Class of stable multistate time-reversible cellular automata with rich particle content

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An alternative class of stable multistate cellular automata (CA) is derived. They serve to significantly extend the two-state, time-irreversible CA proposed by Park, Steiglitz, and Thurston [Physica D **19**, 423 (1986)]. These CA are time-reversible dynamical systems possessing an enormous array of coherent particlelike structures. The particle-interaction picture is surprisingly rich and includes particle production as well.

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In recent years, there has been a significant interest in the various properties and applications of cellular automata (CA). From a general point of view, CA can be described as a dynamical system in discrete space and time whose field variables take only finitely many values (say 0 and 1, or values in a finite field). Many workers now believe that CA, due to its simplicity and remarkable properties, may play a rather important (if not fundamental) role in describing nature (cf. [1] for a review). Important applications include such fields as fluid dynamics, chemical and biological modeling, etc., and there already is a large body of literature discussing numerous computer simulations of stochastic and intrinsic properties of CA (cf. [2]). However, many of these properties were not studied theoretically due to the extreme nonlinearity of the models.

In [3], a remarkable CA, called the parity-rule-filter automata (PRFA), which exhibits a number of important properties, was proposed. These include a wide range of particlelike structures and a subclass of particles with solitonlike behavior. The PRFA has been studied extensively by a number of authors [4–6]. However, the lack of time reversibility in the PRFA suggests that there may yet be a more fundamental underlying CA, sharing many of its desirable features.

In the present paper we will show that this is indeed the case. We will derive a new class of CA which significantly extends the PRFA, is time reversible, possesses an enormous particle content (including all particles in the PRFA itself), and exhibits rich interaction phenomena including particle production.

In order to make this paper self-contained, we begin by summarizing the definitions and the basic results for the PRFA. This is a $1+1$ dynamical system with discrete space-time $\mathbb{Z}\times\mathbb{Z}$ (here $\mathbb{Z}$ denotes the set of all integers) with a field variable $x_i^t\in\{0,1\}$, i, $t\in\mathbb{Z}$. It is assumed that the initial data, $C^0=\{x_i^0\}_{i\in\mathbb{Z}}$ at $t=0$, is localized (i.e., C^0 contains finitely many 1's). Then the time evolution (i.e., the equations of motion) is given by the following parity rule:

$$x_i^{t+1}=\begin{cases}1 & \text{if } S_{i,2}^t \text{ is even and nonzero}\\ 0 & \text{if } S_{i,2}^t \text{ is odd or zero ,}\end{cases} \tag{1}$$

where $S_{i,2}^t=\sum_{j=1}^r x_{t-j}^{t+1}+\sum_{j=0}^r x_{i+j}^t$, and the integer parameter $r\geq 1$, called the radius, describes the range of interaction. This rule is clearly implicit and requires sweeping from the left ($x_i^t=0$ for all $i<N$ for some $N\in\mathbb{Z}$), which is consistent with localized initial data.

An equivalent explicit formulation of the PRFA, the so-called fast rule theorem (FRT), was given in [5], and consists of the following. First, for every t define a subset $B(t)\subset\mathbb{Z}$ by the following inductive procedure: (i) sweeping from the left, place the index (spatial coordinate) of the first nonzero site among $C^t=\{x_i^t\}_{i\in\mathbb{Z}}$ in $B(t)$; (ii) in increments of $r+1$, place $i+r+1$ in $B(t)$, if there is at least one nonzero x_i^t in the r sites to the right of the current index in $B(t)$; (iii) if there are r zero consecutive values of x_i^t to the right of the current index in $B(t)$, go on to the next nonzero x_i^t, place the corresponding index in $B(t)$, and repeat steps (ii) and (iii). The time evolution of the automata according to the FRT is then given by

$$x_{i-r}^{t+1}=\begin{cases}x_i^t & \text{if } i\notin B(t)\\ 1-x_i^t & \text{if } i\in B(t)\ .\end{cases} \tag{2}$$

The FRT can be used effectively to demonstrate that the PRFA is a particularly rich but yet simple dynamical system. The field variables are then $x_i^t\in\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$; i.e., a finite field of two elements, 0 and 1. It has the following properties (see [5]–[8]).

(I) *Stability.* For every t the set $B(t)$ is finite, and therefore at every time step there are finitely many 1's. Moreover, as t increases, the position of the rightmost 1 does not move to the right, so that the field configuration $C^t=\{x_i^t\}_{i\in\mathbb{Z}}$ moves to the left. Observe that by Eq. (2), a Galilean shift is always available.

