# Bose-Einstein condensation in low-dimensional traps

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We demonstrate the possibility of Bose-Einstein condensation (BEC) of an ideal Bose gas confined by one- and two-dimensional power-law traps. One-dimensional systems display BEC in traps that are more confining than parabolic: $U(x)\sim x^{\eta}, \eta<2$. Two-dimensional systems display BEC for any finite value of $\eta$. A possible experimental configuration for a two-dimensional trap is described.

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## I. INTRODUCTION

The development of techniques to cool gaseous atoms to extremely low temperature is providing new possibilities for studying gases in the quantum regime [1]. Prominent among the goals of this research is the observation of Bose-Einstein condensation (BEC), a phase transition that occurs when the atomic de Broglie wavelength becomes comparable to the interatomic spacing. For a noninteracting Bose gas of $N$ particles of mass $M$ and confined in a hard-wall continuum of volume $V$, this takes place at the critical temperature $T_c$ given by [2]

$$kT_c^{3\mathrm{D}}=\frac{h^2}{2\pi M}\left[\frac{1}{g_3(0)}\frac{N}{V}\right]^{2/3}, \tag{1}$$

where $g_3(\mu/kT)$ is the three-dimensional Bose function and $\mu$ is the chemical potential

$$g_3(0)=\zeta(\tfrac{3}{2})=2.612\ , \tag{2}$$

where $\zeta(x)$ is the Riemann zeta function. Because strategies for achieving BEC in a gas have so far been elusive, it is natural to inquire whether similar physical phenomena can be observed in other geometries, for instance, in a two-dimensional (2D) system composed of atoms adsorbed on a surface. Hohenberg has shown, however, that BEC cannot occur in an ideal two-dimensional system [3], and for this reason such systems have received relatively little attention. However, this result is true only for a system confined by rigid boundaries—the two-dimensional equivalent of ideal rigid walls. As we shall show, if a system is confined by a spatially varying potential—i.e., a "trapping" potential—BEC can in principle occur. We shall consider both one- and two-dimensional systems, though experimental interest is likely to be limited to the latter. Our analysis is restricted to power-law potentials because these potentials lead to analytical solutions and because most traps display power-law behavior close to their minimum. Our method is semiclassical, as is appropriate for the relatively weak confining potentials of neutral atoms traps that produce energy-level spacings which are generally microscopic compared to the mean energy.

Possibilities for achieving BEC in some special low-dimensional systems have been pointed out by a number of authors [4(a)]. These include particles confined by a gravitational field [4(b)] and by a rotating container [5], and also in an interacting one-dimensional gas [6]. A rotating quantum liquid has also been analyzed [7]. In this case, the inhomogeneous density leads to behavior equivalent to that in a square-law potential, analyzed below. However, to our knowledge, there has been no general treatment of the problem.

## II. BEC IN A ONE-DIMENSIONAL GAS

We consider a one-dimensional gas of particle of mass $M$ confined by a power-law potential $U(x)=U_0(|x|/L)^{\eta}$. The density of states is

$$\rho(\varepsilon)=\frac{\sqrt{2M}}{h}\int_{-l(\varepsilon)}^{l(\varepsilon)}\frac{dx\,1}{\sqrt{\varepsilon-U(x)}}\ , \tag{3}$$

where $2l(\varepsilon)$ is the available length for particles with energy $\varepsilon$. $l(\varepsilon)=L(\varepsilon/U_0)^{1/\eta}$. Equation (1) becomes

$$\rho(\varepsilon)=\frac{\sqrt{2M}}{h}L\frac{\varepsilon^{1/\eta-1/2}}{(U_0)^{1/\eta}}F(\eta)\ , \tag{4}$$

where

$$F(\eta)=\int_0\frac{y^{(1-\eta)/\eta}}{\sqrt{1-y}}dy\ . \tag{5}$$

The total number of particles is given by

$$N=N_0+\int_0^{\infty}\eta_c\rho(\varepsilon)d\varepsilon\ . \tag{6}$$

Here $N_0$ is the number of particles in the ground state which we explicitly retain because $\rho(0)=0$. In this equation, $n_c=[\exp(\varepsilon-\mu/kT)-1]^{-1}$ is the Bose-Einstein occupation number.

