Bound solitons in the nonlinear Schrödinger–Ginzburg-Landau equation

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Interaction of slightly overlapping solitary pulses (SP's) is considered in the cubic nonlinear Schrödinger equation with small pumping and dissipation terms, and in the quintic Ginzburg-Landau equation with small dispersion terms. In both cases, the small perturbing terms render the asymptotic wave form of a SP spatially oscillating. Using the description of the interaction of SP's in terms of an effective potential, it is demonstrated that this fact may give way to formation of two-pulse and multipulse bound states, which are weakly stable.

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The subject of this work is the perturbed nonlinear Schrödinger (NS) equation with pumping and damping terms:

$$iu_t+u_{xx}+2|u|^2u=i\gamma_0u+i\gamma_1u_{xx}-i\gamma_2|u|^2u \quad (1)$$

($\gamma_0,\gamma_1,\gamma_2>0$), which has attracted attention as a dynamical model of plasma physics and hydrodynamics [1–3]. Recently, Eq. (1) has also found an application in the theory of optical solitons in fibers [4,5]. Equation (1) describes a situation when the trivial solution $u=0$ is unstable against small disturbances. In various physical problems, there occurs another situation, when the trivial state is stable against small disturbances, but can be triggered into a nontrivial state by a finite disturbance. The simplest model describing this situation is based upon the quintic perturbed NS equation [6]:

$$iu_t+u_{xx}+2|u|^2u=-i\gamma_0u+i\gamma_1u_{xx}+i\gamma_2|u|^2u-i\gamma_3|u|^4u \ , \quad (2)$$

where the damping ($\gamma_0,\gamma_1,\gamma_3$) and pumping ($\gamma_2$) coefficients are all positive.

One treats Eqs. (1) and (2) as perturbed NS equations if the dimensionless parameters γ_1, γ_2, and $\gamma_0\gamma_3$ are small. In the opposite case, the same equations can be regarded as the Ginzburg-Landau (GL) equations, which also have applications in plasma physics [3] and in hydrodynamics [7,8], and attract a great deal of attention as general models for pattern formation and onset of chaos [9].

An important object governed by Eqs. (1) and (2) is a solitary pulse (SP). It is known [1] that Eq. (1) with $\gamma_2=0$ has an exact SP solution; if $\gamma_2\neq0$ but γ_1 and γ_2 are small, the SP can be found approximately as a solution close to the soliton of the unperturbed NS equation with a fixed amplitude:

$$u=2i\eta\,\mathrm{sech}[2\eta(x-z_0)]\exp(4i\eta^2t-ik|x-z_0|+i\phi_0) \ , \quad (3)$$

$$\eta^2=\tfrac{3}{4}\gamma_0(\gamma_1+2\gamma_2)^{-1} \ , \quad k/\eta=\tfrac{2}{3}(2\gamma_1+\gamma_2) \ , \quad (4)$$

z_0 and ϕ_0 being arbitrary constants. The presence of the small wave number k produced by the perturbing terms implies that the asymptotic wave field of the soliton is oscillating in x, unlike that in the absence of perturbations. As for Eq. (2), in the near-NS regime ($\gamma_1,\gamma_2,\gamma_0\gamma_3\ll1$) it has the soliton solution in the form (3) with $k/\eta=\gamma_1-\gamma_0/4\eta^2$, where [10]

$$\eta^2=(64\gamma_2)^{-1}\{5(2\gamma_2-\gamma_1)+\sqrt{5[5(2\gamma_2-\gamma_1)^2-96\gamma_0\gamma_3]}\} \ . \quad (5)$$

In the opposite (near-GL) regime, Eq. (2) has a stable solution in the form of a broad SP [11,12].

The aim of this work is to demonstrate that, in both regimes, slightly overlapping SP's can form stable bound states (BS's). These can be two-pulse states, multipulse ones, and periodic arrays of SP's. This result may be important in applications. For instance, a casual formation of a two-soliton BS is detrimental for operation of fiber communication lines, therefore it is necessary to know how this can happen.

