those of semiconductor lasers retain a degree of amplitude and phase correlations in the final state.

IV. NUMERICAL ANALYSIS

For reference purposes we show in Fig. 1 the average intensity $\langle I(t)\rangle$ and the variance of the intensity $\langle(\Delta I)^2\rangle \equiv \langle (I(t)-\langle I(t)\rangle)^2\rangle$ as functions of time. These are found by numerically calculating (by a Simpson rule integration routine) the ensemble average of the expression for the intensity in Eq. (15) and its variance. We assume that H is a time-independent Gaussian variable (valid for times larger than $1/\alpha$) whose variance is given by Eq. (14). All calculations for these and other figures in the paper were done with parameter values $\alpha=|\alpha_0|=1$, $\beta=0.1$. In this case $\epsilon=0.01$ was chosen for convenience, large enough to avoid excessively long transients yet small enough to make the QDT approximation valid. The intensity is taken to be dimensionless with a saturation value of 10 (β^{-1}) and time is conveniently scaled to $\alpha^{-1}=1$.

The peak in the variance is often referred to as evi-

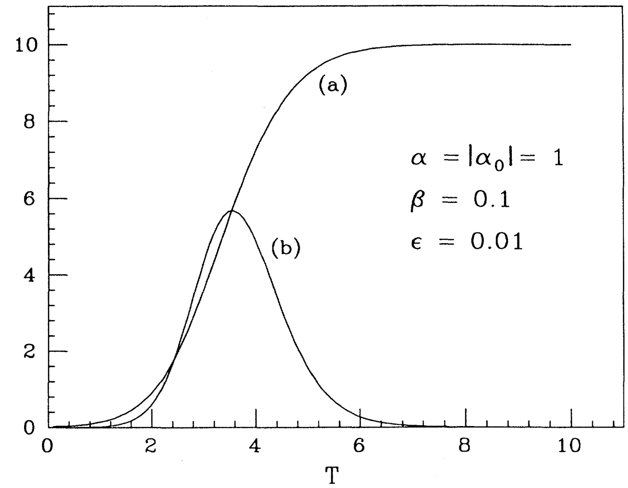

FIG. 1. (a) Mean intensity $\langle I(t)\rangle$ and (b) the variance of the intensity $\langle[\Delta I(t)]^2\rangle$ vs time, where these represent ensemble averages. The parameters $\alpha=|\alpha_0|=1$, $\beta=0.1$, and $\epsilon=0.01$, shown in the inset, have been used for all calculations in this paper. The intensity is taken to be dimensionless with a saturation value of 10 (β^{-1}) and time is conveniently scaled to $\alpha^{-1}=1$.

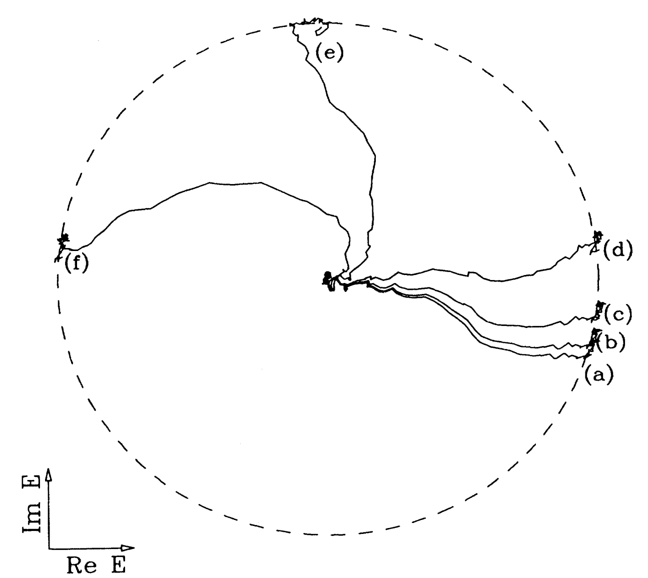

FIG. 2. Evolution of the complex electric field for several values of detuning: (a) $\Theta=0.0$, (b) $\Theta=10^{-2}$, (c) $\Theta=3.5\times10^{-2}$, (d) $\Theta=0.1$, (e) $\Theta=0.5$, (f) $\Theta=1$. The dotted circle indicates where $|E|=(10)^{1/2}$.

dence for "anomalous intensity fluctuations." It arises from the indeterminacy of the switching time which occurs because the signal originates from randomly fluctuating noise. In the overall evolution for a given noise level $\epsilon = 10^{-2}$, we see that the transient is divided into three regions. For small times (less than $T_1 \sim 2.0$) we have linear gain. T_1 can be identified with the mean first passage time, the time at which the intensity reaches a

