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Modified Riccati approach to partially solvable quantum Hamiltonians. II. Morse-oscillator-related family

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We extend the scope of the modified Riccati approach to partial solubility in quantum mechanics introduced in a previous work [L.D. Salem and R. Montemayor, Phys. Rev. A **43**, 1169 (1991)]. With the use of adequate mappings $u(x)$, we show the convenience of the modified Riccati approach to analyze potentials that can be written as rational functions on u. The necessary conditions for a Hamiltonian to be solvable are discussed in detail. By considering the exponential mapping $u = e^{-x}$, we construct a family of potentials related to the exactly solvable Morse oscillator. Within this family, we have identified a three-parameter quasiexactly solvable potential, which, depending on the value of its coupling constants, leads to a symmetric or asymmetric confining potential, with a single-well or a double-well structure. Explicit expressions for the energies and eigenfuctions are given for particular cases. The analytic continuation of the symmetric subset gives rise to a quasiexactly solvable periodic potential.

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I. INTRODUCTION

Since the advent of quantum mechanics exactly solvable Schrödinger equations have been the subject of intense research. Historically, however, the most fruitful formalism for obtaining exact solutions for second-order differential equations was stated before the quantum era by Darboux over a century ago [1]. The Darboux factorization method was rediscovered by Schrödinger himself in the quantum-mechanics context [2], and some years later thoroughly exploited by Infeld and Hull in a remarkable paper [3], where they classified most of the exactly solvable (ES) Hamiltonians. Two decades ago Natanzon obtained the most general form of the potential for which the Schrödinger equation reduces to hypergeometric form [4]. Group-theory techniques have also been applied to study the solubility problem, providing an algebraic approach [5] to obtain the spectrum and matrix elements for solvable potentials from the dynamical groups. Another interesting contribution for constructing exactly solvable potentials is based on the Gelfand and Levitan equation [6]. On this basis Abraham and Moses [7] and Pursey and collaborators [8] developed a procedure to insert and delete some bound states from a given Hamiltonian. More recently, supersymmetric (SUSY) theories motivated the extensive use of SUSY transformations in quantum mechanics [9]. Several authors recognized that SUSY transformations are related to the Schrödinger factorization method and therefore to the Darboux procedure. It was Gendenshtein who introduced the concept of shape invariance in SUSY quantum mechanics and showed [10] that this is a sufficient condition to have exact solubility. Later on it was proved [11] that shape invariance and the condition of factorization of Infeld and Hull are equivalent.

In summary, exact solubility is found only for very few potentials. Most of them are contained in the Infeld and Hull classification [3], and have as a common feature that they can be written as simple rational functions in terms of $u(x)$, where $u(x)$ is an exponential, circular, or hyperbolic function or, at most, a simple combination of them.

On the other hand, in recent years there has been a growing interest in a wider category of potentials, with only a solvable subset of eigenfunctions. There are some examples of such potentials [12], but it is not obvious how to state the general conditions to guarantee this restricted solubility. Certainly the understanding of this problem would allow us to gain insight into the whys of the exactly solvable models and, what is more attractive, to construct new solvable potentials with *a priori* specified features.

This is the second paper of a series dealing with *partial solubility* (PS) and *quasiexact* (QES) solubility in quantum mechanics. In a previous work [13] (hereafter to be referred to as paper I), we presented a general formalism to analyze solubility based on a modified Riccati equation for the regular component of the logarithmic derivative of the wave function. There [13] we fully developed the simplest version of the method, which proved adequate to investigate solubility in the family of finite Laurent-series-type potentials. These include, among others, the Coulomb potential, one-dimensional (1D) and three-dimensional (3D) harmonic and anharmonic polynomial oscillator potentials, inverse power potentials, and the Coulomb plus polynomial terms.

Generalizing what was anticipated in paper I, the modified Riccati approach is also suitable for investigating a more general class of potentials, namely, those which can be written as rational functions of $u(x)$, provided that

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$(du/dx)^2$ is a polynomial in u. In fact, this is the only restriction we shall consider on the transformation $u(x)$. These two conditions are certainly satisfied by the exactly solvable potentials discussed by Infeld and Hull [3] and Natanzon [4] which then should emerge as particular cases in the solubility analysis considered here. An example of such a mapping was explicitly considered in paper I, where we chose $u = x^2$ $[(du/dx)^2 = 4u]$, a transformation that is adequate to consider reflection symmetric potentials $V(x) = V(-x)$. In this family we recognized both the ES members, i.e., the 1D and 3D harmonic oscillators, and the QES members, i.e., the 1D and 3D sextic polynomial oscillators [12]. Besides these we also found an infinite number of partially solvable (PS) potentials. From this perspective each of the well-known ES potentials [3] turns out to be a member of a family characterized by one of the simplest transformations $u(x)$ satisfying the previously mentioned restriction.

In this and forthcoming papers we shall use the generalized modified Riccati approach to construct the potential families related to well-known exactly solvable potentials [3] identifying the corresponding QES members in each family.

This paper is organized as follows. In Sec. II we extend the scope of the modified Riccati approach for a quite general transformation $u(x)$. In Sec. III we discuss the conditions for solubility. In Sec. IV we consider the family of potentials related to the Morse oscillator, using the exponential mapping $u = e^{-x}$, and analyze in detail the ES (Morse oscillator) and QES members. Finally, in Sec. V we summarize and discuss our results.

II. EXTENSION OF THE MODIFIED RICCATI APPROACH

In the following we extend the scope of the modified Riccati approach presented in paper I. When considering the quantum description of a particle in a potential $V(x)$, we are interested in closed-form solutions of the 1D Schrödinger equation

$$\frac{d^2\psi}{dx^2} + [E - V(x)]\,\psi = 0\ . \tag{2.1}$$

The wave function $\psi(x)$ satisfies the appropriate boundary conditions, e.g., if $V(x)$ is a potential well, the bounded eigenfunction vanishes at the boundaries $x_\pm$ of the x-physical domain [14] $x_- < \mathrm{Re}(x) < x_+$.