Note that the soliton solution of Eq. (1), given by Eqs. (3) and (4), is unstable as, at $|x|=\infty$, it coincides with the trivial unstable solution $u\equiv0$. However, this circumstance is not so important, at least in application to the optical solitons in fibers [4,5]. Anyway, the soliton solution of Eq. (2) given by Eqs. (3) and (5) is stable, and the general results obtained below apply as well to these stable solitons.

The interaction of the slightly overlapped solitons in the *unperturbed* NS equation was analyzed by means of the perturbation theory in Ref. [13]. To obtain an effective potential of the soliton-soliton interaction, it is sufficient to insert the linear superposition of the two unperturbed solitons into an exact expression for the energy of the system, and calculate a term produced by overlapping of each soliton with the "tail" of another one. It has been found [13] that the interaction potential in the unperturbed NS equation has no local minimum, so that it cannot give rise to a stable bound state of the two solitons. This inference accords with the well-known fact that the exact solution of the NS equation admits only unstable two-soliton and multisoliton states with zero binding energy [14].

The circumstance that drastically alters the situation for the slightly perturbed equation is that the tail of the soliton (3) contains the oscillating factor $\exp(-ik|x-z_0|)$. Let us reproduce the calculation of the effective potential, taking account of this factor. We will consider the interaction of two solitons (3) with equal amplitudes η. The solitons are separated by a large distance $z\equiv z_0^{(1)}-z_0^{(2)}$ $(z\eta\gg 1)$ and have a phase shift $\phi\equiv\phi_0^{(1)}-\phi_0^{(2)}$. The soliton-soliton interaction is accounted for by the term

$$H_{\rm int}=-\int_{-\infty}^{+\infty}|u(x)|^4dx \tag{6}$$

in the full Hamiltonian of the unperturbed NS equation. Following Ref. 13, one inserts the linear superposition of the two slightly overlapping solitons, $u=u_1+u_2$, into Eq. (6). It is easy to see that in the first approximation the effective potential U of the soliton-soliton interaction following from Eq. (6) after this substitution is the sum of two symmetric terms,

$$U=-4\int_{-\infty}^{+\infty}|u_1(x)|^2{\rm Re}[u_1(x)u_2^*(x)]dx+(1\rightleftarrows 2)\ , \tag{7}$$

where $u_1(x)$ is realized as the soliton wave form (3) with $z_0=0$. $\phi_0=0$, and $u_2(x)$ is the tail of the second soliton which can be taken in the form

$$u_2(x)=2i\eta\exp(4i\eta^2t-2\eta|x-z|-ik|x-z|+i\phi)\ . \tag{8}$$

Note that, when inserting the tail of the first soliton into the second term of the potential (7), one must change signs in front of z and ϕ in Eq. (8). Subsequent straightforward calculations yield the expression

$$U=-256\eta^3\exp(-2\eta z)\cos\phi\cos(kz)\ , \tag{9}$$

which coincides with the unperturbed effective potential if $k=0$.

The potential (9) has two sets of the stationary points:

$$\cos\phi=0\ ,\quad \cos(kz)=0\ , \tag{10}$$

$$\sin\phi=0\ ,\quad \cos(kz)+(k/2\eta)\sin(kz)=0\ . \tag{11}$$

The stationary states (10) are unstable (saddles) as their binding energy is exactly equal to zero, see Eq. (9). However, the states (11) are stable, provided $\cos\phi\cos(kz)>0$. Thus the oscillating potential (9), unlike the unperturbed one with $k=0$, gives rise to the set of stable two-soliton BS's with the distances between the solitons

$$z_n\approx(2n-1)\pi/2|k|\ ,\quad n=1,2,3,\cdots\ . \tag{12}$$

Note that the underlying assumption γ_1, γ_2, $\gamma_0\gamma_3\ll 1$ implies $|k|/\eta\ll 1$ [see, e.g., Eq. (4)], so that the bound solitons are indeed slightly overlapped, as it was presumed: $\eta z_n\gg 1$. By the same reason, the binding energy E of the BS is exponentially small:

$$E_n\equiv-U(z=z_n)$$
$$\approx 128|k|\eta^2\exp[-(2n-1)\pi\eta/|k|]\ . \tag{13}$$