The system will display BEC if the integral of Eq. (6) has a finite value at $\mu=0$ for some $T=T_c$. Below this

 

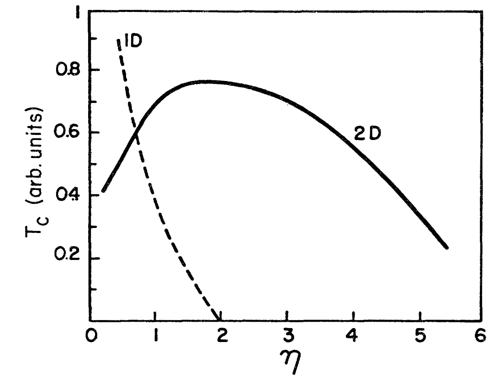


FIG. 1. Evolution of the critical temperature with the potential parameter $\eta$ for one- and two-dimensional traps.

temperature the ground state becomes heavily populated. Inserting Eq. (4) into Eq. (6) yields

$$N=N_0+\frac{\sqrt{2M}}{h}\frac{F(\eta)}{(U_0)^{1/\eta}}(kT)^{1/\eta+1/2}g_1(\eta,\mu/kT) \tag{7}$$

where

$$g_1(\eta,x)=\int_0^\infty \frac{y^{-1/\eta-1/2}}{e^{(y-x)}-1}dy \tag{8}$$

is the one-dimensional Bose function. As high temperatures, Eq. (5) is satisfied by some negative value of $\mu$ and $N_0 \sim 0$. As $T$ is reduced $\mu$ increases, reaching $\mu=0$ at some $T_c$. For $T<T_c$, $\mu$ remains zero and the last term in Eq. (7) decreases. Consequently, $N_0$ increases.

The function $g_1(\eta,0)$ can be written in terms of the $\gamma$ and Riemann $\zeta$ functions [8]:

$$g_1(\eta,0)=\Gamma(1/\eta+\tfrac{1}{2})\zeta(1/\eta+\tfrac{1}{2})\ . \tag{9}$$

From the properties of the Riemann $\zeta$ function, $g_1(\eta,0)$ is finite only if $\eta<2$. Because $\zeta(s)$ diverges for $s<1$, the one-dimensional gas will display BEC only if the potential power is less than 2, i.e., only if the external potential is more confining than a parabolic potential.

The critical temperature is obtained setting $N_0=0$ in Eq. (7). In this case

$$kT_c^{1\mathrm{D}}=\left[\frac{Nh}{\sqrt{2M}}\frac{U_0^{1/2}}{F(\eta)}\frac{1}{g_1(\eta,0)}\right]^{2\eta/(2+\eta)}\ . \tag{10}$$

The variation of $kT_c^{1\mathrm{D}}$ with $\eta$ is shown in Fig. 1. The critical temperature increases monotonically as $\eta$ is decreased below 2.

## III. BEC IN A TWO-DIMENSIONAL BOSE GAS

Next, we consider a two-dimensional Bose gas confined by a power-law potential. The most general potential is $U(x,y)=U_1(x/b)^m+U_2(y/c)^n$, but for simplicity we assume that the potential is isotropic: $U(r)=U_0(r/a)^\eta$. In analogy to Eqs. (4) and (7) we obtain

$$\rho(\varepsilon)=\frac{2\pi M}{h^2}\int_0^{r^*}2\pi r\,dr=\frac{2\pi^2Ma^2}{h^2}\left[\frac{\varepsilon}{U_0}\right]^{2/\eta}\ , \tag{11}$$

where $r^*=(\varepsilon/U_0)^{1/\eta}$, and

$$N=N_0+\frac{2\pi^2Ma^2}{h^2(U_0)^{2/\eta}}(kT)^{2/\eta+1}g_2\left[\eta\frac{\mu}{kT}\right]\ . \tag{12}$$

The two-dimensional Bose function $g_2(\eta,x)$ is given by

$$g_2(\eta,x)=\int_0^\infty\frac{y^{2/\eta}}{e^{(y-x)}-1}dy\ . \tag{13}$$

For $\mu=0$, we obtain

$$g_2(\eta,0)=\Gamma(2/\eta+1)\zeta(2/\eta+1)\ . \tag{14}$$

Unlike the 1D case, $g_2(\eta,0)$ remains finite for all positive values $\eta$. Consequently, BEC, an ideal two-dimensional gas confined by a power-law trap, can, in principle, always occur. A rigid box corresponds to the limit $\eta\rightarrow\infty$. Since $g_2(\infty,0)$ diverges, BEC does not occur, in agreement with Hohenberg's finding ($\zeta$). For BEC to occur in a nonisotropic 2D potential, the requirement is that $n^{-1}+m^{-1}$ be finite. From Eq. (12), the critical temperature in 2D trap is

$$kT_c^{2\mathrm{D}}=\left[\frac{Nh^2U_0^{2/\eta}}{2\pi^2Ma^2g_2(\eta,0)}\right]^{\eta/(2+\eta)}\ . \tag{15}$$

The dependence of $T_c^{2\mathrm{D}}$ on $\eta$ is shown in Fig. 1. $T_c$ has a broad maximum in the vicinity of $\eta=2$.

These results are valid only for the ideal Bose gas. The weakly interacting Bose gas can be treated using the mean-field approximation [9], though at the low densities likely to be of experimental interest, the corrections are not expected to be important.