Let us now consider a mapping $u(x)$, regular in the physical domain. It is useful to define the u-physical domain as the image of the x-physical domain, excluding the zeros of (du/dx). Eventually such a mapping will allow us to obtain closed-form solutions of the Schrödinger equation (2.1) for a given potential $V(x) = W(u)$, or alternatively, will allow us to construct potentials from given closed-form eigenfunctions. Both goals can be easily achieved within the formalism developed below. For $\psi(x) = \Psi(u)$ the Schrödinger equation (2.1) reads

$$\left(\alpha(u)\frac{d^2}{du^2} + \beta(u)\frac{d}{du} + E - W(u)\right)\Psi(u) = 0\ , \tag{2.2}$$

where

$$\alpha(u) = \left(\frac{du}{dx}\right)^2\ , \qquad \beta(u) = \frac{d^2u}{dx^2} = \frac{1}{2}\frac{d\alpha}{du}\ . \tag{2.3}$$

We can always decompose

$$\Psi(u) \equiv C(u)\bar{\Psi}(u)\ , \tag{2.4}$$

where the factor $\bar{\Psi}(u)$ is defined to have no zeros in the u-physical domain and satisfies the appropriate boundary conditions at $u(x_-)$ and $u(x_+)$. On the other hand, the nodal structure of the wave function is embodied in $C(u)$. For a given transformation $u(x)$, the number of nodes of $\psi(x)$ is related to the number of simple zeros that $C(u)$ has in the u-physical domain [15]. For instance if $V = W(u = x^2)$ and $\psi(x)$ has n nodes in the x-physical domain spanning the whole real axis $-\infty < \mathrm{Re}(x) < \infty$, the factor $C(u)$ has $[n/2]$ simple zeros in $0 < \mathrm{Re}(u) < \infty$. The node at the origin that odd-parity eigenfunctions have is outside the u-physical domain and thus must be provided by the factor $\bar{\Psi}(u)$.

We shall assume that the nodal structure of the eigenfunction $\psi(x)$ considered can be represented by a finite number of simple zeros in the u-physical domain. Under this assumption one can always write the nodal factor $C(u)$ as a polynomial of degree N in u:

$$C(u) = \prod_{j=1}^{N}(u - u_j) \equiv C(u;N)\ . \tag{2.5}$$

We should remark that some of the zeros of $C(u;N)$ may fall outside the u-physical domain and thus, only $m \leq N$ of those zeros correspond to nodes of the wave function $\psi(x)$. In that sense, for a fixed value of N, Eqs. (2.4) and (2.5) can represent more than one wave function depending on where the zeros u_j are located.

From Eqs. (2.4) and (2.5), it follows that

$$\begin{aligned}\Psi(u) &\propto \bar{\Psi}(u)\prod_{j=1}^{N}(u - u_j)\\ &= \exp\int^{u} du'\left(\bar{G}(u') + \sum_{j=1}^{N}\frac{1}{u' - u_j}\right)\ ,\end{aligned} \tag{2.6}$$

where $\bar{G}(u)$ is the logarithmic derivative of $\bar{\Psi}(u)$ and thus is, by definition, regular in the u-physical domain. The substitution of Eq. (2.6) into Eq. (2.2) leads to the equation [13]

$$\alpha\left(\bar{G}^2 + \frac{d\bar{G}}{du}\right) + \frac{1}{2}\frac{d\alpha}{du}\bar{G} + 2\alpha F = W - E\ , \tag{2.7}$$

where

$$F(u) \equiv \sum_{j=1}^{N}\frac{\bar{G}(u) + \dfrac{1}{4}\dfrac{d}{du}\ln\alpha - \Gamma_{Nj}}{u - u_j} \tag{2.8}$$

and

$$\Gamma_{Nj} \equiv \sum_{\substack{k=1\\ k\neq j}}^{N}\frac{1}{u_k - u_j}\ , \qquad j = 1, 2, \ldots, N \geq 2\ . \tag{2.9}$$

To be consistent we define $F(u) \equiv 0$ for $N = 0$ and $\Gamma_{Nj} \equiv 0$ for $N = 0, 1$. Some useful relations involving the Γ_{Nj} are derived in the Appendix.

Provided that $W(u)$ is regular in the u-physical domain, the left-hand side of Eq. (2.7) must also be regular in this domain. Therefore, as both $\bar{G}(u)$ and $\alpha(u)$ are regular there, *the following relation must hold*:

$$\left(\bar{G}(u) + \frac{1}{4}\frac{d}{du}\ln\alpha(u)\right)_{u=u_j} = \Gamma_{Nj}\ , \qquad j = 1, 2, \ldots, N\ . \tag{2.10}$$

This set of nonlinear equations relates the zeros $\{u_j\}$ of $C(u; N)$ to the regular component $\bar{G}(u) = d[\ln\bar{\Psi}(u)]/du$.

Let us introduce the function

$$\tilde{G}(u) \equiv \bar{G}(u) + \frac{1}{4}\frac{d}{du}\ln\alpha\ , \tag{2.11}$$

which is also regular in the u-physical domain, as $\alpha(u)$ has no zeros there. Equation (2.7) now takes the form of the modified Riccati equation [see Eq. (2.9) in paper I],

$$\tilde{G}^2 + \frac{d\tilde{G}}{du} + 2F = \frac{W - E}{\alpha} + \gamma^2 + \frac{d\gamma}{du}\ , \tag{2.12}$$

where $\gamma(u) = \frac{1}{4}d[\ln\alpha(u)]/du$. From Eq. (2.10)

$$\tilde{G}(u_j) = \Gamma_{Nj}\ , \qquad j = 1, 2, \ldots, N \tag{2.13}$$

and therefore, $F(u)$ becomes

$$F(u) = \sum_{j=1}^{N} \frac{\tilde{G}(u) - \tilde{G}(u_j)}{u - u_j}\ . \tag{2.14}$$

Finally, in terms of $\tilde{G}(u)$, the wave function reads

$$\psi(x) \propto [\alpha(u)]^{-1/4}\, C(u; N)\, \exp\int^{u} du'\,\tilde{G}(u')\ . \tag{2.15}$$

This particular form of writing the wave function is reminiscent of the uniform semiclassical approximation as proposed by Miller and Good [16]. In that sense, the modified Riccati approach provides a clear understanding about the relationship between the Coulomb, Morse oscillator, and harmonic oscillator with centripetal barrier problems [17], and thus sheds light on the origin of the Langer correction [18].

We note in passing that Eqs. (2.12)–(2.15) reduce to the plain modified Riccati approach presented in Sec. II of paper I when $u(x) = x$.