Note that the potential energy was analyzed in the system which, strictly speaking, had no potential at all as it contained small dissipative terms. However, analysis of full equations of motion for the solitons' parameters, which is an exercise on the perturbation theory, leads to an effective equation of motion for a particle in the potential (9) in the presence of friction, so that the minima of the potential are stable indeed.

Alongside the two-soliton BS's, there may as well exist multisoliton ones, and also the BS's in the form of periodic or irregular arrays of solitons, with distances z_n between neighboring solitons.

Let us proceed to the GL regime. In this case it is convenient to rewrite Eq. (2) in the form of the GL equation proper:

$$v_t=-(1-\epsilon)v+(1+i\beta)v_{xx}+(1+i\alpha)|v|^2v$$
$$-\tfrac{3}{16}(1+i\delta)|v|^4v\ , \tag{14}$$

where α and β are small dispersive parameters, the additional one δ has been added for generality (it is implied $\alpha\sim\beta\sim\delta$), and ϵ is assumed small too. At $\alpha=\beta=\delta=\epsilon=0$, Eq. (14) has the exact kink solution.

$$v_0(x)=2\exp(i\phi_0)\{1+\exp[-2\sigma(x-z_0)]\}^{-1/2}\ ,$$
$$\sigma=\pm 1\ , \tag{15}$$

where ϕ_0 and z_0 are arbitrary constants, cf. Eq. (3). As has been demonstrated in Ref. 12 (see also Ref. 11), when the perturbing parameters in Eq. (14) are different from zero, the kink ($\sigma=+1$) and antikink ($\sigma=-1$) can form a large-size SP, provided $\epsilon\sim\alpha^2$. The frequency of the SP is $\omega=4\alpha-3\delta$, and the local wave number in a vicinity of each kink is

$$k(x)=\sigma[A+B|v_0(x)|^2]\ ,$$
$$A=2\alpha-(\beta+3\delta)/2\ ,\quad B=3(\beta-\delta)/16\ . \tag{16}$$

Thus the kink has the SP on one side of it, and on another side the wave field falls off exponentially, cf. Eq. (8):

$$v(x)=2\exp(i\phi_0)\exp[i\omega t-(1+iA)|x-z_0|]\ . \tag{17}$$

Again, Eq. (17) tells us that the small perturbing terms render the tail of the SP oscillating.

Let us now recollect that, in the case $\alpha=\beta=\delta=0$, Eq. (14) can be presented in the gradient form, $v_t=-\delta L/\delta v$, where the Lyapunov functional is

$$L=\int_{-\infty}^{+\infty}dx[(1-\epsilon)|v|^2+|v_x|^2-\tfrac{1}{2}|v|^4+\tfrac{1}{16}|v|^6]\ . \tag{18}$$

Stable configurations correspond to minima of L. We will consider the interaction of two SP's with a relative phase ϕ, which are separated by a large distance $z\gg 1$. One can again insert the linear superposition of the two slightly overlapping SP's into Eq. (18). The lowest-order term that accounts for their interaction is [cf. Eq. (7)]

$$U=2\int_{-\infty}^{+\infty}[\tfrac{3}{16}|v_1(x)|^2-1]|v_1(x)|^2$$
$$\times{\rm Re}[v_1(x)v_2^*(x)]dx+(1\rightleftarrows 2)\ , \tag{19}$$

where $v_1(x)$ is the kink (15) with $z_0=0$, $\phi_0=0$ times $\exp(i\omega t)$, and $v_2(x)$ is the tail (17) of the adjacent antikink (belonging to the other SP), with $z_0=z$, $\phi_0=\phi$. The eventual result is [cf. Eq. (9)]

