Phase distribution of a quantum state without using phase states

W. Vogel
Institut für Physik, Pädagogische Hochschule, Güstrow, O-2600, Germany

W. Schleich
Max-Planck-Institut für Quantenoptik, Garching, W-8046, Germany
and Abteilung für Theoretische Physik III, Universität Ulm, Oberer Eselsberg, W-7900 Ulm, Germany

(Received 24 May 1991)

We use the field-strength eigenstates, that is, the quadrature eigenstates rotated by an angle *ccphi*, to define a phase distribution of a single mode of the radiation field. A measurement procedure lies at the heart of this operational phase distribution: A balanced homodyne-detection scheme measures, in principle, the field-strength probability curve in its dependence on *ccphi*. The probability of finding a zero electric field plotted versus *ccphi* constitutes the proposed distribution. For a wide class of quantum states, this curve is in good agreement with the abstract phase probability curve obtained from phase states, but it is free of the familiar problems accompanying the notion of a Hermitian phase operator. The phase distribution of a quantum state has been achieved without using phase states; this can be summarized as "phase without phase."

PACS number(s): 42.50.Bs, 42.50.Dv

The phase-sensitive phenomenon of the squeezing [1] of the quantum fluctuations of a single mode of the radiation field in the conjugate variables corresponding to the electric and magnetic field, that is, squeezing in the quadrature components rests on solid theoretical foundations. The same holds true for quantum effects related to photon number [2]. Why? Because there exist well-defined Hermitian operators corresponding to these quantities [3,4]. However, in the domain of phase, that is, the variable conjugate to m, we walk on shaky ground, caused by the problems associated with the definition of a Hermitian phase operator [4–8]. In the present article we suggest a phase distribution that is free of these complications and, more importantly, we relate it to a measurement—a balanced homodyne-detection scheme [9]. This is in the spirit of Lamb's operational approach towards nonrelativistic quantum mechanics [10], which forces us to devise an experimental apparatus for every quantum-mechanical operator or wave function we intend to "measure." This operational phase distribution, which we base on well-defined states—the so-called field strength eigenstates or rotated quadrature eigenstates [11]—we compare and contrast to the abstract phase probability curve

$$p(\varphi,|\psi\rangle)=|\langle\varphi|\psi\rangle|^2=\frac{1}{2\pi}\left|\sum_{m=0}^{\infty}\psi_m e^{-im\varphi}\right|^2 , \quad (1)$$

motivated in Ref. [12] from semiclassical considerations.

We assume that the light field is in a normalized quantum state

$$|\psi\rangle=\sum_{m=0}^{\infty}\psi_m|m\rangle , \quad (2)$$

where $|m\rangle$ denotes the mth number state. In the definition of the phase distribution Eq. (1), we have made use of the phase states [5,12,13]

$$|\varphi\rangle=(2\pi)^{-1/2}\sum_{m=0}^{\infty}\exp(im\varphi)|m\rangle . \quad (3)$$

We gain considerable insight into the phase distribution $p(\varphi,|\psi\rangle)$, Eq. (1), by capitalizing on the concept of area of overlap and interference in phase space [14]. According to this semiclassical principle the area of overlap between the appropriate phase-space representation [15] of the state $|\psi\rangle$ indicated in Fig. 1 by a "blob" or a long "cigar" and the phase state $|\varphi\rangle$ shown in Fig. 1(a) by the divergent beam determines the probability $|\langle\varphi|\psi\rangle|^2$. To realize experimentally such a phase state $|\varphi\rangle$ is a difficult enterprise. However, field-strength eigenstates $|E(\varphi)\rangle$, depicted in Fig. 1(b) by the thin phase-space strip, are the essential ingredient of homodyne detection [9,11] and are experimentally accessible. Figures 1(a) and 1(b) suggest that the area of overlap between the state representative $|\psi\rangle$ and the phase state $|\varphi\rangle$ is close to and resembles the zone of crossover between $|\psi\rangle$ and the strip state $|E(\varphi)=0\rangle$. Consequently, the corresponding probabilities $|\langle\varphi|\psi\rangle|^2$ and $|\langle E(\varphi)=0|\psi\rangle|^2$ are closely related. We hence expect to approximate the abstract phase distribution $p(\varphi,|\psi\rangle)\equiv|\langle\varphi|\psi\rangle|^2$, Eq. (1), by the operational phase distribution