(II) *Time irreversibility.* As a dynamical system the PRFA contains dissipation, i.e., is nonconservative, and is therefore time irreversible. The simplest initial data, a single 1 in an infinity of zeros, evolves to the zero configuration in one time step. This state is called a prenull. More generally, if the configuration C^t contains prenulls, i.e., there exists (at least one) $i\in B(t)$ such that $x_i^t=1$ and $x_{i+j}^t=0$, $j=1,2,\ldots,r$, then the one-step evolution $C^t\rightarrow C^{t+1}$ is irreversible; i.e., the state C^t cannot be uniquely determined from C^{t+1} by "going backwards." Call $\mathcal{C}_r$ the (reversible) subset of set $\mathcal{C}$ of all initial data consisting of those states in which there appears no prenull in evolution forward and backward in time, and let $\mathcal{C}_i$ (irreversible) be its complement in $\mathcal{C}$. Then it follows from the FRT that the restriction of the PRFA to C_r is time reversible, whereas its restriction to C_i is irreversible. The irreversibility is responsible for the complications discussed in part (IV-2) below.

(III) *Particle content.* The time evolution of PRFA exhibits many different coherent structures. Each is represented by initial data $C^0=\{x_i^0\}_{i\in\mathbb{Z}}$, which evolves in a time-reversible way as a freely moving localized object; i.e., there are integers p, called the period, and d, called the displacement, such that for all $i\in\mathbb{Z}$, $x_{i-d}^{t+p}=x_i^t$, such modes are called particles. The time evolution of a particle is translation to the left by d units after p successive time steps. Its velocity is defined as $v=d/p$, where $0\le v\le r-1$. When $p=1$ the particle moves like a pointlike object; a localized wave retaining its "shape." Particles with $p>1$ have additional internal structure (inner degrees of freedom) that exhibits itself during the period. There exist many particles with various p and d, and a systematic way of constructing them via linear difference equations is given in [8]. Among all particles, one can distinguish basic particles that behave like solitons [7]. These correspond to initial data C^0 localized in the space interval of length $r+1$ units; the zero configuration and prenull are excluded.

(IV) *Interaction and scattering of particles.* This concept refers to the large-time behavior of the initial data consisting of spatially separated particles. The corresponding asymptotic picture is described by different coherent structures—asymptotic states—which depend on the interaction between particles. There are two major cases.

(IV-1) *Solitonic (time-reversible and elastic) scattering.* This process describes scattering of basic particles and is similar to the scattering in $1+1$ integrable solitonic dynamical systems, namely, the initial state C^0 consisting of spatially separated basic particles $A_1, A_2, \ldots, A_n$ ordered by their velocities $v_1<v_2<\ldots,v_n$ evolves through a complicated interaction into an asymptotic state C^∞ describing the free motion of the same particles $A_n,\ldots,A_2,A_1$ arranged in the opposite order.

(IV-2) *Nonsolitonic (time-irreversible) scattering.* This phenomenon is due to the existence of the set $\mathcal{C}_i$ of time-irreversible initial data [property (II) above]. In the process of evolution, the states from $\mathcal{C}_i$ produce "false" states, containing prenulls, which die in the next time step (the prenull state is responsible for the nonconservation of the energy-type functional introduced in [6]). There are three typical examples of the nonsolitonic scattering.

(a) *Gluing.* particles A and B interact and form another particle C (see Fig. 1);

(b) *Inelastic.* Scattering of two particles A and B results in two different particles A' and B', neither of which is A or B;

(c) *Irreversible solitonic.* Scattering of two particles A and B results in the same particles B and A, but this process, due to the appearance of prenulls, is not time reversible.

Thus we have seen that the PRFA possesses many remarkable properties, but, fundamentally speaking, it lacks time reversibility, an issue we address next.

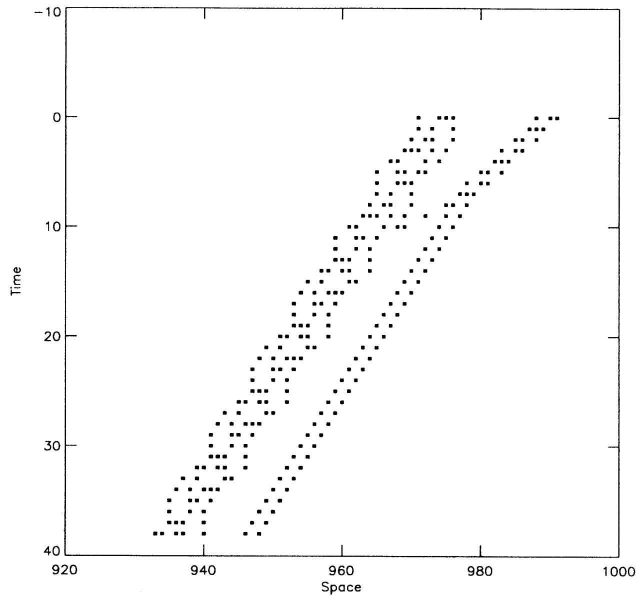

FIG. 1. Evolution depicting a bound-state irreversible interaction of Eq. (1) or (2) with $r=3$ (40 time steps).