To conclude this section it is very important to point out that, given the regular component $\bar{G}(u)$ and N, the system of nonlinear equations (2.13) can have more than one solution set $\{u_j\}$. A simple visualization of this fact relies on an electrostatic analogy. The set of Eqs. (2.13) can be read off as the equilibrium condition of a system of N particles with a potential energy

$$\begin{aligned} \mathcal{U}(u_1, u_2, \ldots, u_N) &= \sum_{k\neq j} \ln|u_k - u_j| + \sum_{j=1}^{N}\int^{u_j} du'\,\tilde{G}(u') \\ &= \sum_{k\neq j} \mathcal{U}_{\text{pair}}(k, j) + \sum_{j=1}^{N} \mathcal{U}_{\tilde{G}}^{\text{ext}}(u_j)\ , \end{aligned} \tag{2.16}$$

and constrained to move along a line [the image of the open interval (x_-, x_+) by the transformation $u(x)$]. The N positive "unit charges" with coordinates $u_1, u_2, \ldots, u_N$ interact through a *two-body* repulsive force in the presence of the *external* attractive field $\int^u du'\,\tilde{G}(u')$. Hence, the number of solution sets to Eq. (2.13) is equal to the number of equilibrium configurations of this N particle system. This very intuitive picture was first noticed by Stieltjes in relation to the position of the zeros of Jacobi polynomials [19]. Our Eq. (2.13), as well as the interpretation given in (2.16), generalizes such a finding for polynomials not so well known in mathematical physics [13, 20].

III. SOLUBILITY ANALYSIS

From now on, and generalizing the discussion in Sec. II of paper I, we focus our study on potentials for which a mapping $u(x)$ exists such that (i) $V(x) = W(u)$ is a rational function of u, and (ii) $\alpha(u) = (du/dx)^2$ is a polynomial in u. In such a case, the modified Riccati equation (2.12) is particularly well suited to relate the functional form of $\tilde{G}(u)$ and $W(u)$ by simple inspection.

According to Eq. (2.15), given the transformation $u(x)$, the regular component $\tilde{G}(u)$, and N, one can always construct a wave function $\psi(x)$. This is so because the polynomial $C(u; N)$ (or equivalently its zeros $\{u_j\}$) is specified by $\tilde{G}(u)$ and N through the set of Eqs. (2.13). The wave function constructed in this way, is an eigenfunction of the Schrödinger equation with the potential $W(u)$ given by

$$W(u) = E + \alpha\left[\tilde{G}^2 + \frac{d\tilde{G}}{du} + 2F - \left(\gamma^2 + \frac{d\gamma}{du}\right)\right]\ . \tag{3.1}$$

As remarked at the end of Sec. II, given $\tilde{G}(u)$ and N, more than one wave function in the form of (2.15) can be constructed, provided that the nonlinear equations (2.13) can have more than one solution set $\{u_j\}$. Let us suppose for the moment that, given $\tilde{G}(u)$ and N, we have already found the $\{u_j^{(m)}\}$, where the superscript m labels the mth solution set of Eqs. (2.13). In principle each set $\{u_j^{(m)}\}$ can lead to a different function $F(u; N, \{u_j^{(m)}\}) \equiv F_m(u)$ and hence the left-hand side of Eq. (3.1) can accordingly change. There are three possibilities.

(a) If the functions αF_m corresponding to distinct solution sets $\{u_j^{(m)}\}$ differ from each other in their u dependence, the resulting wave functions correspond to different potentials. These potentials belong to the class of PS Hamiltonians.

(b) Another much more interesting possibility is that the αF_m corresponding to distinct solution sets $\{u_j^{(m)}\}$ differ from each other only by an additive constant. In this case the different wave functions correspond to eigenstates of the same Hamiltonian, with different energies. We have then a QES potential. The solution sets $\{u_j^{(m)}\}$ differ from each other in the number of zeros inside the u-physical domain. Correspondingly, the different wave functions have a different number of nodes. At most there can be $N+1$ of these eigenfunctions.

(c) Finally if the difference between the αF_m (corresponding to two possible solutions sets) vanishes, both eigenfunctions would have the same energy. Being both eigenstates of the same Hamiltonian, this is in contradiction with the well-known fact that degeneracy cannot occur for 1D bound states due to time reversal symmetry. Hence, there must be only one solution set $\{u_j\}$ for each value of N. This is the case for ES potentials.

In summary, the recognition of PS, QES, or ES of the Schrödinger equation (2.1) with $V(x) = W(u)$ derives from the analysis of αF_m. The necessary conditions for the three cases just discussed require the possibility of writing

$$2\alpha F_m = \begin{cases} \mathcal{F}(u;N,\{u_j^{(m)}\}) - \mathcal{K}(N,\{u_j^{(m)}\}) & \text{for PS ,} \\ \mathcal{F}(u;N) - \mathcal{K}(N,\{u_j^{(m)}\}) & \text{for QES,} \\ \mathcal{F}(u;N) - \mathcal{K}(N) & \text{for ES ,} \end{cases} \tag{3.2}$$

where we have made explicit the following.

(i) The u-independent term of $2\alpha F_m$, namely, $-\mathcal{K}$. This term is, up to an additive constant, the energy of the eigenfunction with zeros at $\{u_j^{(m)}\}$.

(ii) The dependence of the u-dependent term of $2\alpha F_m$, namely, $\mathcal{F}$, on the different solution sets $\{u_j^{(m)}\}$ of Eqs. (2.13) for a given $\tilde{G}(u)$ and N.

From a computational perspective, in order to specify completely the eigenfunctions and eigenergies there is a more convenient alternative to the nonlinear equations (2.13) for the $\{u_j\}$. This is true at least for the more interesting cases, namely, the QES and ES potentials. It consists in finding the coefficients $\{c_k\}$ of the polynomial $C(u;N) \equiv \sum_{k=0}^{N} c_k\, u^k$. Replacing $\Psi(u) = C(u;N)\bar{\Psi}(u)$ in the Schrödinger equation (2.2), we obtain

$$\alpha\bar{\Psi}\,\frac{d^2C}{du^2} + \left(2\alpha\frac{d\bar{\Psi}}{du} + \frac{1}{2}\frac{d\alpha}{du}\bar{\Psi}\right)\frac{dC}{du} + \left(\alpha\frac{d^2\bar{\Psi}}{du^2} + \frac{1}{2}\frac{d\alpha}{du}\frac{d\bar{\Psi}}{du} + (E-W)\bar{\Psi}\right)C = 0\ . \tag{3.3}$$

Factorizing $\bar{\Psi}(u) = \exp\int^u du'\,\bar{G}(u')$ and taking into account Eqs. (2.7) and (2.11), we obtain a second-order differential equation for the polynomial $C(u;N)$,

$$\alpha\left(\frac{d^2C}{du^2} + 2\tilde{G}\,\frac{dC}{du} - 2F\,C\right) = 0\ . \tag{3.4}$$

The result of the analysis of solubility summarized in Eq. (3.2) is pertinent here. When the potential is PS, the coefficient $2F$ depends on the particular solution to be found (see Appendix B in paper I), and the determination of the coefficients c_k from (3.4) is a formidable task due to the high nonlinearity of the problem, unless N is small.