$$p_E(\varphi,|\psi\rangle)\equiv\mathcal{N}|\langle\mathcal{O}_{E(\varphi)}|\psi\rangle|^2 . \quad (4)$$

For the sake of simplicity in notation we here and in the remainder of the article subscribe to the abbreviation $|E(\varphi)=0\rangle\equiv|\mathcal{O}_{E(\varphi)}\rangle$ for the field eigenstate of zero electric field at phase angle φ. Moreover, in the definition of $p_E(\varphi,|\psi\rangle)$, Eq. (4), we have introduced a normalization factor [16] $\mathcal{N}$ to ensure that

$$\int_0^{\pi}d\varphi\, p_E(\varphi,|\psi\rangle)=1 .$$

The above normalization condition and Fig. 1 suggest already that this distribution is π-periodic in contrast to the abstract phase distribution $p(\varphi,|\psi\rangle)$ for the same state being 2π-periodic. This effect results from the phase-space strip of the field-strength state extending from minus infinity to plus infinity in contrast to the one-sidedness of the divergent beam phase state $|\varphi\rangle$. This, however, is not a serious deficiency of this distribution [17]. Moreover, when $|\psi\rangle$ extends along a radial direction as illustrated in Figs. 1(c) and 1(d) the similarity in zones of crossover fails and the two distributions $p(\varphi,|\psi\rangle)$, Eq. (1), and $p_E(\varphi,|\psi\rangle)$, Eq. (4), are quite different.

These geometrical considerations serve as a motivation for the analysis pursued in the remainder of this article.

