Coherent radiation from spatiotemporally modulated gyrating electron beams

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(Received 5 June 1991)

An electron beam from a cyclotron autoresonant accelerator passing adiabatically along a down-tapered axisymmetric static magnetic field can propagate as an axis-encircling helix. This helix rotates temporally at the fundamental pump frequency of the accelerator, and varies spatially with an axial pitch that depends upon the difference between the pump frequency and the local gyrofrequency. We show that the pitch can thus be freely adjusted so as to allow both frequency and wave-number matching between the beam and a guided fast wave at each harmonic of the pump frequency. As a result, power transfer from beam to wave can occur in a process that is first order in the wave amplitude. Examples of a harmonic converter based on this concept are given for 94- and 1000-GHz sources employing magnetic fields no stronger than 10 kG. Since the electrons can maintain phase coherence during the interaction, one can speculate that the power transfer can be fairly efficient. Practical advantages of this harmonic converter for high-power applications may include alleviation of mode competition, direct output coupling, lower wall heat loading, and simpler beam collectors than for corresponding cavity gyrotron oscillators and noncryogenic magnets.

PACS number(s): 41.70.+t, 52.75.Ms, 41.80.Ee

I. INTRODUCTION

Interactions of electromagnetic radiation with electron beams having well-defined velocity-space distributions are at the heart of a number of research investigations and practical devices. The need for well-characterized beams has led to the development of sophisticated sources for producing annular beams with narrow spreads in velocity both parallel and perpendicular to the axial magnetic field in gyrotrons and cyclotron autoresonant masers. For these applications, magnetron injection guns can produce beams that are nearly monoenergetic, axisymmetric, and low in pitch-angle spread [1].

Free-electron laser research [2] and analysis of the wiggler-free free-electron laser [3] have shown that fundamental features enter the physics of wave-particle interactions once a beam acquires spatially coherent modulation. Furthermore, recent work has been reported on the interaction of a beam with temporal modulation produced by a short rf accelerating cavity [4]. In this latter work, clear experimental evidence was offered for efficient conversion of energy from the temporally modulated beam to radiation, without the customary starting current threshold for an oscillator. Although the interaction in that work was labeled a harmonic gyrotron, the authors showed that the mechanism responsible for generating the radiation was not as in a conventional gyrotron. It was also pointed out that, since the beam received most of its energy and all of its temporal modulation from an external rf source, the device was essentially a harmonic converter.

Interest in a harmonic converter as a source of tunable high-power radiation at millimeter and sub-millimeter wavelengths could be significant for at least four reasons. First, the required magnetic field strength would be lower than that for a fundamental harmonic interaction. Second, phase stability at the harmonic could be achieved by locking to the fundamental source frequency, and mode competition could be alleviated. Third, the intense heat load which must be dissipated on the walls or mirrors of cavity oscillators might not be an issue since, as shall be shown below, a waveguide coupler is superior to a resonator for a converter. And fourth, a simplified beam collector design might result if the potential can be realized for very high converter efficiency.

This paper describes and analyzes a process in which harmonic power can flow cumulatively from a spatiotemporally modulated electron beam to a guided fast electromagnetic wave. This process could form the basis for devices for the efficient production of millimeter and sub-millimeter wavelength power as alternatives to gyrotrons and free electron lasers. The term "spatiotemporal modulation" is used in this paper to stress the fact that coupled space-time variations are initially imposed on the beam, as occurs at the first cavity of a gyroklystron amplifier. Related means for imposing the modulation are by cyclotron autoresonance acceleration [5] or, under some conditions, by the short cavity acceleration mechanism described in Ref. [4]. After such a beam is accelerated, the electrons can follow helical trajectories characterized by the phase variable $\phi_0+\xi z-pt$, where ϕ_0 is the initial phase value, ξ is the axial pitch number for the electron orbit gyrations (i.e., the pitch is $2\pi/\xi$), z is the axial coordinate, p is the temporal radian frequency, and t is the time. For cyclotron autoresonant acceleration, p would be the pump frequency at which the accelerator is driven. A uniform axial magnetic field B is present in the interaction region and, as shall be shown below, $\xi=\gamma(p-\Omega)/u$, where u is the electron axial momentum divided by the rest mass, and $\Omega=eB/m\gamma$ is the gyration frequency for an electron of charge $-e$, mass m, and relativistic energy factor γ. We shall assume for simplicity that the accelerator is excited in circular polarization and that, by virtue of undergoing phase focusing in the accelerator, all electrons experience the same magnitude of acceleration; only their phases differ.