$$U=-32\exp(-z)\cos\phi\cdot\cos(Az)\ , \tag{20}$$

where A is defined by Eq. (16). Evidently, the effective *pseudopotential* (20) has the set of stable equilibria with $\sin\phi=0$, $\cos(Az)+A\sin(Az)=0$ [cf. Eq. (11)], i.e., with

$$z_n=(2n-1)\pi/2|A|\ ,\quad n=1,2,3,\cdots \tag{21}$$

[cf. Eq. (12)]. The "margin of stability" of the bound states, characterized by the value of the pseudopotential (20) at $z=z_n$, is again exponentially small [cf. Eq. (13)]:

$$E_n\equiv -U(z=z_n)$$
$$\approx 32|A|\exp[-(2n-1)\pi/2|A|]\ . \tag{22}$$

Like the solitons in the NS regime, in the GL regime the SP's may form multipulse BS's and periodic or irregular arrays alongside the pairwise BS's.

To analyze the interaction of two SP's, it was assumed $z\gg 1$, but z need not be large as compared to the proper size l of the SP. l is large too, but it is governed by another small parameter and it is uniquely determined [12] unlike the distance z_n which depends on the arbitrary integer n.

The analytical treatment made it possible to reveal the BS's whose "margin of stability" was exponentially narrow, see Eqs. (13) and (22). The fundamental reason for this was that the dimensionless parameters γ_1, γ_2, and $\gamma_0\gamma_3$ in the underlying Eqs. (1) and (2) had to be assumed either small or large. However, in the intermediate case γ_1, γ_2, $\gamma_0\,\gamma_3\sim 1$ (when the SP's can only be investigated numerically [15]) the BS's, if any, could be more robust.

In many cases, the generalized NS equation must include additional terms which account for higher dispersion. For instance, the nonlinear optical fibers usually operate in a spectral range near the zero of the first dispersion; in this case, the second dispersion must be taken into account, i.e., the term $i\zeta u_{xxx}$ with real ζ must be added to the right-hand side of Eq. (1) [16]. This term gives rise to the additional oscillating factor $\exp[iq(x-z)]$, $q=2\zeta\eta^2$, in the soliton's asymptotic (8). The crucial difference from the previous factor $\exp(ik|x-z|)$ is that we have $(x-z)$ instead of $|x-z|$. After straightforward calculations, one can see that all the difference introduced by the new factor is the change of $\cos\phi$ in the potential (9) to $\cos(\phi+qz)$. Eventually, this amounts to the fact that the value of ϕ in the stationary states is determined by the equation $\sin(\phi+qz)=0$ instead of $\sin\phi=0$, see Eq. (11). The higher dispersion does not influence the stability and binding energies of the BS's; in particular, the BS's are absent if $q\neq 0$ but $k=0$. The same pertains to the "skew" terms, like $i\zeta u_{xxx}$, added to the basic (this time, dissipative) part of the GL equation (14)

In conclusion, let us briefly discuss feasible experimental manifestations of the effect revealed. A plausible object that could be interpreted as a soliton in a nonlinear system combining the dispersion and dissipation is the quasi-one-dimensional (strongly scratched) localized spot of convection in a layer of a liquid crystal heated from below, discovered in Ref. [17]. One might try to interpret a stationary pattern of the spots observed in Ref. [17] as a multipulse BS. Another interesting object is the localized convection pulse observed in a binary liquid filling a narrow annular channel [18]. Interaction of two pulses in this system was recently studied in Ref. [19]. It was demonstrated that, when the two pulses are not far from each other, they suffer a slow fusion into one pulse. It remains to be understood if this interaction can be described within the framework of the approach developed in the present paper.