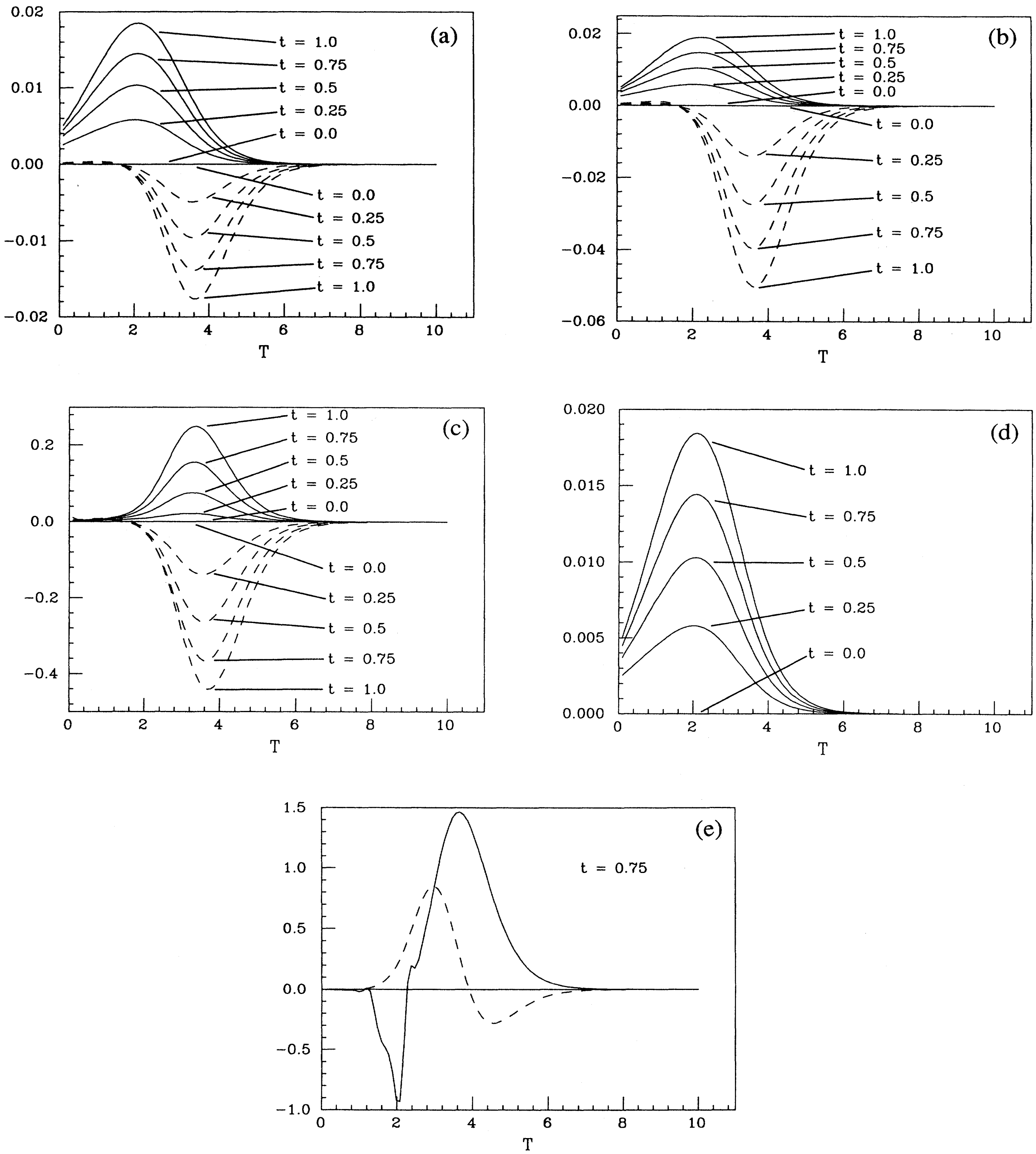

FIG. 3. ReC (solid line) and ImC (dashed line) vs T for different values of $t = 0.25$, 0.5, 0.75, and 1.0 as indicated. (a) $\Theta = 0.035$, (b) $\Theta = 0.1$, (c) $\Theta = 1.0$, (d) $\Theta = 0.0$ (note that in this case Im$C = 0$), (e) $\Theta = 5.0$.

macroscopic value. The second time interval $T_1 < T < T_2$ is bounded from above by $T_2 \sim 4.0$, which falls in the middle of the region where evolution is governed by nonlinear gain. Finally, for $T > 6.0$ the intensity in nearly all transients has come close to its final value.