For the QES case, Eq. (3.4) can be cast as a finite eigenvalue problem for the coefficients c_k [see Eq. (3.2)], and thus the determination of energy levels and wave functions is completely algebraic.

Lastly, when the potential is ES, Eq. (3.4) results in one of the well-known differential equations of mathematical physics, e.g., a confluent or a hypergeometric differential equation with polynomial solutions.

In the next section we shall use the modified Riccati equation (2.12) to construct families of partially solvable potentials starting from a given form for $\tilde{G}(u)$. Using the solubility analysis here presented we shall determine whether the necessary conditions for PS, QES, or ES hold.

IV. MORSE-OSCILLATOR-RELATED FAMILY

We propose here to investigate a family of potentials that include as a special case the Morse oscillator [21],

$$V_{\mathrm{MO}}(x) = S^2(e^{-2x} - e^{-x})\ , \tag{4.1}$$

for which the transformation

$$u(x) = e^{-x} \tag{4.2}$$

provides closed form solutions [21, 22]. For the mapping (4.2) we have $\alpha(u) = u^2$, and $\gamma(u) = (2u)^{-1}$. Hence, the modified Riccati equation (2.12) [more conveniently written as in (3.1)] reads

$$W(u) = E + \tfrac{1}{4} + u^2\left(\tilde{G}^2 + \frac{d\tilde{G}}{du} + 2F\right)\ . \tag{4.3}$$

Following Sec. VI of paper I, we construct a family of potentials that has as a particular member the Morse oscillator. On the basis of the mapping (4.2), we consider

$$\tilde{G}(u) = \tilde{G}_{\mathrm{MO}}(u) + \delta\tilde{G}(u)\ , \tag{4.4}$$

where $\tilde{G}_{\mathrm{MO}}(u) = a_0 + a_{-1}u^{-1}$ has the functional form that solves Eq. (4.3) for the Morse oscillator (4.1), and $\delta\tilde{G}(u)$ is a finite Laurent series in u, regular in the u-physical domain. Therefore,

$$\tilde{G}(u) = \sum_{k=-L}^{M} a_k\, u^k \tag{4.5}$$

with both $L, M \geq 0$. From Eq. (2.14) and with the particular finite Laurent-type form of $\tilde{G}(u)$ given in Eq. (4.5), we find $F(u)$ to have also a finite Laurent series form [13]. Explicitly

$$F(u) = \sum_{k=-L}^{M-1} f_k^N\, u^k\ , \tag{4.6}$$

where the coefficients f_k^N are given by [13]

$$f_k^N = \begin{cases} N \displaystyle\sum_{m=k+1}^{M} a_m \langle\!\langle u^{m-k-1} \rangle\!\rangle \ \text{ for } 0 \le k \le M-1 \ , \\ -N \displaystyle\sum_{m=-L}^{k} a_m \langle\!\langle u^{m-k-1} \rangle\!\rangle \ \text{ for } -L \le k \le -1 \ , \end{cases} \tag{4.7}$$

and we have introduced the kth moment of the set $\{u_j\}$,

$$\langle\!\langle u^k \rangle\!\rangle \equiv \frac{1}{N} \sum_{j=1}^{N} (u_j)^k \ . \tag{4.8}$$

Using Eqs. (4.5)–(4.8) in (4.3), we obtain a family of potentials related to the Morse oscillator. This family comprises the finite Laurent potentials in u,

$$W(u) = \sum_{k=-2L}^{2M} w_k \, u^{k+2} = e^{-2x} \sum_{k=-2L}^{2M} w_k \, e^{-kx} \ , \tag{4.9}$$

where the coupling constants w_k, for $k \neq -2$, are given by [13]

$$w_k = \begin{cases} \displaystyle\sum_{j=-L}^{k+L} a_j \, a_{k-j} \ , \quad -2L \le k \le -L-2 \ , \\ -La_{-L} + \displaystyle\sum_{j=-L}^{-1} a_j \, a_{-L-1-j} \ , \quad k = -L-1 \ , \\ 2f_k^N + (k+1)a_{k+1} + \displaystyle\sum_{j=-\min(L,M-k)}^{\min(L+k,M)} a_j \, a_{k-j} \ , \quad -L \le k \le M-1 \ , \\ \displaystyle\sum_{j=k-M}^{M} a_j \, a_{k-j} \ , \quad M \le k \le 2M \ . \end{cases} \tag{4.10}$$

In principle, the constant w_{-2} can be arbitrarily chosen as it only fixes the energy origin. Thus, for $k = -2$, we obtain a relation between the energy and the coefficients $\{a_k\}$ and f_{-2}^N:

$$E = w_{-2} - \tfrac{1}{4} - 2f_{-2}^N + a_{-1} - \sum_{j=-\min(L,M+2)}^{j=\min(L-2,M)} a_j \, a_{-j-2} \ . \tag{4.11}$$

It is clear from (4.9) that the potentials thus constructed have no singularities in $-\infty < \mathrm{Re}(x) < \infty$ and thus the u-physical domain is the open interval $0 < \mathrm{Re}(u) < \infty$. Therefore, if $C(u;N)$ has m zeros on $\mathrm{Re}(u) > 0$, the wave function has m nodes and thus corresponds to the mth excited level. Explicitly

$$\psi_m(x;N) = C(u;N) \, u^{a_{-1}-1/2} \exp \sum_{\substack{k=-L \\ k\neq -1}}^{M} \frac{a_k}{k+1} \, u^{k+1} \ , \tag{4.12}$$

where the coefficients a_k can be chosen with the only requirement of making the wave function (4.12) normalizable.