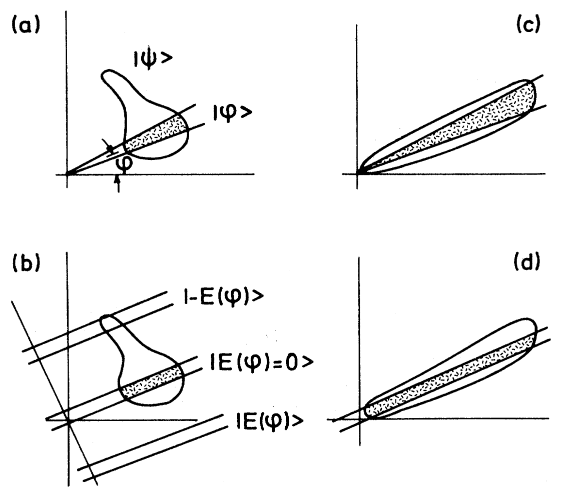

FIG. 1. Phase-state overlap compared and contrasted to field-strength eigenstate overlap. Phase information about a state $|\psi\rangle$ springs from either the phase probability $|\langle\varphi|\psi\rangle|^2$, that is, from the scalar product between a phase state $|\varphi\rangle$ and $|\psi\rangle$, or from the zero-field-strength curve $|\langle\mathcal{O}_{E(\varphi)}|\psi\rangle|^2$ corresponding to the scalar product between the zero-field-strength state $|E(\varphi)=0\rangle\equiv|\mathcal{O}_{E(\varphi)}\rangle$ and $|\psi\rangle$. The area-of-overlap principle associates with $|\langle\varphi|\psi\rangle|^2$ the phase-space domain of crossover between the diverging beam and a "blob" or an extended "cigar," that is, between the corresponding phase-space representatives of $|\varphi\rangle$ and $|\psi\rangle$. Likewise the stripe cut out of the $|\psi\rangle$-blob or $|\psi\rangle$-cigar by the phase-space strip, that is, by the zero-field-strength state representative governs the quantity $|\langle\mathcal{O}_{E(\varphi)}|\psi\rangle|^2$. The blob state $|\psi\rangle$ shown on the left is well located in radial direction. It has a narrow photon distribution $|\langle m|\psi\rangle|^2$ and is not influenced by the spreading of the diverging beam. In this case the beam acts almost like the phase-space strip: The two quantities $|\langle\varphi|\psi\rangle|^2$ and $|\langle\mathcal{O}_{E(\varphi)}|\psi\rangle|^2$ are closely related. In contrast the elongated cigar state on the right side of the figure containing a multitude of number states brings out the very difference between $|\varphi\rangle$ and $|E(\varphi)=0\rangle$. For such a state the idea of approximating the phase probability distribution $|\langle\varphi|\psi\rangle|^2$ by the zero-field-strength curve $|\langle\mathcal{O}_{E(\varphi)}|\psi\rangle|^2$ fails. The phase-space strip extends from $-\infty$ over the origin of phase space to $+\infty$ and hence the field-strength eigenstates for $E=0$ are π-periodic, whereas the phase states $|\varphi\rangle$ display 2π periodicity. Hence the two distributions differ in their periods.

The eigenstates $|E(\varphi)\rangle$ of the appropriately scaled field operator

$$\hat{E}(\varphi)=i[\hat{a}\exp(-i\varphi)-\hat{a}^{\dagger}\exp(i\varphi)], \tag{5}$$

read [11]

$$|E(\varphi)\rangle=(2\pi)^{-1/4}\sum_{m=0}^{\infty}[\Gamma(m+1)2^m]^{-1/2}\times H_m(E(\varphi)/\sqrt{2})\times\exp[-E^2(\varphi)/4]\times\exp[im(\varphi-\pi/2)]|m\rangle\ . \tag{6}$$

Here $\hat{a}$, $\hat{a}^{\dagger}$, H_m, and Γ denote the annihilation and creation operator of the field mode [3,4], the mth Hermite polynomial [18], and the gamma function, respectively. The use of the gamma function here rather than the usual factorial will be useful when we replace the discrete summation variable m by a continuous integration variable. We emphasize that the phase φ of the field appears in the eigenstate $|E(\varphi)\rangle$ as a parameter and not as the eigenvalue of a phase operator.

The probability of observing the field strength value E for a given phase φ of the field reads [11,19]

$$p(E(\varphi),|\psi\rangle)=|\langle E(\varphi)|\psi\rangle|^2\ . \tag{7}$$

Hence the operational phase distribution defined via Eq. (4) is the curve obtained by depicting the value of the field-strength probability distribution at the particular field value $E(\varphi)=0$ in its dependence on the parameter φ as illustrated geometrically in Fig. 2. We, therefore, cal-