Some of the features of individual trajectories and the effect of detuning on their evolution can be identified in Fig. 2 which shows single realizations of the trajectories obtained by numerically integrating the real and imaginary parts of Eq. (3) for different detunings using a first-order Euler method generalized for the stochastic equations. (More stable methods usually required for deterministic differential equations are not needed here because of the stabilizing influence of the noise.) The same sequences of random numbers for the noise were used for each of the transients. We plot the trajectories in the complex plane of the electric field. As a result we see trajectories which originate from the same historical record of noise fluctuations, permitting us to see clearly the effects of detuning. The initial evolution is phase diffusion in the vicinity of the origin where phase noise dominates any deterministic effects. For small detunings the evolution occurs with almost constant phase once the trajectory leaves the vicinity of $|E|=0$. In contrast, for larger detunings we see the trajectories spiral out away from the origin. When the detuning is large enough that the deterministic rotation of the phase exceeds the phase diffusion variations along a single transient, we have a large enough detuning to have a measurable deterministic correlation of phase and amplitude fluctuations. As the final value of the intensity is approached, the phase rotation slows and in the final state there is only the ordinary phase diffusion indicated by the spread along the circumference of a circle that represents the set of all possible "deterministic" equilibrium states [see Eq. (6a)].

Figures 3–9 contain different characteristics of the correlation function, Eq. (31), which we have calculated numerically. We recall that the average in Eq. (31) implies an average over phase noise, which is carried out analytically, and the ensemble average over the different QDT intensity trajectories. The field-field correlation function $\langle E(t_1)E^*(t_2)\rangle$ is calculated from Eq. (30) using the approximation for $I_{\rm QDT}$ from Eq. (15). The ensemble average in this case is carried out by numerically averaging (using a Simpson's rule integration procedure) over the Gaussian distribution of the variable H in (15). We suppose that both t_1 and t_2 are much greater than $1/2\alpha$ and that H is a time-independent Gaussian variable whose variance is given by Eq. (14). Here again it is convenient to have chosen a value of the noise that is not too small since the relatively short switch-on time prevents very large exponents from appearing in the integrals used to evaluate the averages in Eq. (31).

To evaluate the complex function C defined in Eq. (31) we find it useful to numerically calculate several other

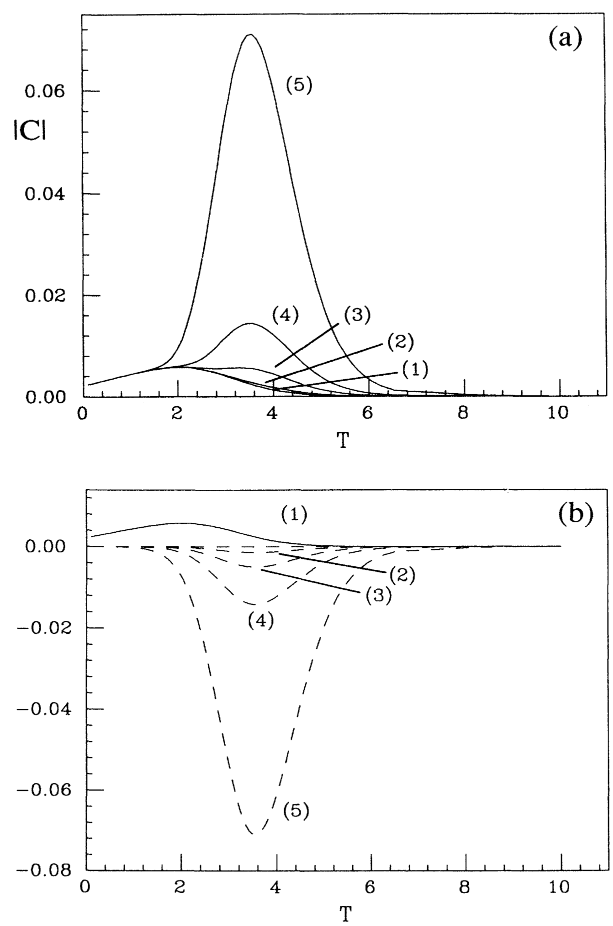

FIG. 4. (a) $|C|$ vs T for $t=0.25$ and for several values of the detuning as indicated: (1) $\Theta=0.0$, (2) $\Theta=10^{-2}$, (3) $\Theta=3.5\times10^{-2}$, (4) $\Theta=0.1$, (5) $\Theta=0.5$. (b) ReC and ImC vs T for $t=0.25$ for the detunings of (a).