Note added in proof. In recent work by P. Kolodner [Phys. Rev. A **44**, 6448 (1991); **44**, 6466 (1991)], results of a more accurate experimental study of collisions between counterpropagating pulses in the annular convection channel were reported. It has been demonstrated that the collision may result in the formation of a stable bound state of the pulses similar to the one described in the present work, provided the relative velocity of the colliding pulses is sufficiently small. Although the counterpropagating waves should be described by a system of *two* coupled NS-GL equations, the mechanism analyzed here, i.e., that based on the interaction of the pulse with the spatially oscillating tail of the mate pulse, is fairly universal, and it must as well account for formation of the BS in the system of two coupled equations. It seems also worthy to mention the ac-driven damped NS model [D. J. Kaup and A. C. Newell, Phys. Rev. B **18**, 5162 (1978)] $iu_t+u_{xx}+2|u|^2u=-i\gamma_0u+\epsilon\cos(\omega t)$. As is well known, this model admits stable solitons with the amplitude $\eta=\sqrt{\omega}/2$, phase locked to the ac drive. Following the lines of the analysis developed above, it is straightforward to see that these solitons can form the BS's at the distances $z_n\approx 2(2n-1)\pi\eta/\gamma_0$, the stable ones corresponding to odd n (in this case, the phase difference ϕ between the two solitons is always zero, as both are phase locked to the drive).

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[1] N. R. Pereira and L. Stenflo, Phys. Fluids **20**, 1733 (1977).

[2] N. R. Pereira and F. Y. F. Chu, Phys. Fluids **22**, 874 (1979).

[3] A. L. Fabrikant, Zh. Eksp. Teor. Fiz. **86**, 470 (1984) [Sov. Phys. JETP **59**, 274 (1984)].

[4] K. J. Blow, N.J. Doran, and D. Wood, Opt. Lett. **12**, 1011 (1987).

[5] A. Hoök, D. Anderson, and M. Lisak, Opt. Lett. **13**, 1114 (1988).

[6] V. I. Petviashvili and A. M. Sergeev, Dok. Akad. Nauk SSSR **276**, 1380 (1984) [Sov. Phys. Dokl. **29**, 493 (1984)].

[7] K. Stuartson and J. T. Stuart, J. Fluid Mech. **48**, 529 (1971).

[8] P. C. DiPrima, W. Eckhaus, and L. A. Segel, J. Fluid Mech. **48**, 705 (1971).

[9] Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. **55**, 356 (1976); H. T. Moon, P. Huerre, and L. P. Redekopp, Phy-

sica D **7**, 135 (1983); C. D. Doering, J. D. Gibbon, D. D. Holm, and B. Nicolaenko, Nonlinearity **1**, 279 (1988); B. A. Malomed and A. A. Nepomnyashchy, Phys. Rev. A **42**, 6238 (1990).
[10] B. A. Malomed, Physica D **29**, 155 (1987); S. Fauve and O. Thual, Phys. Rev. Lett. **64**, 282 (1990).
[11] W. van Saarlos and P. Hohenberg, Phys. Rev. Lett. **64**, 749 (1990); V. Hakim, P. Jakobsen, and Y. Pomeau, Europhys. Lett. **11**, 19 (1990).
[12] B. A. Malomed and A. A. Nepomnyashchy, Phys. Rev. A **42**, 6009 (1990).
[13] V. I. Karpman and S. S. Solov'ev, Physica D **3**, 487 (1981).
[14] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].
[15] O. Thual and S. Fauve, J. Phys. (Paris) **49**, 1829 (1988).
[16] P. K. Wai, C. R. Menyuk, Y. C. Lee, and H. H. Chen, Opt. Lett. **11**, 464 (1986).
[17] R. Ribotta and A. Joets, Phys. Rev. Lett. **60**, 2164 (1988).
[18] J. J. Niemela, G. Ahlers, and D. S. Cannell, Phys. Rev. Lett. **64**, 1365 (1990); K. E. Anderson and R. P. Behringer, Phys. Lett. A **145**, 323 (1990).
[19] J. A. Glazier and P. Kolodner, Phys. Rev. A **43**, 4269 (1991).