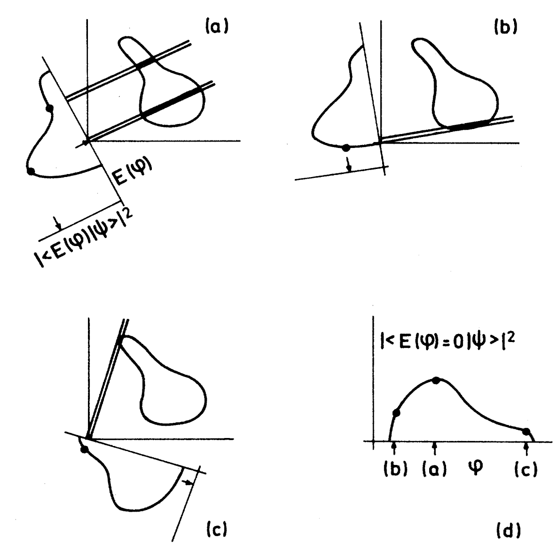

FIG. 2. The operational phase distribution—the zero-field-strength curve—obtained from the field-strength probability distribution. For various angles φ [(a), (b), and (c)] we record the field-strength probability curve $|\langle E(\varphi)|\psi\rangle|^2$ corresponding to the area of overlap between the phase-space strip state $|E(\varphi)\rangle$ and the $|\psi\rangle$-blob. The value of the quantity $|\langle E(\varphi)=0|\psi\rangle|^2$ in its dependence on φ (d) constitutes, when properly normalized, the operational phase distribution.

culate the scalar product $\langle E(\varphi)=0|\psi\rangle \equiv \langle \mathcal{O}_{E(\varphi)}|\psi\rangle$ by combining Eq. (6) for $E(\varphi)=0$ with Eq. (2), which yields

$$\langle \mathcal{O}_{E(\varphi)}|\psi\rangle=(2\pi)^{-1/4}\sum_{m=0}^{\infty} H_m(0)[\Gamma(m+1)2^m]^{-1/2} \times \psi_m \exp[-im(\varphi-\pi/2)] \, .$$

The properties $H_{2m+1}(0)=0$ and $H_{2m}(0) =(-1)^m\Gamma(2m+1)/\Gamma(m+1)$ of the Hermite polynomials [18] reduce the above equation to

$$\langle \mathcal{O}_{E(\varphi)}|\psi\rangle=(2\pi)^{-1/4}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(2m+1)}{\Gamma(m+1)} \times[\Gamma(2m+1)2^{2m}]^{-1/2}\psi_{2m} \times \exp[-i2m(\varphi-\pi/2)] \, ,$$

or

$$\langle \mathcal{O}_{E(\varphi)}|\psi\rangle=(2\pi)^{-1/4}\sum_{m=0}^{\infty}\frac{[\Gamma(2m+1)/2^{2m}]^{1/2}}{\Gamma(m+1)} \times \psi_{2m}\exp(-i2m\varphi) \, . \tag{8}$$

With the help of the Poisson summation formula [20]

$$\sum_{n=0}^{\infty} f(n)=\sum_{\nu=-\infty}^{+\infty}\int_0^{\infty} dn \, f(n)\exp(2\pi i n\nu)+\tfrac{1}{2}f(0) \, , \tag{9}$$

we find [21] from Eq. (8)

$$\langle \mathcal{O}_{E(\varphi)}|\psi\rangle=(2\pi)^{-1/4}\sum_{\nu=-\infty}^{+\infty}\int_0^{\infty} dn\frac{[\Gamma(2n+1)/2^{2n}]^{1/2}}{\Gamma(n+1)}\psi_{2n}\exp[-i2n(\varphi-\pi\nu)]+\tfrac{1}{2}(2\pi)^{-1/4}\psi_0$$

or

$$\langle \mathcal{O}_{E(\varphi)}|\psi\rangle=\tfrac{1}{2}(2\pi)^{-1/4}\sum_{\nu=-\infty}^{+\infty}\int_0^{\infty} dm\frac{[\Gamma(m+1)/2^{m}]^{1/2}}{\Gamma(m/2+1)}\psi_{m}\exp[-im(\varphi-\pi\nu)]+\tfrac{1}{2}(2\pi)^{-1/4}\psi_0 \, . \tag{10}$$

In the last step we have introduced the new integration variable $m=2n$.