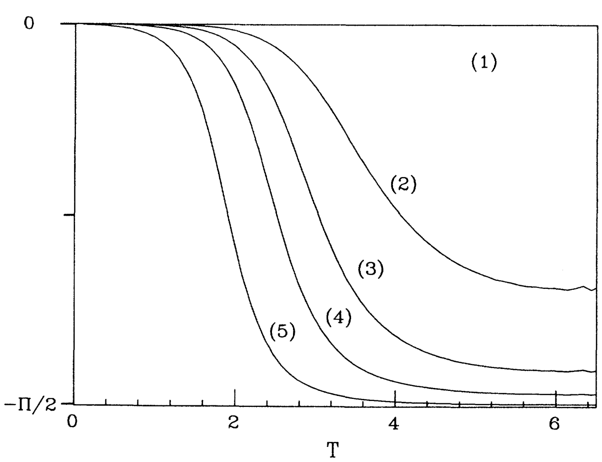

FIG. 5. Writing the complex correlation function $C=|C|e^{i\delta}$ we plot the phase δ vs T for $t=0.25$ and for the same values of Θ as in Figs. 4(a) and 4(b). Note that for curve (1) with zero detuning the phase remains at 0.

correlation functions. The amplitude-amplitude correlation function $\langle R(t_1)R(t_2)\rangle$ is also calculated from the ensemble average of the QDT approximation [Eq. (15)] for $I=R^2$. The phase correlation function $EXP \equiv \langle e^{i[\phi(t_1)-\phi(t_2)]}\rangle = \langle e^{i\Psi(t_1,t_2)} e^{-(\epsilon/2)\chi(t_1,t_2)}\rangle$ is again calculated by ensemble averaging using Eqs. (23) and (24) and the QDT approximation. Here $\langle\ \rangle$ means $<\langle\ \rangle_{\xi_\phi}> I_{\rm QDT}$.

Figure 3 shows various aspects of the correlation function C during the transient evolution of the intensity described in Fig. 1, but for different values of the detuning. Figures 3(a)–3(c) show the real and imaginary parts of C versus time for different values of the delay time t. As the detuning is increased, there is a delay in the peak of the real part of C and an enhancement of the magnitude of the imaginary part of C. Phase diffusion dominates in the first region and it is reflected in the peak of Re C.

Fluctuations of the instantaneous frequency in the ensemble due to anomalous intensity fluctuations dominate in the second region, giving rise to a peak of ImC because of the large coupling of intensity and phase fluctuations when there is sufficient detuning. For reference, Fig. 3(d) shows the function C for zero detuning which is real. With larger values of the detuning there are oscillations in the real and imaginary parts of C during the switch-on. A sample is shown in Fig. 3(e).

Since some of the oscillations in C are due to deter-

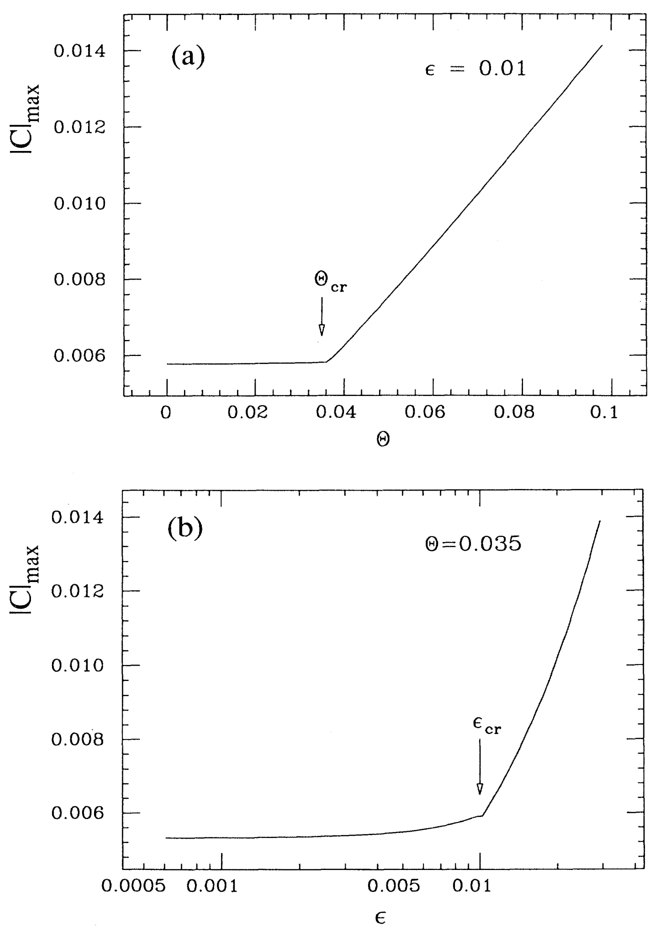

FIG. 6. (a) Maximum value of $|C|$ ($|C|_{\rm max}$) vs Θ for $\epsilon=0.01$. (b) Maximum value of $|C|$ vs ϵ for $\Theta=0.035$.