The solubility analysis presented in Sec. III relies on the particular functional dependence of αF in terms of the zeros $\{u_j\}$. In fact, to perform such an analysis, there is no need to determine the set $\{u_j\}$ but only its moments $\langle\!\langle u^k \rangle\!\rangle$ on which the coefficients f_k^N depend, according to Eq. (4.7). Taking the rth moments of (2.13), one obtains

$$\begin{aligned} \langle\!\langle u^r \tilde{G} \rangle\!\rangle &= \frac{1}{N} \sum_{j=1}^{N} (u_j)^r \, \tilde{G}(u_j) \\ &= \frac{1}{N} \sum_{j=1}^{N} (u_j)^r \, \Gamma_{Nj} \equiv \langle\!\langle u^r \Gamma \rangle\!\rangle, \qquad r \ge 0 \ . \end{aligned} \tag{4.13}$$

The weighted moments $\langle\!\langle u^r \Gamma \rangle\!\rangle$ are evaluated in the Appendix. For the particular finite Laurent-type form of $\tilde{G}(u)$ given in Eq. (4.5), the set of Eqs. (2.13) is equivalent to

$$\sum_{k=-L}^{M} a_k \, \langle\!\langle u^{k+r} \rangle\!\rangle = \begin{cases} 0 \ \text{ for } r = 0 \ , \\ -(N-1)/2 \ \text{ for } r = 1 \ , \\ \dfrac{r}{2} \langle\!\langle u^{r-1} \rangle\!\rangle - \dfrac{N}{2} \displaystyle\sum_{s=0}^{r-1} \langle\!\langle u^s \rangle\!\rangle \langle\!\langle u^{r-1-s} \rangle\!\rangle \\ \hfill \text{for } r \ge 2 \ . \end{cases} \tag{4.14}$$

Let us now isolate the more interesting members of the family, namely, the ES and the QES potentials.

(a) If we were able to eliminate the dependence of αF on the moments $\langle\!\langle u^k \rangle\!\rangle$, then we would have satisfied the necessary conditions to have an exactly solvable potential [see Eqs. (3.2)]. From the system of Eqs. (4.14) one can solve the moments $\langle\!\langle u^k \rangle\!\rangle$ completely in terms of the coefficients a_k, only when

$$\begin{aligned} &\text{(i)} \ a_{-1} \neq 0, \ \ a_0 \neq 0, \ \ \text{and } a_k = 0 \text{ otherwise,} \\ &\text{(ii)} \ a_{-1} \neq 0, \ \ a_{-2} \neq 0, \ \ \text{and } a_k = 0 \text{ otherwise,} \\ &\text{(iii)} \ a_0 \neq 0, \ \ a_1 \neq 0, \ \ \text{and } a_k = 0 \text{ otherwise.} \end{aligned} \tag{4.15}$$

Case (iii) is a spurious situation that gives rise to nonnormalizable eigenfunctions. For the first two cases [(i) and (ii)], the function αF becomes

$$\begin{aligned} &\text{(i)}\quad \alpha F = N a_0 u \ , \\ &\text{(ii)}\quad \alpha F = \tfrac{1}{2}N(N-1) + N a_{-1} \ , \end{aligned} \tag{4.16}$$

corresponding to the Morse oscillator potential as given by Eq. (4.1), and its mirror image $V_{\mathrm{MO}}(-x)$, respectively.

(b) Let $a_{-2} \neq 0$, $a_{-1} \neq 0$, $a_0 \neq 0$, and $a_k = 0$ otherwise; then, from the system of Eqs. (4.14) we find

$$\alpha F = N a_0 u + \tfrac{1}{2}N(N-1) + N a_{-1} + N a_0 \langle\langle u \rangle\rangle \ , \tag{4.17}$$

and hence, according to Eq. (3.2), we have a QES potential.

(c) Any other set of values for the a_k leads to a PS potential.

Until now, we have applied the solubility analysis to identify the ES ($M = 0, L = 1$) and the QES ($M = 0, L = 2$) members of the family defined by Eq. (4.9). In the following subsections we pursue the analysis of such cases provided that the functional form of $\tilde{G}_{\mathrm{ES}}$ and $\tilde{G}_{\mathrm{QES}}$ is known.

A. The Morse oscillator ($M = 0, L = 1$)

First we consider the best known representative of this family, the Morse potential ($M = 0, L = 1$). Although its features are well known [22], we find it instructive to analyze it from the present perspective. In this case we have $\tilde{G}(u) = a_0 + a_{-1}u^{-1}$, and thus

$$F(u) = N a_0\, u^{-1} \tag{4.18}$$

does not depend on the $\langle\langle u^k \rangle\rangle$ with $k \neq 0$. From Eqs. (4.9) and (4.10) the resulting potential is

$$\begin{aligned} W(u) = \sum_{k=-2}^{0} w_k\, u^{k+2} &= (a_0)^2\, u^2 + 2a_0(N + a_{-1})\, u \\ &= (a_0)^2\, e^{-2x} + 2a_0(N + a_{-1})e^{-x} \ , \end{aligned} \tag{4.19}$$

where we have fixed the energy origin so that $w_{-2} = 0$. We remark that this is the only potential within the family that has no constraining relations between the coefficients of $W(u)$.

The closed form eigenfunctions and corresponding energy levels [see Eqs. (4.11) and (4.12)] are

$$\psi(x) \propto C(u; N)\, u^{a_{-1}-1/2}\, e^{a_0 u} \ , \tag{4.20}$$

$$E = -(a_{-1} - \tfrac{1}{2})^2 \ . \tag{4.21}$$

The wave function given in Eq. (4.20) can only represent a bound state because it has only a finite number of nodes, and therefore must vanish at the boundaries $x \to \pm\infty$. This requires

$$a_0 < 0 \ , \qquad N + a_{-1} > \tfrac{1}{2} \ . \tag{4.22}$$

From Eq. (4.19) the potential has a minimum at $x_0 = \ln[-a_0/(N + a_{-1})]$. We can always fix the origin of coordinates at x_0 so that

$$N + a_{-1} + a_0 = 0 \ . \tag{4.23}$$

Therefore, the potential has *only one free parameter*, e.g., $S > 0$ in terms of which $a_0 = -S$ and $a_{-1} = S - N$ are defined. The potential (4.19) then takes the usual expression for the Morse oscillator with a minimum at $x = 0$ and dissociation energy S^2,

$$V(x) = S^2\left(e^{-2x} - 2e^{-x}\right) \ , \tag{4.24}$$

with an energy level at [see Eq. (4.21)]

$$E_N = -(S - N - \tfrac{1}{2})^2 \ . \tag{4.25}$$

For a given value of N we have obtained *only one* discrete level which *must* be nondegenerate, since it corresponds to a bound state. Thus, *only one* wave function in the form $\psi_N(x) \propto C(u; N)\, u^{S-N-1/2} \exp(-Su)$ can be obtained associated with that level.