We now consider this exact result for a state $|\psi\rangle$ with a coherent amplitude as suggested in Fig. 1. In this case the dominant contribution to the integration over the variable m in Eq. (10) arises from m values that allow the application of the improved Stirling formula [22]

$$\Gamma(m+1)\cong(2\pi)^{1/2}(m+\tfrac{1}{2})^{m+1/2}\exp(-m-\tfrac{1}{2}) \, .$$

With the help of this relation we find

$$\frac{[\Gamma(m+1)/2^m]^{1/2}}{\Gamma(m/2+1)}\cong\pi^{-1/4}(m+\tfrac{1}{2})^{-1/4} \, ,$$

which simplifies Eq. (10)

$$\langle \mathcal{O}_{E(\varphi)}|\psi\rangle=\frac{1}{\sqrt{2}}(2\pi)^{-1/2}\sum_{\nu=-\infty}^{+\infty}\int_0^{\infty} dm\,[2(m+\tfrac{1}{2})]^{-1/4}\psi_m\exp[-im(\varphi-\pi\nu)]+\tfrac{1}{2}(2\pi)^{-1/4}\psi_0 \, .$$

For field states such as the blob of Figs. 1(a) and 1(b), the function $(m+1/2)^{-1/4}$ is slowly varying, that is, the photon distribution $|\psi_m|^2$ is narrow compared to the m scale on which this fourth root varies. Hence for such states we may replace in this term the integration variable m by an effective value $\overline{m}$ and factor it out of the integral, which yields

$$\langle \mathcal{O}_{E(\varphi)}|\psi\rangle=\frac{1}{\sqrt{2}}(2\pi)^{-1/2}[2(\overline{m}+\tfrac{1}{2})]^{-1/4}\sum_{\nu=-\infty}^{+\infty}\int_0^{\infty} dm\,\psi_m\exp[-im(\varphi-\pi\nu)]+\tfrac{1}{2}(2\pi)^{-1/4}\psi_0 \, . \tag{11a}$$

The elongated cigar of Figs. 1(c) and 1(d) would not have allowed such a slowly varying approximation.

In order to compare this result to the corresponding expression

$$\langle \varphi|\psi\rangle=(2\pi)^{-1/2}\sum_{m=0}^{\infty}\psi_m e^{-im\varphi}$$

obtained from the state $|\psi\rangle$, Eq. (2), and the phase state $|\varphi\rangle$, Eq. (3), we again make use of the Poisson summation equation (9) and arrive at

$$\langle \varphi|\psi\rangle=(2\pi)^{-1/2}\sum_{\nu=-\infty}^{+\infty}\int_0^{\infty} dm\,\psi_m\exp[-im(\varphi-2\pi\nu)] +\tfrac{1}{2}(2\pi)^{-1/2}\psi_0 \, . \tag{11b}$$

Equations (11a) and (11b) bring out most clearly the similarities and differences between the phase distributions, Eqs. (1) and (4), resulting from these probability amplitudes.

(i) When $\psi_0\cong 0$, as is the case, for example, when the state $|\psi\rangle$ has a large coherent amplitude, the difference in Eq. (11) due to the rest term in the Poisson summation formula is minute.

(ii) The probability amplitude $\langle \varphi|\psi\rangle$ is periodic in φ with period 2π in contrast to $\langle \mathcal{O}_{E(\varphi)}|\psi\rangle$ which, in full agreement with the geometrically motivated result discussed earlier, shows π-periodicity.

(iii) The prefactors of $\langle \varphi|\psi\rangle$ and $\langle \mathcal{O}_{E(\varphi)}|\psi\rangle$ are different reflecting the difference in the domains of integration: a triangle in the case of $\langle \varphi|\psi\rangle$ versus a rectangular shape in the example of $\langle \mathcal{O}_{E(\varphi)}|\psi\rangle$ [23].