Notice first that the parity rule Eq. (1) is equivalent to the following difference relation in the finite field $\mathbb{F}_2=\mathbb{Z}/2\mathbb{Z}$:

$$\sum_{j=0}^{r} x_{i-j}^{t+1} \equiv \sum_{j=0}^{r} x_{i+j}^{t} + \delta_2(x_i^t)\prod_{j=1}^{r}\delta_2(x_{i-j}^{t+1})\delta_2(x_{i+j}^{t}) - 1 \; . \qquad (3)$$

Here equality ($\equiv$) is understood mod 2, i.e., $(1+1)$ mod $2=1+1\equiv 0$, and $\delta_q(x)$ is a "δ function mod q" ; i.e., $\delta_2(x)=1$ if $x=0$ mod 2 and 0 otherwise. Rewriting Eq. (3) in the form

$$x_i^{t+1} \equiv S_{i,2}^t + \delta_2(x_i^t)\prod_{j=1}\delta_2(x_{i-j}^{t+1})\delta_2(x_{i+j}^{t}) - 1 \; ,$$

we can interpret $S_{i,2}^t$ as a linear term and the rest as a highly nonlinear perturbation.

We note that Eq. (3) admits a natural multistate generalization. Consider the finite ring $\mathbb{F}_q=\mathbb{Z}/q\mathbb{Z}$ consisting of q elements $0,1\ldots,q-1$ ($\mathbb{F}_q$ is a field if q is prime), $x_i^t\in\mathbb{F}_q$ and define its time evolution by

$$\sum_{j=0}^{r} x_{i-j}^{t+1} \equiv \sum_{j=0}^{r} x_{i+j}^{t} + \delta_q(x_i^t)\prod_{j=1}^{r}\delta_q(x_{i-j}^{t+1})\delta_q(x_{i+j}^{t}) - 1 \; , \qquad (4)$$

where now equality ($\equiv$) is understood mod q. Alternatively, (4) is equivalent to the following q-state parity rule:

$$x_i^{t+1} = \begin{cases} (S_{i,q}^t-1)\bmod q & \text{if } S_{i,q}^t\neq 0 \\ 0 \;\; \text{if } S_{i,q}^t=0 \; , \end{cases} \qquad (5)$$

where now $S_{i,q}^t=(q-1)\sum_{j=1}^{r}x_{i-j}^{t+1}+\sum_{j=0}^{r}x_{i+j}^t$, and (5) is equivalent to the following q-state fast rule theorem:

$$x_{i-r}^{t+1} = \begin{cases} x_i^t \;\; \text{if } i\notin B(t) \\ (x_i^t-1)\bmod q \;\; \text{if } i\in B(t) \; , \end{cases} \qquad (6)$$

where the set $B(t)$ was defined earlier by (i)–(iii).

This multistate CA has all of the properties (I)–(IV) of the PRFA (with yet a wider particle content); but as with the PRFA it is time irreversible; i.e., a prenull vanishes at the next time step. An inspection of Eq. (3) shows that the reason for dissipation and time irreversibility is due to the presence of the factor $\delta_2(x_i^t)$ in the product, which violates the symmetry $x_{i+j}^t \leftrightarrow x_{i-j}^{t+1}$. Thus we will define a new two-state CA by the following difference relation:

$$\sum_{j=0}^{r} x_{i-j}^{t+1} \equiv \sum_{j=0}^{r} x_{i+j}^{t} + \prod_{j=1}^{r} \delta_2(x_{i-j}^{t+1})\delta_2(x_{i+j}^{t}) - 1 \ , \qquad (7)$$

which has the symmetry $x_{i+j}^t \leftrightarrow x_{i-j}^{t+1}$, and therefore is time reversible. Backward evolution in time, from C^{t+1} to C^t, is given by the same Eq. (7) which we solve by sweeping from the right. Equivalently, the time evolution in this new CA can be described by the reversible parity rule

$$x_i^{t+1} = \begin{cases} 1 & \text{if } S_{i,2}^t \text{ is even and nonzero or } x_i^t = S_{i,2}^t = 1 \\ 0 & \text{otherwise,} \end{cases} \qquad (8)$$

and alternatively by the reversible fast rule theorem (RFRT), given exactly by Eq. (2), where the set $B(t)$ should be replaced by the set $\tilde{B}(t)$ defined as follows: steps (i) and (ii) in the definition of $B(t)$ remain unchanged, whereas step (iii) is replaced by the following:

(iii)′ If $i \in \tilde{B}(t)$ and $x_{i+j}^t = 0$, $j = 1, \ldots, r$, then either $i + r \in \tilde{B}(t)$ if $x_i^t = 1$ or, if $x_i^t = 0$, the index of the next nonzero x_i^t in C^t belongs to $\tilde{B}(t)$.

In particular, the RFRT shows that the new CA is stable. From Eqs. (7) and (8) it follows that restriction of the new CA to the subset $\mathcal{C}_r$ of initial data coincides with the PRFA, therefore it possesses all the desirable properties (I)–(IV-1). Moreover, due to time reversibility, it "cures" the bad property (IV-2) of the PRFA. Namely, the time evolution according to Eqs. (7) and (8) shows that in this new CA the prenulls do not vanish, but instead are now elementary basic particle solutions stationary in time. They represent the simplest of all particles, which we call prebasic particles. With these particles, the reversible rule possesses an infinite collection of particles, with highly nontrivial interaction properties.

For instance, even in the simplest case $r = 1$ (which is totally trivial for the PRFA), appropriate initial data produce particles with zero velocity and arbitrarily large periods exhibiting interesting and complicated internal motion. In the case $r \geq 2$, the situation is more dramatic: in particular, initial data consisting of certain sequences of consecutive 1's (with lengths belonging to certain infinite arithmetic progressions depending on r) give rise to particles with arbitrarily large periods.

In addition to the solitonic interactions covered by case (IV-1), the new CA exhibits highly non-trivial particle interactions. We note the change from the "gluing" situation that occurs in the irreversible rule to particle production in the reversible rule. In Figs. 1 and 2 we take the same initial conditions; namely AOB: 100 111, $z(11)$, 1011, with radius $r = 3$ [$z(11)$ denotes 11 zeroes]. This represents two particles: A, $v = 1$, $p = 6$, and B, $v = \frac{5}{3}$,

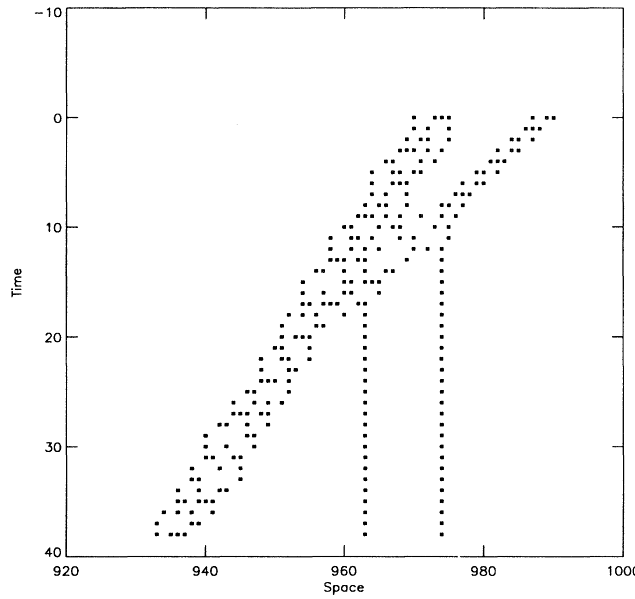

FIG. 2. Evolution depicting particle production defined by Eq. (7) or (8), or the RFRT with $r = 3$ (40 time steps).

$p = 3$, which eventually collide. The irreversible rule produces a final "glued" state of a single particle of velocity 1, period 6, whereas the reversible rule results in *three* particles; a large one with $v = 1$, $p = 18$ moving away from two stationary (and well separated) prebasic particles.

This example illustrates that scattering of particles A and B results in a new particle C and a number of emitted prebasic particles. Reversing the time order implies that particle C bombarded by a beam of prebasic particles splits into two particles A and B. Needless to say, this picture is very suggestive: prebasic particles playing various roles, in particular being responsible for particle production. In a sense, this new CA might serve as a "toy" model of fundamental physics.