Notice that the potential thus written does not depend on the particular value chosen for N. Hence, we are free to choose the value of N in the range $0 \leq N \leq N_{\max} < S - \frac{1}{2}$ in order to satisfy Eqs. (4.22) and (4.23). The Morse oscillator potential has then *all* its discrete energy levels given by Eq. (4.25).

To completely determine the wave functions, we have to look for the polynomial solution $C(u; N)$ of the differential equation (3.4)

$$u\frac{d^2C}{du^2} + (2S - 2N - 2Su)\frac{dC}{du} + 2SNC = 0 \ . \tag{4.26}$$

This is the well-known Kummer's equation [23] and its regular solution is proportional to the Laguerre polynomial $L_N^{2S-2N-1}(2Su)$ with *all* its N zeros in $\mathbb{R}(u) > 0$. In summary, the Nth excited wave function of the Morse oscillator potential is

$$\psi_N(x) \propto L_N^{2S-2N-1}(2Se^{-x})\, \exp[-(S - N - \tfrac{1}{2})x - Se^{-x}] \ , \tag{4.27}$$

with energy given by Eq. (4.25) for $N = 0, 1, \ldots, N_{\max} < S - \frac{1}{2}$.

B. The QES member ($M = 0, L = 2$)

The other example that we will develop here corresponds to the case $M = 0, L = 2$, e.g.,

$$\tilde{G}(u) = a_0 + a_{-1}u^{-1} + a_{-2}u^{-2} \ , \tag{4.28}$$

and thus from Eq. (4.17)

$$F(u) = N a_0 u^{-1} + [\tfrac{1}{2}N(N-1) + N a_{-1} + N a_0 \langle\langle u \rangle\rangle]u^{-2} \ , \tag{4.29}$$

which explicitly depends on the first moment $\langle\langle u \rangle\rangle$. According to Eqs. (4.9) and (4.10) the associated potential is

$$\begin{aligned} W(u) &= (a_0)^2u^2 + 2a_0(N+a_{-1})u \\ &\quad +2a_{-2}(a_{-1}-1)u^{-1} + (a_{-2})^2u^{-2} \\ &= (a_0)^2e^{-2x} + 2a_0(N+a_{-1})e^{-x} \\ &\quad +2a_{-2}(a_{-1}-1)e^{x} + (a_{-2})^2e^{2x} \, . \end{aligned} \tag{4.30}$$

As before we have chosen the energy origin fixing $w_{-2} = 0$. In particular for $a_{-2} = 0$ or $a_0 = 0$ we recover the Morse-oscillator potential.

The closed-form eigenfunctions and corresponding energy levels are

$$\psi(x) \propto C(u;N)\, u^{a_{-1}-1/2} \exp(a_0u - a_{-2}u^{-1}) \, , \tag{4.31}$$

$$\begin{aligned} E &= -\tfrac{1}{4} - a_{-1}(a_{-1}-1) - 2a_0a_{-2} - N(N-1) \\ &\quad -2Na_{-1} - 2Na_0\langle\!\langle u \rangle\!\rangle \, . \end{aligned} \tag{4.32}$$

The potential (4.30) is a confining one as $V(|x| \to \infty) \to \infty$ and thus, all the eigenstates in such a potential are bound. Therefore, the eigenfunctions (4.31) *must* vanish at $|x| \to \infty$. This imposes a restriction on the values that a_0 and a_{-2} can take, say $a_0 < 0$ and $a_{-2} > 0$. As was previously noted, for a given value of N the closed-form solutions mentioned above can represent, at most, the first $N+1$ states.

Unlike what happened with the Morse oscillator, there is no way to avoid the explicit dependence on N in the potential (4.30), and hence $W(u) = W(u;N)$. In other words, among all the potentials $\sum_{k=-4}^{0} w_k u^{k+2}$, we can obtain closed-form solutions as given by (2.4) and (2.5), only when the constraint relation

$$\frac{w_{-3}}{\sqrt{w_{-4}}} + \frac{w_{-1}}{\sqrt{w_0}} + 2(N+1) = 0 \tag{4.33}$$

between the coupling constants is satisfied. As in the case of the QES potential considered in paper I, e.g., the symmetric sextic polynomial potential, the constraint (4.33) on the couplings depends on N, but not on a particular solution set $\{u_j\}$. Hence, if Eqs. (2.13) have more than one solution, all the eigenstates (4.31) and (4.32) correspond to the same potential.

Choosing the origin of coordinates such that the potential becomes asymptotically symmetric $[(a_0)^2 = (a_{-2})^2]$, we introduce a parametrization

$$a_{-2} = -a_0 \equiv \eta > 0 \, , \qquad a_{-1} \equiv \tfrac{1}{2}(\Delta - N + 1) \, . \tag{4.34}$$

The three-parameter potential (4.30) now reads

$$V(x;N) = 2\eta^2\cosh 2x - 2\eta(N+1)\cosh x + 2\eta\Delta\sinh x \, . \tag{4.35}$$

The parameter Δ clearly gives a direct measure of the asymmetry of the potential which results in a single well or a double well, depending on the values of the parameters. In particular, when $\Delta = 0$ (symmetric case) the potential develops two wells when $\eta < \eta_c \equiv (N+1)/4$, and only one otherwise [24]. The wave functions (4.31) now read

$$\begin{aligned} \psi(x) &\propto C(u;N)\, u^{-(N-\Delta)/2} \exp[-\eta(u^{-1}+u)] \\ &= C(e^{-x};N)\, e^{(N-\Delta)x/2} \exp[-2\eta\cosh(x)] \, . \end{aligned} \tag{4.36}$$

As given in Eq. (4.32), the energy level of this wave function depends both on N and the first-order moment $\langle\!\langle u \rangle\!\rangle$ of the zeros of $C(u;N)$.

To specify completely the wave functions and energy levels, we need in addition the polynomial solutions $C(u;N)$ of the second-order differential equation (3.4), which in this case reads

$$\begin{aligned} u^2\frac{d^2C}{du^2} &+ [2\eta - (N-1-\Delta)u - 2\eta u^2]\frac{dC}{du} \\ &+ [E - 2\eta^2 + \tfrac{1}{4}(N-\Delta)^2 + 2\eta Nu]C = 0 \, . \end{aligned} \tag{4.37}$$

For the very particular symmetric case $\Delta = 0$, Eq. (4.37) was first studied by Ince before the Schrödinger formulation of quantum mechanics, in relation with the vibrational modes of a membrane with variable density [20]. Thus, our Eq. (4.37) gives rise to a generalization of the Ince polynomials.

