Equilibrium behavior of a Brownian particle in a random environment

S. Stepanow and T. Kraft
Technische Hochschule Merseburg, Fachbereich Physik, O-4200 Merseburg, Germany
(Received 11 June 1991)

Behavior of a Brownian particle confined in a harmonic potential that is disturbed by a random environment is considered. The effect of the environment on the mean-square displacement, the relaxation time, and the equilibrium distribution function are studied within the renormalization-group method to the first order of $\epsilon=2-d$ (where d is the space dimensionality).

PACS number(s): 05.40.+j, 02.50.+s

In recent years there has been an increasing interest in studying the diffusive behavior in random environments [1–9]. Little attention has been paid to the study of the equilibrium properties of random walks in random environments. To our knowledge, there are only the papers of Refs. [12] and [13] to this subject. In this Brief Report we present the results of the study of the behavior of a Brownian particle confined in a harmonic potential that is disturbed by a random environment. This problem differs from that considered in [13] where the particle is confined in a box. The advantage of the harmonic potential is that it enables one an analytical consideration analogous to that in the diffusive regime. The problem considered here is connected to the diffusion problem: If the strength of the harmonic potential tends to zero, we return to the diffusion problem. We will show the differences between these two related problems. The study of the equilibrium behavior of the Brownian particle enables one to make conclusions about the relaxation behavior of the latter.

The probability distribution $P(x,t,0,0)$ for the particle obeys the equation

$$\partial_t P=D_0\Delta P=\nabla(FP) \tag{1}$$

with

$$F=\mu_0 x+F(x) \tag{2}$$

being the force in unity of kT. $F(x)$ is a random force with the correlation function

$$\langle F^{\mu}(x)F^{\nu}(x')\rangle=c^{\mu\nu}(x-x') , \tag{3}$$

where μ and ν denote the Cartesian coordinates of the force. In the present paper we use the short-ranged force-force correlation function whose Fourier transform is given by

$$C^{\mu\nu}(q)=\Delta_0\delta^{\mu\nu} . \tag{4}$$

The transition probability $P(x,t,0,0)$ averaged over the random force can be represented by means of diagrams [10,11]. The computation of the quantities under consideration is straightforward. In this paper, we will compute the following quantities. The mean-square displacement of the particle

$$x^2=\lim_{t\to\infty}\langle x^2(t)\rangle , \tag{5a}$$

the mean of the product of the displacements of two particles

$$x_{12}=\lim_{t\to\infty}\langle x_1^{\mu}(t)x_2^{\nu}(t)\rangle , \tag{5b}$$

and the Fourier transform of the transition probability averaged over the disorder

$$P(p)=\lim_{t\to\infty}\int_x P(x,t)e^{-ipx} . \tag{5c}$$

The positions of the particles at time $t=0$ are supposed to be $x_1(0)=x_2(0)=0$.

The result of the computation of these quantities up to first order in strength of the disorder is

$$x^2=(d/\mu_0)\left[1+\frac{1}{\epsilon}\Delta_0\mu_0^{-\epsilon/2}\right] , \tag{6}$$

$$x_{12}=(d/\mu_0)\Delta_0\mu_0^{-\epsilon/2}[1+O(\epsilon)] , \tag{7}$$

$$P(p)=e^{-p^2/(2\mu_0)}\left[1-\frac{1}{\epsilon}\Delta_0\mu_0^{-\epsilon/2}\frac{p^2}{2\mu_0}\right] , \tag{8}$$

with $\epsilon=2-d$.

It is interesting that the average of the positions of two particles is not zero. This correlation appears because the particles prefer and avoid the same space regions. It is interesting to compare (6) and (7) with the corresponding quantities when μ_0 is zero. In this case the infrared singularities are controlled by the time and as a consequence the limit $t\to\infty$ cannot be carried out. When μ_0 is not zero, then the infrared singularities would be controlled by μ_0 and the limit $t\to\infty$ is regular. In this sense the problem with the harmonic potential is complementary to the diffusion problem.

The quantity $1/\mu_0$ describes the relaxation of the particle as it can be seen from the following relation:

$$\langle x^2(t)\rangle=\langle x^2\rangle_{\text{eq}}(1-e^{-\mu_0 t})+x(0)^2e^{-\mu_0 t} . \tag{9}$$

$1/\mu_0$ is the relaxation time.

The analysis of the bare perturbation expansions (6) and (7) can be performed by means of the renormalization group [11,14]. In accordance with the general receipt we must eliminate the $1/\epsilon$ poles from the perturbation expansions. This can be achieved by the redefining of the parameter μ_0 and Δ_0. Up to the first order of Δ_0 we obtain from (6) and (7)

$$\frac{1}{\mu'}=\frac{1}{\mu_0}\left[1+\frac{1}{\epsilon}\Delta_0\left[\frac{1}{\mu_0}\right]^{\epsilon/2}\right], \tag{10}$$

$$\Delta=\Delta_0\left[1-\frac{1}{\epsilon}\Delta_0\left[\frac{1}{\mu_0}\right]^{\epsilon/2}\right]. \tag{11}$$

In order to make (10) and (11) finite for $d=2$ we introduce the cutoff λ as follows:

$$\Delta_0\frac{1}{\epsilon}\mu_0^{-\epsilon/2}\rightarrow\frac{\Delta^0}{\epsilon}(\mu_0^{-\epsilon/2}-\lambda^{\epsilon}). \tag{12}$$