We now outline a strategy to measure the phase distribution $p_E(\varphi,|\psi\rangle)$. In the first step we record the probability distribution $p(E(\varphi),|\psi\rangle)$ for various values of φ as suggested in Fig. 2. For this purpose we use the balanced homodyne-detection scheme shown in Fig. 3. Here we superpose the signal field E_s to be measured with the classical field of a local oscillator [24] of well-defined phase φ and well-defined amplitude $\mathcal{E}_L$. The operators for the total intensities of the two output ports are

$$\hat{I}_+ \equiv \tfrac{1}{2}(\hat{E}_s^{(-)} + \mathcal{E}_L e^{-i\varphi})(\hat{E}_s^{(+)} + \mathcal{E}_L e^{+i\varphi})$$

and

$$\hat{I}_- \equiv \tfrac{1}{2}(\hat{E}_s^{(-)} - \mathcal{E}_L e^{-i\varphi})(\hat{E}_s^{(+)} - \mathcal{E}_L e^{+i\varphi}) ,$$

where $\hat{E}_s^{(+)}$ and $\hat{E}_s^{(-)}$ denote the positive and negative frequency components of the slowly varying signal field operator, respectively [3,4]. Therefore, the operator of the difference in these intensities $\Delta\hat{I} \equiv \hat{I}_+ - \hat{I}_-$ reads

$$\begin{aligned}\Delta\hat{I} &\equiv \mathcal{E}_L(\hat{E}_s^{(+)} e^{-i\varphi} + \hat{E}_s^{(-)} e^{+i\varphi}) \\ &= \mathcal{E}_L \mathcal{E}_0 i(\hat{a}e^{-i\varphi} - \hat{a}^{\dagger} e^{i\varphi}) \\ &= \mathcal{E}_L \mathcal{E}_0 \hat{E}(\varphi) . \end{aligned} \tag{12}$$

In the last step we have made use of the relation $\hat{E}_s^{(+)} = i\mathcal{E}_0 \hat{a}$, where $\mathcal{E}_0$ denotes the electric field per photon [3,4], and the definition of $\hat{E}$, Eq. (5). Hence apart from an unessential scaling factor, the operator $\Delta\hat{I}$ defined in Eq. (12) corresponds to the field operator $\hat{E}$ given by Eq. (5). The statistical behavior of this measured difference signal shown in the lower part of Fig. 3, that is, its probability distribution, is essentially the field-strength probability distribution we seek [25]. In the second step we now repeat this procedure for various φ values. From each probability curve $p(E(\varphi),|\psi\rangle)$ for fixed φ we only need the value of this distribution at $E=0$. A plot of those values as a function of φ shown in Fig. 2(d) provides this operational phase distribution. We emphasize that the squeezing experiments reported so far measure the second moment of this field-strength distribution only. However the present scheme involves the total distribution and hence all moments. Hence the measurement of the full field-strength distribution revealing the internal structure of the state would in itself constitute a novel measurement scheme.

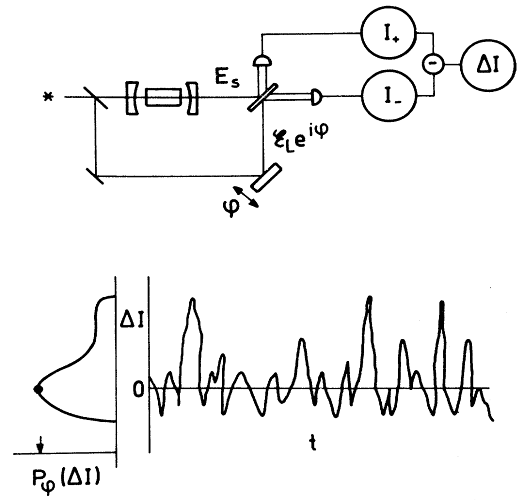

FIG. 3. Experimental scheme to measure the field-strength probability curve based on balanced homodyne detection. We superpose on a beam splitter the signal field E_s to be measured and created, for example, by laser light sent through a nonlinear crystal in a resonator, with the original local oscillator laser field. The difference $\Delta I \equiv I_+ - I_-$ between the intensities $I_+ = |E_s^{(+)} + \mathcal{E}_L e^{i\varphi}|^2$ and $I_- = |E_s^{(+)} - \mathcal{E}_L e^{i\varphi}|^2$ measured at the two output ports of the beam splitter shown in its time dependence in the lower part of the figure is essentially the field-strength variable (provided the laser field is considered to be a classical field). The probability curve $P_\varphi(\Delta I)$ for this difference intensity ΔI (for a field measurement time) is the field strength distribution. We vary the phase between the local oscillator and the field and record the field-strength distribution for a different phase.