In terms of the coefficients c_k of the polynomial $C(u;N) = \sum_{k=0}^{N} c_k\, u^k$, the second-order differential equation (4.37) can be read off as a finite eigenvalue problem of order $(N+1)$:

$$\sum_{l=0}^{N} H_{k,l}\, c_l = E\, c_k \, , \tag{4.38}$$

where the only nonzero elements of the matrix $H_{k,l}$ are

$$\begin{aligned} H_{k,k-1} &= -2\eta(N+1-k) \, , \\ H_{k,k} &= 2\eta^2 - [k - \tfrac{1}{2}(N-\Delta)]^2 \, , \\ H_{k,k+1} &= -2\eta(k+1) \, , \end{aligned} \tag{4.39}$$

for $k = 0, 1, \ldots, N$. The coefficients of the tridiagonal matrix satisfy $H_{k,k+1}H_{k+1,k} > 0$. This property allows us to transform the matrix $H_{k,l}$ into a symmetric real one by a similarity transformation, and therefore, *all* the $(N+1)$ eigenvalues are real. From each eigenvector we can reconstruct a polynomial $C(u;N)$ or equivalently a wave function $\psi(x)$ as given in (4.36). These $(N+1)$ distinct eigenfunctions have at most N nodes and therefore, the associated eigenvalues are the $N+1$ lowest-energy levels of the spectrum.

In other words, the potential given in Eq. (4.35) is QES, with its ground and N first excited states belonging to the solvable Hilbert subspace. The complete specification of closed-form eigenfunctions and corresponding energies only involves a finite-matrix diagonalization.

In the general (nonsymmetric) case the eigenvalue problem (4.38) can be solved explicitly up to $N = 3$. For larger values of N we must resort to a numerical diagonalization.

In the symmetric situation $V(x) = V(-x)$, or equivalently $W(u) = W(u^{-1})$, the parity of a wave function with m nodes (m zeros in $0 < \mathbb{R}(u) < \infty$) is $(-1)^m$. From Eq. (4.36) with $\Delta = 0$, we have $C(1/u) = (-1)^m u^N C_m(u)$ and hence $c_k^{(m)} = (-1)^m c_{N-k}^{(m)}$. This symmetry allows us to reduce the order of the eigenvalue problem, and explicit analytical expressions can be obtained up to $N = 7$.

In the following we obtain explicitly the closed-form eigensolutions for the potential (4.35) with (a) $N = 1$

and (b) $N = 3$. In the first case ($N = 1$) the potential reads

$$V(x; N = 1) = 2\eta^2 \cosh 2x - 4\eta \cosh x + 2\eta\Delta \sinh x \ . \tag{4.40}$$

The 2×2 matrix H to be diagonalized is

$$H = \begin{pmatrix} 2\eta^2 - \frac{1}{4}(\Delta - 1)^2 & -2\eta \\ -2\eta & 2\eta^2 - \frac{1}{4}(\Delta + 1)^2 \end{pmatrix} \ . \tag{4.41}$$

This leads to the ground-state and first excited energy levels,

$$E_{0,1} = 2\eta^2 - \tfrac{1}{4}(1 + \Delta^2) \mp (4\eta^2 + \tfrac{1}{4}\Delta^2)^{1/2} \ ; \tag{4.42}$$

the corresponding eigenfunctions are

$$\begin{aligned} \psi_0(x) &\propto \{2\eta \, e^{x/2} + [\tfrac{1}{2}\Delta + (4\eta^2 + \tfrac{1}{4}\Delta^2)^{1/2}] \, e^{-x/2}\} \\ &\quad \times \Phi(x; \eta, \Delta) \ , \\ \psi_1(x) &\propto \{[\tfrac{1}{2}\Delta + (4\eta^2 + \tfrac{1}{4}\Delta^2)^{1/2}] \, e^{x/2} - 2\eta \, e^{-x/2}\} \\ &\quad \times \Phi(x; \eta, \Delta) \ , \end{aligned} \tag{4.43}$$

where $\Phi(x) = \exp[-2\eta \cosh x - (\Delta/2)x]$. The well-defined even- and odd-parity eigenstates are recovered when $\Delta = 0$. Moreover, if we let $\eta \rightarrow 0$ (so the condition $\eta < \eta_c$ is fulfilled and the potential has a double-well structure) a pair of nearly degenerate levels results, as expected [24].

Let us now consider $N = 3$ for which the QES potential (4.35) becomes

$$V(x; N = 3) = 2\eta^2 \cosh 2x - 8\eta \cosh x + 2\eta\Delta \sinh x \ . \tag{4.44}$$

In this case, the matrix H is

$$H = \begin{pmatrix} 2\eta^2 - \frac{1}{4}(\Delta - 3)^2 & -2\eta & 0 & 0 \\ -6\eta & 2\eta^2 - \frac{1}{4}(\Delta + 1)^2 & -4\eta & 0 \\ 0 & -4\eta & 2\eta^2 - \frac{1}{4}(\Delta - 1)^2 & -6\eta \\ 0 & 0 & -2\eta & 2\eta^2 - \frac{1}{4}(\Delta + 3)^2 \end{pmatrix} , \tag{4.45}$$

leading to a quartic secular equation for the solvable energy levels. The explicit expressions for the eigenvalues and eigenfunctions, although algebraic, are rather cumbersome, and we do not present them here. For the particular symmetric case ($\Delta = 0$) the problem decouples in the even- and odd-parity sectors, and the secular equation factorizes into two quadratic equations, whose solutions are

$$\begin{aligned} E_{0,2} &= 2\eta(\eta - 1) - \tfrac{5}{4} \mp (1 - 4\eta + 16\eta^2)^{1/2} \ , \\ E_{1,3} &= 2\eta(\eta + 1) - \tfrac{5}{4} \mp (1 + 4\eta + 16\eta^2)^{1/2} \ . \end{aligned} \tag{4.46}$$

As before, when $\eta \rightarrow 0$ the energy levels are degenerate in pairs. Finally, the corresponding eigenfunctions are

$$\begin{aligned} \psi_{0,2}(x) &\propto [2\eta \ \cosh \tfrac{3}{2}x \\ &\qquad -(E_{0,2} + \tfrac{9}{4} - 2\eta^2) \cosh \tfrac{1}{2}x] e^{-2\eta \cosh x} \ , \\ \psi_{1,3}(x) &\propto [2\eta \ \sinh \tfrac{3}{2}x \\ &\qquad -(E_{1,3} + \tfrac{9}{4} - 2\eta^2) \sinh \tfrac{1}{2}x] e^{-2\eta \cosh x} \ , \end{aligned} \tag{4.47}$$

with apparent well-defined parity.