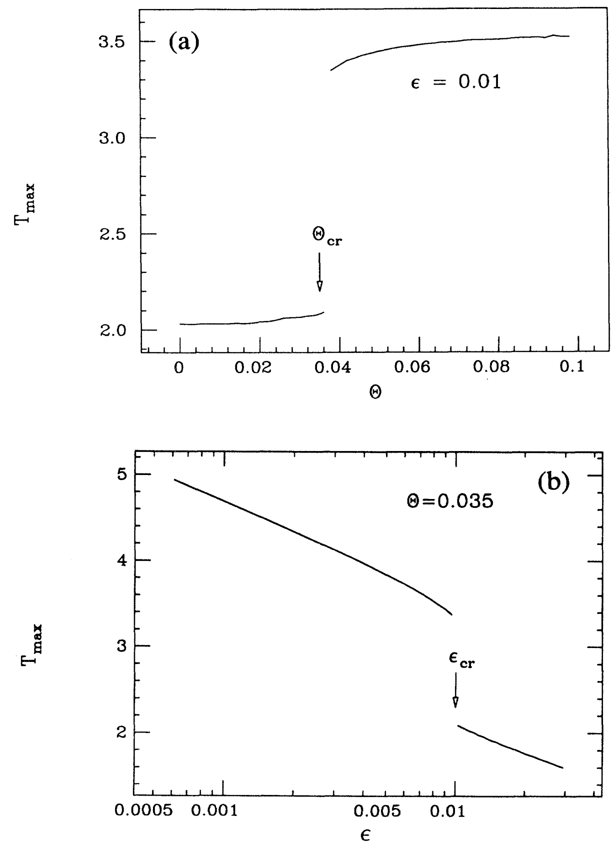

FIG. 7. (a) Time at which $|C|$ has its maximum ($T_{\rm max}$) plotted vs detuning Θ for $\epsilon=0.01$. (b) Time at which $|C|$ has its maximum plotted vs noise intensity ϵ for $\Theta=0.035$.

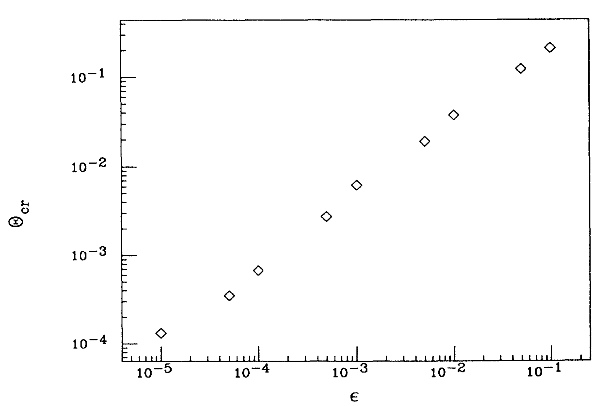

FIG. 8. Dependence of the critical value of Θ on the noise intensity ϵ.

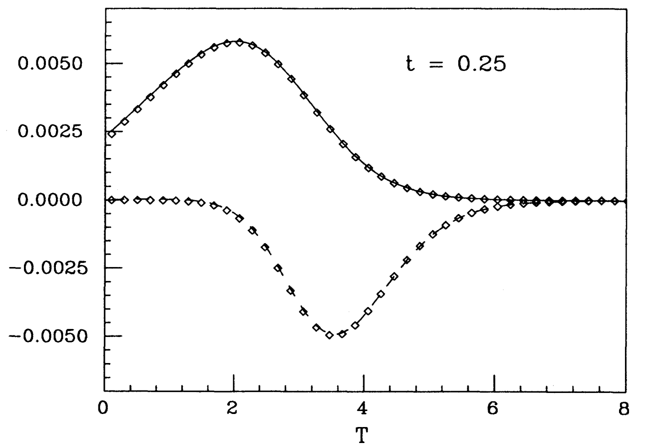

FIG. 9. ReC and ImC as obtained by the small t approximation [Eqs. (A18) and (A8b)] (diamonds) compared with results from numerical evaluation of the averages in Eq. (A1) (solid line). Parameters are the same as those of Fig. 4(a).

ministic phase rotation during the transient, it is useful to look at the modulus of C. This is plotted in Fig. 4(a) for different detunings using $t=0.25$ since the time T at which the peaks occur is relatively independent of t. For reference, the results of Fig. 3 for $t=0.25$ are summarized in Fig. 4(b). We see that when the detuning is zero, there is a single peak at about $T=2.1$ which is approximately the mean first passage time (the time when the intensity becomes appreciably larger than the spontaneous-emission noise). This indicates that for zero detuning the strongest amplitude and phase correlations occur just as the trajectory leaves the phase diffusion region.