In conclusion we have defined in this article a phase distribution based on the field-strength eigenstates. We give a prescription for the measurement of this distribution using a balanced homodyne-detection scheme. For a wide class of physical states this operational phase distribution is in good agreement with the phase probability distribution derived from ordinary phase states.

ACKNOWLEDGMENTS

We thank R. Blatt, C. Fabre, J. Kimble, A. Schenzle, A. Sizmann, R. Slusher, and B. Yurke for fruitful and stimulating discussions. One of us (W.V.) gratefully acknowledges the hospitality extended to him at the Max-Planck-Institut für Quantenoptik at Garching.

[1] For a review of squeezed states of the electromagnetic field see the special issues J. Opt. Soc. Am. B **4**, 10 (1987) and J. Mod. Opt. **34**, 6 (1987).

[2] M. C. Teich and B. A. E. Saleh, *Photon Bunching and Antibunching,* Vol. XXVI of *Progress in Optics,* edited by E. Wolf (Elsevier, North-Holland, Amsterdam, 1988).

[3] M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974).

[4] R. Loudon, *The Quantum Theory of Light,* 1st ed. (Oxford University, Oxford, 1973).

[5] For a review of this problem, see P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968); H. Paul, Fortschr. Phys. **22**, 657 (1974); for the most recent discussion of this problem see J. H. Shapiro and S. R. Shepard, Phys. Rev. A **43**, 3795 (1991).

[6] S. M. Barnett and D. T. Pegg, J. Phys. A **19**, 3849 (1986).

[7] D. T. Pegg and S. M. Barnett, Europhys. Lett. **6**, 483 (1988); Phys. Rev. A **39**, 1665 (1989); S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989).

[8] J. Bergou and B.-G. Englert, Ann. Phys. (N.Y.) **209**, 479 (1991).

[9] H. P. Yuen and V. W. S. Chan, Opt. Lett. **8**, 177 (1983); B. L. Schumaker, *ibid.* **9**, 189 (1984); J. H. Shapiro, IEEE J. Quantum Electron. **21**, 237 (1985); see also Sec. V of B. Yurke, W. Schleich, and D. F. Walls, Phys. Rev. A **42**, 1703 (1990); for a discussion of a homodyne detection measurement using a new quantum state, the so-called optimal phase state, see J. H. Shapiro, S. R. Shepard, and N. C. Wong, Phys. Rev. Lett. **62**, 2377 (1989).

[10] W. E. Lamb, Jr., Phys. Today **22**, (4), 23 (1969).

[11] M. Schubert and W. Vogel, Wiss. Z. Friedrich Schiller Univ. Jena Math. Naturwiss. Reihe **27**, 179 (1978); W. Vogel, Phys. Rev. A **42**, 5754 (1990); for applications to phase operators and squeezed states, see M. Schubert, Phys. Lett. **27A**, 698 (1968) and M. Schubert and W. Vogel, *ibid.* **68A**, 321 (1978).

[12] W. Schleich, R. J. Horowicz, and S. Varro, Phys. Rev. A **40**, 7405 (1989); in *Quantum Optics V*, edited by J. D. Harvey and D. F. Walls (Springer, Heidelberg, 1989); W. Schleich, J. P. Dowling, R. J. Horowicz, and S. Varro, in *New Frontiers in Quantum Optics and Quantum Electrodynamics*, edited by A. Barut (Plenum, New York, 1990).