## IV. EXPERIMENTAL REALIZATION OF A TWO-DIMENSIONAL BOSE GAS

A possible configuration for a two-dimensional system for spin-polarized hydrogen is a pillbox-shaped container in a uniform field, with its axis parallel to the field axis, as shown in Fig. 2. A thin coil wound around the perimeter provides the inhomogenous trapping field. The inner surface is covered with liquid helium, and atoms in the "high-field seeking" state enter through a thin tube. The field of the coil varies radially according to

$$B(r)\cong -B_t\left[1+\left[\frac{3}{4}\frac{r}{a}\right]^2\right]\ , \tag{16}$$

where $a$ is the radius of the pillbox and $B_t$ is the field at the center of the coil. The trapping potential is given by

$$U(r)=U_0\left[\frac{r}{a}\right]^2\ , \tag{17}$$

where $U_0=(3/4)\mu_0B_t$ and $\mu_0$ is the Bohr magneton. The total number of particles trapped on either end surface is

$$N_S=\int_0^a 2\pi r\sigma_0e^{-U(r)/kT}dr=\pi a^2\frac{kT}{U_0}\sigma_0\ , \tag{18}$$

where $\sigma_0$ is the surface density on axis and we have assumed that $\exp[-U(a)/kT]\approx 0$. From Eq. (15), we obtain

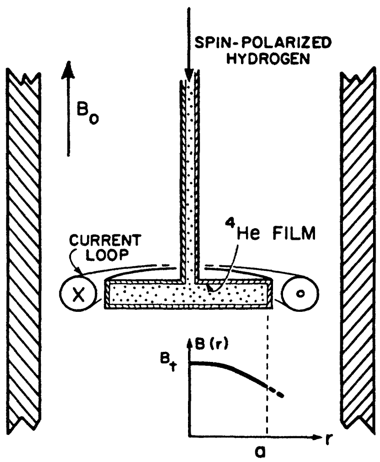


FIG. 2. Geometry for a two-dimensional trap for spin-polarized hydrogen. A gas of spin-polarized hydrogen is confined in a solenoid by the field $B_0$. Atoms in contact with the $^4$He surface produce a two-dimensional adsorbed gas that is trapped by the radial potential created by the field of the current loop $B_t(r)$.

$$kT_c^{2D} = \frac{1}{2\pi} \frac{h^2 \sigma_0}{M g_2(2,0)} . \tag{19}$$

The surface and volume density for a weakly interacting adsorbed gas are related by [10]

$$\sigma_0 = \lambda_D n_0 e^{E_b/kT} \tag{20}$$

where $\lambda_D = h/(2\pi M kT)^{1/2}$ is the thermal de Broglie wavelength, $E_b$ is the surface adsorption energy, and $n_0$ is the volume density on axis. The total number of atoms in the gas phase is

$$N = \int n_0 e^{-U(r)/kT} dV = n_0 V (kT/U_0) . \tag{21}$$

To compare the critical temperature for the 2D and the 3D system, one can combine Eqs. (1) and (19)–(21) to yield

$$(kT_c^{2D})^{5/2} = 1.588 (kT_c^{3D})^{3/2} U_0 e^{E_b/kT_c^{3D}} , \tag{22}$$

where we have introduced the numerical values $g_2(2,0)=1.645$ and $g_3(0)/g_2(2,0)=1.588$. Because $(U_0 E_b) \gg kT_c$, Eq. (22) predicts an enormous enhancement of the critical temperature for the surface compared to the bulk.

In practice, three-body recombination on the surface limits the useful surface density. The three-body recombination rate constant is [11] $L_s = 1.2\times10^{-24}$ cm$^4$ s$^{-1}$. This value was measured in the temperature range 0.3–0.6 K and was found to be approximately temperature independent. One expects $L_s$ to vary as $T^{1/2}$, but to be conservative we shall assume that $L_s$ is temperature independent. A surface density on axis of $\sigma_0 = 2\times10^{11}$ cm$^{-2}$ would yield a characteristic recombination decay time $\tau_s = (L_s \sigma^2)^{-1} = 20$ s, which is an acceptable value. From Eq. (19), $T_c^{2D} = 4$ mK, a temperature that is low but achievable. The value of $U_0$ is fixed by the requirement that $\exp(-U_0/kT_c^{2D} \ll 1$. Taking $U_0 = 10kT_c^{2D}$, a conservative value, leads to a trapping field of $B_t = 400$ G, which is easily achieved. The total number of atoms, Eq. (18), is $6.3\times10^{10}$. Note that the interparticle separation under these conditions is larger than the hydrogen $S$-wave scattering length by $\sim 4\times10^3$, so that the independent-particle approximation is expected to be realistic.

Although the experimental conditions appear to be favorable, we point out that removing the heat recombination may present a challenge and that methods for studying the surface gas remain to be developed. Nevertheless, this analysis suggests that low-dimensional systems are of potential experimental interest.

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