V. CONCLUDING REMARKS

In this paper we have extended the scope of the modified Riccati approach developed in paper I, to treat a more general class of potentials by introducing a coordinate transformation $u(x)$. The modified Riccati equation was shown to be particularly well suited to investigate, as well as to construct, solvable potentials which can be written as rational functions of the mapping $u(x)$, provided that $(du/dx)^2$ is a polynomial in u. We have refined the solubility analysis for this kind of potentials by looking at the functional form taken by $2\alpha F$ [see Eq. (3.2)].

Having noticed that all the exactly solvable potentials classified by Infeld and Hull [3] and Natanzon [4] satisfy the above-mentioned requirements, we have proposed [13] to construct PS and QES potentials classified in families related to the already well-known ES ones [3]. These families are generated by the addition of regular terms to the logarithmic derivative of the wave functions of the exactly solvable potentials, for which the identification of the mapping $u(x)$ is evident [13].

In particular, we have explicitly constructed and analyzed the Morse-oscillator-related family using the exponential mapping $u = e^{-x}$. Within this family we have identified a three-parameter QES potential which, depending on the value of its coupling constants, leads to a symmetric ($\Delta = 0$) or asymmetric ($\Delta \neq 0$), and a single-well or a double-well potential.

We want to emphasize that the existence of exact and closed form solutions to such a versatile model, with a symmetric or asymmetric bistability and a phase transition to monostability when the *control parameters* η, Δ and N are varied, is of great interest. Certainly, it allows us to check and compare different approximations schemes to tunneling rates in such systems. This has motivated a much more detailed analysis of the QES potential found here. Our research along these lines is outside the scope of this paper and is planned to be given elsewhere [24].

To conclude we address some interesting points related to the same QES problem. First, the symmetric subset of the QES potential given in (4.35) suggests another possible mapping. Instead of the transformation $u = e^{-x}$ we could have used

$$u = \cosh x \,, \qquad \alpha(u) = (u-1)(u+1) \,. \tag{5.1}$$

The mapping (5.1), like $u = x^2$ treated in paper I, is only appropriate for considering reflection symmetric potentials. With this mapping the potential (4.35) in the symmetric case takes the form

$$V(x; N, \eta, \Delta = 0) = 4\eta^2\, u^2 - 2\eta(N+1)\, u - 2\eta^2 \,, \tag{5.2}$$

which is also suitable for treatment by means of the modified Riccati approach.

Second, a related interesting problem arises from the analytic continuation of the QES member (or any other symmetric member in the family). It is easy to show that, if $\psi_E(x)$ is an eigenfunction of the Schrödinger equation (2.1) with $V(x)$, then $\phi_\varepsilon(x) \equiv \psi_E(ix)$ is an eigenfunction of the same equation with the potential $U(x) \equiv -V(ix)$ and the corresponding eigenvalue is $\varepsilon = -E$. When this transformation is applied to the QES potential given in Eq. (4.35), we find that the periodic potential

$$U(x; N) = 2\eta(N+1)\cos x - 2\eta^2 \cos 2x \tag{5.3}$$

is also QES. The energy levels obtained from the eigenvalue problem (4.38) correspond to some of the band edges of the spectrum [24]. The analytic continuation procedure is equivalently found when considering the mapping $u = \cos x$. Consequently, the band-edge eigenfunctions in the periodic potential (5.3) are also obtained from Eq. (4.38) with $\Delta = 0$. It can be shown that the two parameter (η,N) periodic potential (5.3) has a finite number of gaps and thus can be of interest in solid-state physics as a model which, while still simple, is much more realistic than the well-known Kronig-Penney potential [25].

We postpone the use of the transformations $u = \cosh x$ and $u = \cos x$, for the analysis of the family related to the Pöschl-Teller potential, to a planned future work [24]. The symmetric QES member found here, and the QES periodic potential (5.2) will be obtained there, as particular cases of the QES member in the new families.

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APPENDIX

In this appendix we derive some simple and useful relations satisfied by the coefficients

$$\Gamma_{Nj} \equiv \sum_{k\neq j} \frac{1}{u_k - u_j} \,, \qquad j = 1, 2, \ldots, N \geq 2 \,, \tag{A1}$$

defined in Sec. II. From symmetry considerations, we can evaluate the successive *weighted moments* defined as

$$\langle\!\langle u^r \Gamma \rangle\!\rangle \equiv \frac{1}{N} \sum_{j=1}^{N} (u_j)^r \Gamma_{Nj} = -\frac{1}{2N} \sum_{j=1}^{N} \sum_{k\neq j} \frac{(u_k)^r - (u_j)^r}{u_k - u_j} \,. \tag{A2}$$

In particular, for $r = 0$ and $r = 1$, we have

$$\langle\!\langle \Gamma \rangle\!\rangle = 0 \,, \tag{A3}$$

$$\langle\!\langle u\Gamma \rangle\!\rangle = -(N-1)/2 \,, \tag{A4}$$

respectively. These relations have been used in Sec. IV to simplify the expressions in evaluating αF in order to determine PS, QES, or ES solubility in the potentials there considered.

For the higher moments ($r \geq 2$), we have

$$\begin{aligned} \langle\!\langle u^r \Gamma \rangle\!\rangle &= -\frac{1}{2N} \sum_{j=1}^{N} \sum_{k\neq j} [(u_j)^{r-1} + (u_j)^{r-2} u_k + \cdots + (u_k)^{r-1}] \\ &= \frac{1}{2N} \left[r \left(\sum_{j=1}^{N} (u_j)^{r-1} \right) - \sum_{s=0}^{r-1} \left(\sum_{j=1}^{N} (u_j)^s \right) \left(\sum_{k=1}^{N} (u_k)^{r-s-1} \right) \right] \\ &= \frac{r}{2} \langle\!\langle u^{r-1} \rangle\!\rangle - \frac{N}{2} \sum_{s=0}^{r-1} \langle\!\langle u^s \rangle\!\rangle \, \langle\!\langle u^{r-s-1} \rangle\!\rangle \,, \end{aligned} \tag{A5}$$

where $\langle\!\langle u \rangle\!\rangle \equiv (1/N) \sum_{k=0}^{N} (u_k)^r$ are the *nonweighted moments* as defined in Eq. (4.8).

In summary, only the $r = 0$ and $r = 1$ weighted moments *do not* depend on the position of the zeros $\{u_j\}$ of $C(u; N)$.

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