Introducing $y=1/\mu$ and the dimensionless interaction constant $g=\Delta\lambda'^{\epsilon}$ we obtain the differential equations of the renormalization group (RG) as follows:

$$\frac{\partial \ln y}{\partial \ln\lambda}=g, \tag{13}$$

$$\frac{\partial q}{\partial \ln\lambda}=\epsilon g-g^2. \tag{14}$$

It is interesting to compare (13) and (14) with the corresponding RG equations in the diffusion limit ($\mu_0\rightarrow 0$). The Gell-Mann–Low equation (14) coincides with that in the diffusion problem. The renormalization of x^2 is in contrast with the renormalization of $x^2(t)$ of the diffusion problem. This occurs because both problems have different symmetry. The fixed point value of g is $g^*=\epsilon$. The solution of (13) and (14) is

$$\Delta=\frac{\Delta_0}{1+\frac{1}{\epsilon}\Delta_0\lambda'^{\epsilon}}, \tag{15}$$

$$y=y_0\left[1+\frac{1}{\epsilon}\Delta_0\lambda'^{\epsilon}\right]. \tag{16}$$

The final value of the parameter of the RG, λ_m, has to fulfill the matching condition [15]

$$\lambda_m^2=\frac{1}{\mu(\lambda_m)}. \tag{17}$$

By using (16) and (17) we obtain in the scaling limit ($\Delta_0\rightarrow\infty, \mu_0\rightarrow 0$)

$$\frac{1}{\mu}=\left[\frac{1}{\epsilon}\Delta_0\right]^{1+\epsilon/2}\left[\frac{1}{\mu_0}\right]^{1+\epsilon/2}\simeq\left[\frac{1}{\mu_0}\right]^{2\nu} \tag{18}$$

with $2\nu=1+\epsilon/2$.

The latter shows that the random environment weakens the strength of the harmonic potential. By remembering that $1/\mu_0$ is the relaxation time, we conclude that the disorder defined by the correlation function (3) increases the relaxation time.

The quantities $\langle x^2\rangle$ and $\langle x_1^{\mu}x_2^{\mu}\rangle$ are given up to first order in ϵ as

$$\langle x^2\rangle\simeq\frac{d}{\mu}\simeq\left[\frac{1}{\mu_0}\right]^{2\nu}, \tag{19}$$

$$\langle x_1x_2\rangle\simeq\frac{d}{\mu}g\simeq d\epsilon\left[\frac{1}{\mu_0}\right]^{2\nu}. \tag{20}$$

It is surprising that the average (20) differs from zero. In the diffusion problem such an average has been considered in [11]. It is interesting if such correlations can be measured.

There appear interesting questions if one attempts to draw a parallel between the Brownian particles and electrons in a random environment. The correlation effects in behavior of different particles in the same environment seems to be independent of the dynamics governing the behavior of the particles. As well as the Brownian particles, the electrons would prefer (avoid) the same places in the space. Therefore, in case of (delocalized) electrons one would expect an effective attraction between the electrons caused by the disorder. Nevertheless the direct calculations are necessary in order to check this conjecture.

The renormalized distribution function $P(p)$ is obtained to the first order of ϵ as

$$P(p)=e^{-\frac{p^2}{2\mu}} \tag{21}$$

with the only change being the strength of the harmonic potential. Therefore the equilibrium distribution function of the Brownian particle in the disorder under consideration remains Gaussian. We note that it was not possible for us to compute analytically the Renyi entropies considered in [13]. We note that the Renyi entropy

$$H_q=\langle(1-q)^{-1}\ln\int_{-\infty}^{\infty}dx\,P^q(x)\rangle$$

associated with (21) is given by

$$H_q=\tfrac{1}{2}\ln q/(q-1)+\tfrac{1}{2}\ln(2\pi/\mu).$$

The dependence of H_q on q is in agreement with the result obtained in [13].

In conclusion we considered the Brownian particle confined in harmonic potential which is disturbed by a random environment. We have shown that the only result of the disorder is the renormalization of strength of the harmonic potential. The random environment under consideration weakens the strength of the potential. As a consequence of this the relaxation time increases. The equilibrium distribution function of the particle remains Gaussian.

[1] Y. Sinai, Teor. Veroyatn. Ee Prilosh. **27**, 247 (1982).
[2] E. Marinari, G. Parisi, D. Ruelle, and P. Windey, Phys. Rev. Lett. **50**, 1223 (1983).
[3] D. S. Fisher Phys. Rev. A **30**, 960 (1984).
[4] D. S. Fisher, D. Friedan, Z. Qiu, S. J. Shenker, and S. H. Shenker, Phys. Rev. A **31**, 3841 (1985).
[5] S. Obukhov, Zh. Eksp. Teor. Fiz. **39**, 21 (1984) [JETP Lett. **39**, 23 (1984)].

[6] J. Honkonen and E. Karjalainen, Phys. Lett. A **129**, 338 (1988).
[7] V. E. Kravtsov, I. V. Lerner, and V. I. Youdson, J. Phys. A **18**, L703 (1985).
[8] J. Honkonen and Y. M. Pis'mak (unpublished).
[9] V. K. Dugaev and K. D. Tovstyuk, Phys. Status Solidi B **160**, 415 (1990).
[10] S. Stepanow and J. -U. Sommer, J. Phys. A **23**, L541 (1990).
[11] S. Stepanow, Phys. Rev. A **43**, 2771 (1991).
[12] H. E. Roman, A. Bunde, and S. Havlin, Phys. Rev. A **38**, 2185 (1988).
[13] P. Grassberger and R. Leuverink, J. Phys. A **23**, 773 (1990).
[14] D. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (McGraw-Hill, New York, 1978).
[15] A. D. Bruce and D. J. Wallace, J. Phys. A **9**, 1117 (1976).