For larger detunings a second peak emerges at $T=3.8$, which is at the time of the peak of the anomalous intensity fluctuations. This peak reflects the rapid phase rotation induced by the deterministic switch-on of the phase. The strong correlation of the phase evolution with the intensity evolution gives fluctuations in the phase in correspondence with intensity fluctuations arising from different switching times.

The phase of C for the parameters considered in Fig. 4 is shown in Fig. 5. As the detuning is increased, the phase shifts from zero (the value at zero detuning) to a value which saturates at $-\pi/2$ for large detunings since in the intermediate region the imaginary part is much larger than the real part.

The three time regions identified earlier are evidently relevant to the evolution of C. Phase diffusion dominates in the first region and it is reflected in the peak of ReC. Fluctuations of the instantaneous frequency in the ensemble due to anomalous intensity fluctuations dominate in the second region, giving rise to a peak of ImC because of the large coupling of intensity and phase fluctuations when there is sufficient detuning. The phase of C (Fig. 5) measures the relative importance of the types of fluctuations which separately govern ReC and ImC. We see a smooth transition in time from the dominance of phase diffusion (where the phase of C is zero) to the dominance of anomalous intensity fluctuations (where the phase of C is $-\pi/2$).

It is instructive to study the correlation function in the same limit that we have considered in obtaining Eq. (29) (i.e., $\alpha T \gg 1$ and $\alpha t \ll 1$ [expanding $C(T,t)$ up to terms of the form $(t/T)^2\ln(t/T)$]). We have devoted the Appendix to this calculation and we shall quote here only the results. The real and the imaginary parts of the correlation function defined in Eq. (31) are given by

$$\mathrm{Re}C(T,t)=\langle I(T)e^{-\epsilon t/2I(T)}\rangle - \langle I(T)\rangle\langle e^{-\epsilon t/2I(T)}\rangle \ , \tag{32a}$$

$$\mathrm{Im}C(T,t)=-i\Theta\beta t[\langle I(T)^2\rangle-\langle I(T)\rangle^2] \ , \tag{32b}$$

where $\langle I\rangle$ and $\langle I^2\rangle$ are given in the QDT approximation by special functions [see Eqs. (A16) and (A17)].

This approximation reveals why the real part of C has its peak at the mean first passage time and the imaginary part of C has its peak at the time of the peak in the anomalous intensity fluctuations. It also explains why the imaginary part dominates for large Θ.

A useful way to summarize these results is to plot the characteristics of the peak value of the norm of C for different levels of noise. Figures 6 and 7 give the value of the norm of C at its maximum ($C_{\max}$) and the time at which the maximum occurs ($T_{\max}$), respectively. In Figs. 6(a) and 7(a) these are plotted for a particular value of the noise strength $\epsilon=10^{-2}$, while in Figs. 6(b) and 7(b) the detuning is fixed ($\Theta=3.75\times10^{-2}$). In both of these cases we see that there are relatively minor changes of both quantities as the noise strength is varied for small noise strengths, but above a critical value (ϵ_{cr}) there is an abrupt shift in the time at which the maximum occurs and a corresponding increase in the rate of growth of $C_{\max}$ with noise strength. For Figs. 6(b) and 7(b) these are plotted versus ϵ for fixed Θ to find ϵ_{cr}.

To interpret Fig. 6(b), note that the lower branch of the values of $T_{\max}$ corresponds to the first passage times and the upper branch corresponds to the time of the peak of the anomalous intensity fluctuations. Both of these times vary as $1/\ln\epsilon$, with the passage time always preceeding the time of the peak in the anomalous intensity fluctuations. For weak noise, the peak is at the same time as the peak in the anomalous intensity fluctuations. This indicates that the noise is weak enough for the deterministic phase and frequency evolution during the transient switch-on to be unmasked. For larger noise, the noise-induced phase diffusion masks the deterministic effects and the evolution is the same as for zero detuning. Hence the important characteristic determining whether the system differs significantly from the zero-detuning case is whether the mean-square phase diffusion rate $\epsilon/2I(T)$ is small in comparison with the square of the "deterministic" frequency $\alpha\Theta-\beta\Theta I(T)$.

It is useful to plot the critical value of the detuning at which this transition occurs as a function of the noise. Nearly linear scaling is shown in Fig. 8. For a fixed detuning, the critical value of ϵ separates two situations. for $\epsilon > \epsilon_{cr}$, the peak in the norm of C for $t=0.25$ is governed by noise (phase diffusion), while for $\epsilon < \epsilon_{cr}$ the peak is governed by the anomalous intensity fluctuations.