The next example demonstrates that both rules admit (different) nonsolitonic interactions. We take for both the initial conditions AOB: 11 000 001 011, $z(11)$, 1011, with $r = 5$. This represents two particles, with A having $v = \frac{11}{4}$ and $p = 8$, and B having $v = 3$, $p = 3$. The interaction results in two different (well separated) particles $A'O'B'$. In the irreversible rule A' has $V = \frac{11}{4}$, $p = 4$ and B' has $v = 2$, $p = 2$, whereas in the reversible rule A' has $v = \frac{54}{21}$, $p = 42$, and B' has $v = 2$, $p = 30$.

The final example is startling. We take as initial conditions AOB: 101 101 101, $z(14)$, 1111 010 111, with $r = 4$. This represents two particles: A, $v = \frac{17}{8}$, $p = 8$, B, $v = \frac{5}{2}$, $p = 10$. The interaction picture is entirely different. In the irreversible rule we have an "almost" solitonic picture $A'O'B'$ where $A' = B$, $B' = A$; however, we *cannot* reverse time and recover the initial conditions, since a prenull is produced in the evolution. This is an example of an irreversible solitonic interaction (in a sense there is a loss of phase information, i.e., a "phase shock"). On the other hand, the reversible rule produces 20 particles (we calculated this evolution to 23 000 time steps): eight nontrivial ones and 12 prebasic particles. The eight well-separated particles actually consist of six different particles. The eight particles have the following characteristics:

$$v=\tfrac{7}{3}\ ,\quad p=3 \text{ for } P_1,\quad v=\tfrac{33}{16}\ ,\quad p=32 \text{ for } P_2\ ,$$
$$v=\tfrac{3}{2}\ ,\quad p=28 \text{ for } P_3,\quad v=\tfrac{3}{2}\ ,\quad p=2 \text{ for } P_4\ ,$$
$$v=1\ ,\quad p=6 \text{ for } P_5 \text{ and } P_6$$

and

$$v=\tfrac{3}{4}\ ,\quad p=4 \text{ for } P_7 \text{ and } P_s\ .$$

This behavior is unexpected and further attests to the complexity of the reversible rule.

Finally, we present a multistate generalization of the proposed CA; the set $\mathbb{F}_q=\{0,1,\ldots q-1\}$ of values x_i^t might be considered as higher spins ("color") or perhaps other internal degrees of freedom. We define a new difference relation [compare with Eq. (4)] as follows:

$$\sum_{j=0}^{r} x_{i-j}^{t+1} \equiv \sum_{j=0}^{r} x_{i+j}^{t} + [\delta_q(x_i^t)-\delta_q(x_i^t-1)] \times \prod_{j=1}^{r} \delta_q(x_{i-j}^{t+1})\delta_q(x_{i+j}^{t})-1\ , \tag{9}$$

where $x_i^t \in \mathbb{F}_q$ for all $i,t \in \mathbb{Z}$. The corresponding q state RFRT takes the form

$$x_{i-r}^{t+1}=\begin{cases} x_i^t & \text{if } i \notin \widetilde{B}(t) \\ (x_i^t-1)\bmod q & \text{if } i \in \widetilde{B}(t)\ , \end{cases} \tag{10}$$

where the set $\widetilde{B}(t)$ was defined earlier by (i)–(iii)′. Qualitatively, it has the same properties as the two-state CA, with even richer particle content and interactions due to the "color."

The FRT for the backwards time evolution is given by

$$x_{i+r}^{t}=\begin{cases} x_i^{t+1} & \text{if } i \notin \mathcal{B}(t) \\ (x_i^{t+1}+1)\bmod q & \text{if } i \in \mathcal{B}(t)\ . \end{cases} \tag{11}$$

where the set $\mathcal{B}(t)$ is defined in the same manner as $\widetilde{B}(t)$ but with the sweeping going from right to left.

Thus this novel class of proposed time-reversible multistate CA's exhibits intriguing regularity and a large variety of coherent structures. Further investigation is highly desirable, and will certainly lead to a better understanding of this fascinating phenomenon. Given this vast array of coherent particlelike solutions, we believe that this new CA, and models like this one, may find valuable applications in many areas of science (e.g., nonlinear physics, information and computation theory, etc.) These models may be viewed as basic (perhaps integrable, i.e., and arbitrarily large number of solutions may be constructed analytically) time-reversible dynamical systems over a finite field $\mathbb{F}_q$. In $1+1$ integrable solitonic systems, the infinite number of conservation laws and properties of the system prohibit particle production. However, over a finite field, these properties are less restrictive, allowing the CA to be consistent with nonelastic particle interaction.

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