[13] Reference [5] shows that these states are only normalized to δ functions and are not mutually orthogonal. This could be cured by truncating the Fock space as discussed in Ref. [7]. However, the expansion coefficients ψ_m of a physical state decay as to make the infinite sum in Eq. (1) convergent. Hence for such a physical state this truncation procedure is not necessary.

[14] J. A. Wheeler, Lett. Math. Phys. **10**, 201 (1985); J. P. Dowling, W. Schleich, and J. A. Wheeler, Ann. Phys. (Leipzig) **48**, 423 (1991); for applications to squeezed and coherent states see W. Schleich and J. A. Wheeler, Nature **326**, 574 (1987) and W. Schleich, H. Walther, and J. A. Wheeler, Found. Phys. **18**, 953 (1989).

[15] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121 (1984).

[16] The phase distribution Eq. (1) is properly normalized, that is,

$$\int_0^{2\pi} d\varphi\, p(\varphi,|\psi\rangle)=(2\pi)^{-1}\sum_{m,n=0}^{\infty}\psi_m^*\psi_n\int_0^{2\pi}d\varphi\, e^{-i\varphi(n-m)}$$
$$=\sum_{m=0}^{\infty}|\psi_m|^2=1\ ,$$

whereas the operational phase probability curve, Eq. (4), needs the normalization factor $\mathcal{N}$. The origin of this difference in normalization lies in the fact that according to Ref. [11] the field strength distribution $p(E(\varphi),|\psi\rangle)\equiv|\langle E(\varphi)|\psi\rangle|^2$ is normalized with respect to $E(\varphi)$, that is

$$\int_{-\infty}^{\infty}dE\, p(E(\varphi),|\psi\rangle)=\int_{-\infty}^{\infty}dE\langle\psi|E(\varphi)\rangle\langle E(\varphi)|\psi\rangle$$
$$=\langle\psi|\psi\rangle=1\ .$$

Hence we cannot expect the distribution $p_E(E(\varphi)=0,|\psi\rangle)$, that is a portion of the field strength distribution, to be normalized with respect to the parameter φ.

[17] Since the operational phase distribution is always π-periodic regardless of the true phase-space behavior of the state it seems to contain less information than the abstract phase distribution. However in the proposed measurement strategy we record the total field strength distribution $p(E(\varphi),|\psi\rangle)$ for all values of E and φ. This quantity is 2π-periodic with respect to φ and reveals the periodicity properties of the state $|\psi\rangle$. Moreover, according to the work of K. Vogel and H. Risken [Phys. Rev. A **40**, 2847 (1989)] we can invert uniquely this distribution to find the corresponding s-parametrized phase-space distribution determining the state $|\psi\rangle$ and hence its periodicity properties.

[18] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, 1939).

[19] Such field distributions have also been calculated by K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989) and in the context of generalized squeezed states by S. L. Braunstein and C. M. Caves, *ibid.* **42**, 4115 (1990).

[20] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Wiley Interscience, New York, 1953).

[21] Here we assume that the expansion coefficients ψ_m in the discrete variable m allow an extension to a continuous variable.

[22] See, for example, C. Leubner, Eur. J. Phys. **6**, 299 (1985).

[23] The triangular phase-space element corresponding to the phase-state contribution is roughly $\frac{1}{2}[2(\overline{m}+\frac{1}{2})]^{1/2}[2(\overline{m}+\frac{1}{2}]^{1/2}d\varphi$ in contrast to $[2(\overline{m}+\frac{1}{2})]^{1/2}dp$ resulting from a field strength eigenstate.

[24] We do realize that no such classical oscillator exists. However, for an oscillator in a coherent state of large coherent amplitude α the influence of its fluctuations $\delta\alpha$ can be neglected since $\delta\alpha/\alpha\ll 1$.

[25] The present treatment neglects electronic noise in the detectors. In the standard squeezing experiments measuring the second moment of the distribution the second moment of the electronic noise is measured separately and then subtracted. In the present scheme the total noise distribution has to be recorded initially and then subtracted appropriately.