V. SUMMARY

We have developed here an extension of the QDT formalism to study phase fluctuations in the presence of detuning in the transient statistics of the laser after switch-on. The theory predicts a rather sharp crossover phenomenon in the modulus-phase cross correlation function. As experiments have recently used heterodyne techniques to measure the instantaneous phase of a laser signal [31], it should be possible to measure these effects and to study their relevance for transient laser processes. This may give rise to an important form of self-phase modulation in the generation of Q-switched pulses in detuned lasers.

When the deterministic effects are important there would be measurable chirping and detectable deterministic phase motion during the transient. Rapid variations of the intensity from large values to zero and back to large values that were presumed to involve shifts of the phase by π (sign changes of the field) would have additional phase shifts associated with the deterministic evolution during the transient.

These additional variations in the phase and frequency evolution during transient evolution of the laser also will give additional contributions to the linewidth during transients. The deterministic frequency has the same statistical properties as the intensity in our model. Since the range of frequency evolution (chirp) is tied to the intensity transient, we will have an enhancement of the intrinsic linewidth due to fluctuations of the chirp.

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APPENDIX

In this appendix we show how to calculate analytically an approximation for the correlation function defined in Eq. (31). The equation can be rewritten in terms of T and t. Introducing the intensity $I=R^2$,

$$C(T,t) \equiv \langle E(T+t)E^*(T)\rangle - \langle I^{1/2}(T+t)I^{1/2}(T)\rangle\langle e^{i[\phi(T+t)-\phi(T)]}\rangle \; . \tag{A1}$$

We want to obtain an expression for Eq. (A1) when $\alpha T \gg 1$, while $\alpha t \ll 1$ (i.e., keeping only the first order in t). Using Eqs. (21)–(24) we perform first of all the averages over the phase noise, which yields an expression containing only the intensity process $I_{\rm QDT}$ given by Eq. (15) (in the following we will drop the index QDT). Then using Eqs. (27)–(29) together with Eq. (14) we get

$$C(T,t) = \langle I(T)e^{i\Theta[\alpha-\beta I(T)]t-\epsilon t/2I(T)}\rangle - \langle I(T)\rangle\langle e^{i\Theta[\alpha-\beta I(T)]t-\epsilon t/2I(T)}\rangle \; , \tag{A2}$$

in which $\langle\;\rangle$ means

$$\langle\;\rangle = \int dx\, e^{-x}\cdots \; , \tag{A3}$$

and we have normalized the integration variable (intensity noise) as follows:

$$I(T)=2u(T)x/[1+2v(T)x] \; , \tag{A4}$$

where

$$u(T)=\exp(2\alpha T\sigma^2) \; , \tag{A5}$$

$$v(T)=\beta(u-\sigma^2)/\alpha \; , \tag{A6}$$

$$\sigma^2=\langle |H|^2\rangle = \epsilon/2\alpha+\epsilon/2\alpha_0 \; . \tag{A7}$$

Since the function $e^{-\epsilon t/2I(T)}$ is not analytic in the neighborhood of $I=0$, we cannot interchange the averages and the series expansion in t/T. Thus let us leave the exponentials of this kind as they are and perform the series expansion on the analytic part of C inside the integrals:

$${\rm Re}C(T,t)=\langle I(T)e^{-\epsilon t/2I(T)}\rangle - \langle I(T)\rangle\langle e^{-\epsilon t/2I(T)}\rangle \; , \tag{A8a}$$

$${\rm Im}C(T,t)=-i\Theta\beta t[\langle I(T)^2\rangle - \langle I(T)\rangle^2] \; . \tag{A8b}$$

These formulas show us how the real part of C is related to the ordinary (detuning independent) phase diffusion. The imaginary part depends linearly on the detuning. And it is, to first order in t, also proportional to the intensity fluctuations.

Now we have to perform the integrals appearing in Eqs. (A8) and then expand the result in series of t. Using Eq. (A4) the first term in Eq. (A8a) may be rewritten as

$$\langle I(T)\exp[-\epsilon t/2I(T)]\rangle = \exp[-\epsilon v(T)t/2u(T)] \times \int_0^\infty dx\,\exp\left[-x-\frac{\epsilon t}{4u(T)x}\right]\left[\frac{2u(T)x}{1+2v(T)x}\right] . \tag{A9}$$

This integral can be separated into two pieces in order to evaluate correctly the contributions around $x=0$ and $x\to\infty$:

$$\int_0^\infty dx\,\exp(-x-\epsilon t/4ux)\left[\frac{2ux}{1+2vx}\right] = 2u\int_0^{1/2v} dx\,x\,\exp\left[-x-\frac{\epsilon t}{4ux}\right]\sum_{k=1}^{\infty}(-2vx)^k + \int_{1/2v}^{\infty} dx\,e^{-x}\left[\frac{2u(T)x}{1+2v(T)x}\right]\left[1-\frac{\epsilon t}{4ux}\right] . \tag{A10}$$

It is possible to expand the integrand in the second term in t. After some rearrangements we end up with

$$\int_0^\infty dx \exp(-x-\epsilon t/4ux)\left[\frac{2ux}{1+2vx}\right]$$
$$=2u\left[\int_0^\infty dx\, x \exp\left[-x-\frac{\epsilon t}{4ux}\right]\sum_{k=1}^{\infty}(-2vx)^k-\int_{1/2v}^{\infty}dx\left[\sum_{k=1}^{\infty}(-2vx)^k\right]\left[x-\frac{\epsilon t}{4u}\right]\right]$$
$$+\int_{1/2v}^{\infty}dx\, e^{-x}\left[\frac{2u(T)x}{1+2v(T)x}\right]\left[1-\frac{\epsilon t}{4ux}\right]. \qquad (A11)$$

We can also rewrite Eq. (A11) in a more convenient form:

$$\int_0^\infty dx \exp(-x-\epsilon t/4ux)\left[\frac{2ux}{1+2vx}\right]=2u\left\{\left[\int_0^\infty dx\, x\exp\left[-x-\frac{\epsilon t}{4ux}\right]\sum_{k=1}^{\infty}(-2vx)^k\right]-(k+1)!+\left[\frac{\epsilon t}{4u}k!\right]\right\}$$
$$+\int_0^\infty dx\, e^{-x}\left[\frac{2u(T)x}{1+2v(T)x}\right]\left[1-\left[\frac{\epsilon t}{4ux}\right]\right]. \qquad (A12)$$

The results for the integrals appearing in Eq. (A12) may be found [37]:

$$\int_0^\infty dx \exp\left[-\left[x+\frac{\epsilon t}{2ux}\right]\right]\sum_{k=1}^{\infty}x^{k+1}$$
$$=2\left[\frac{\epsilon t}{4u}\right]^{k/2+1}K_{k+2}\left[2\left[\frac{\epsilon t}{4u}\right]^{1/2}\right], \qquad (A13)$$

$$\int_0^\infty dx\, e^{-x}\left[\frac{2u(T)x}{1+2v(T)x}\right]\left[1-\frac{\epsilon t}{4ux}\right]$$
$$=\frac{u}{v}-\left[\frac{u}{v}+\frac{\epsilon t}{2}\right]\frac{\exp(1/2v)}{2v}E_1\left[\frac{1}{2v}\right], \qquad (A14)$$

where K_n is a modified Bessel function of order n and E_1 is the exponential integral. The series appearing in Eq. (A12) seems to be rather formal, but expanding the Bessel function we get a cancellation of divergent terms. Up to terms of order $t^2\ln(t)$:

$$\left\langle I(T)\exp\left[-\frac{\epsilon t}{2I(T)}\right]\right\rangle$$
$$\cong\left\{\frac{u}{v}\left[1-\frac{\exp(1/2v)}{2v}E_1\left[\frac{1}{2v}\right]\right]-\frac{\epsilon t}{2}\right\}. \qquad (A15)$$

Other terms in Eqs. (A8a) and (A8b) are given by special functions [37,38]:

$$\langle I(T)\rangle=\left\{\frac{u}{v}\left[1-\frac{\exp(1/2v)}{2v}E_1\left[\frac{1}{2v}\right]\right]\right\}, \qquad (A16)$$

$$\langle I(T)^2\rangle=2\left[\frac{u}{2v^2}\right]^2\exp(1/4v)W_{-2,1/2}(1/2v), \qquad (A17)$$

$$\left\langle\exp\left[-\frac{\epsilon t}{2I(T)}\right]\right\rangle$$
$$=\exp\left[-\frac{\epsilon v(T)t}{2u(T)}\right]2\left[\frac{\epsilon t}{4u(T)}\right]^{1/2}K_1\left[2\left[\frac{\epsilon t}{4u(T)}\right]^{1/2}\right], \qquad (A18)$$

where $W_{-2,1/2}$ is the Whittaker function and K_1 is the modified Bessel function.

Finally using the expansion for the Bessel functions it is possible to give an approximate formula for ReC up to order $t^2\ln(t)$:

$$\mathrm{Re}C\cong\frac{\epsilon t}{4u}\left\{\langle I\rangle\left[\ln\left[\frac{4u}{\epsilon t}\right]-2v-0.154\right]-2u\right\}. \qquad (A19)$$

For ImC we have used Eq. (A8b) together with Eqs. (A16) and (A17). These results are compared with direct computer evaluation of Eq. (A1) in Fig. 9.

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