The interactions studied in this paper are those which emerge in first order in the external wave-field amplitude, rather than in second order, as is customary for linearized wave-particle interactions. This lower-order interaction is possible because the electron beam carries spatiotemporal modulation which can drive the wave fields, even in the absence of field-induced perturbations on the particle motions. The analysis we present shows that quadratic spatial growth for fast electromagnetic waves interacting with such a beam may be obtained, and that oscillations can occur in a cavity surrounding the beam. It shall be shown that synchronous coupling from the particles to the wave can occur only if both frequency and wave-number matching occur. In prior related work [4], the need for wave-number matching was not appreciated, so that significant coupling could only occur at waveguide cutoff or in a short output coupler. In the magnicon interaction [6], strong temporal coherence is used to provide highly efficient generation of cm-wavelength radiation. The results we obtain when both frequency and wave-number matching occur include wave growth and power extraction at a frequency higher than the gyrofrequency or one of its harmonics. In addition, as one is not limited to a cavity output coupler, a long interaction region can be used to extract a large portion of the power from the beam using a tapered magnetic field and/or a tapered waveguide.

This paper is organized as follows. Section II reviews the single-particle equations of motion in an axisymmetric tapered magnetic field to characterize the parameters for spatiotemporal modulation on a beam. Section III develops the equations for growth of a guided wave interacting with a modulated beam, and determines the steady-state power level for a cavity surrounding such a beam. Section IV analyzes the consequences of requiring both frequency and wave-number matching and provides examples relevant for power generation at millimeter and submillimeter wavelengths. Section V restates the contributions of this work and suggests areas for further work.

II. ELECTRON TRAJECTORIES IN AN AXISYMMETRIC TAPERED B-FIELD

In this section, we review the well-known trajectories for electrons orbiting in an axisymmetric static magnetic field. This is to provide a basis for the analysis in Sec. III and to clarify some vague references to so-called "prebunched" beams which occasionally appear in the literature. We shall assume that electrons have been accelerated by an ideal cyclotron autoresonant accelerator [5] and at $z=0$ have entered a drift region free of fields, except for a static axisymmetric magnetic field. As an entrance boundary value, we take the electron momentum divided by the rest mass (hereafter, normalized momentum) to be given by

$$\mathbf{v}(0,t)=\hat{\mathbf{e}}_x w_0\cos(pt)+\hat{\mathbf{e}}_y w_0\sin(pt)+\hat{\mathbf{e}}_z u_0 , \tag{1}$$

where p is the radian frequency at which the accelerator is driven. In order to obtain the circular motion in the x-y plane indicated in Eq. (1), the axial magnetic field B_0 at the exit of the accelerator would normally have to satisfy the equation

$$\Omega_0=eB_0/m\gamma=p , \tag{2}$$

where the energy factor is given by $\gamma^2=1+\mathbf{v}\cdot\mathbf{v}/c^2$. [Equation (2) may not be satisfied during the acceleration itself.] We now wish to characterize the normalized momentum and coordinates of an electron as it drifts along the static magnetic field. For a uniform magnetic field it is only necessary to make the substitutions, for $z>0$,

$$\begin{aligned} &t\rightarrow t-\gamma z/u_0 , \\ &\hat{\mathbf{e}}_x\rightarrow\hat{\mathbf{e}}_x\cos(\gamma\Omega_0 z/u_0)+\hat{\mathbf{e}}_y\sin(\gamma\Omega_0 z/u_0) , \end{aligned} \tag{3}$$

and

$$\hat{\mathbf{e}}_y\rightarrow-\hat{\mathbf{e}}_x\sin(\gamma\Omega_0 z/u_0)+\hat{\mathbf{e}}_y\cos(\gamma\Omega_0 z/u_0) ,$$

where γ is, of course, constant throughout the motion. Substitution of Eqs. (3) into Eq. (1) yields, for the normalized momentum,

$$\begin{aligned} \mathbf{v}(z,t)=&\hat{\mathbf{e}}_x w_0\cos[pt+\gamma(\Omega_0-p)z/u_0] \\ &+\hat{\mathbf{e}}_y w_0\sin[pt+\gamma(\Omega_0-p)z/u_0]+\hat{\mathbf{e}}_z u_0 . \end{aligned} \tag{4}$$

In Eq. (4) the term $\gamma(\Omega_0-p)z/u_0$ in the phases shows the distinction between $2\pi u_0/\gamma\Omega_0$, the spatial pitch arising from gyrations induced by the static B field, and $2\pi u_0/\gamma p$, the spatial pitch arising from temporal variations at the source. One can easily confirm that Eq. (4) satisfies the equation of motion $d\mathbf{v}/dt=\mathbf{\Omega}\times\mathbf{v}$, provided the convective derivative $d/dt=\partial/\partial t+(u_0/\gamma)\partial/\partial z$ is employed. The coordinates $\mathbf{r}(z,t)$ for the particle follow from $d\mathbf{v}/dt=-\gamma\Omega_0^2\mathbf{r}$, from which one concludes that the electron gyration radius $R_0=w_0/\gamma\Omega_0$.

If, as we stated above, one has, at the exit from the accelerator, the condition $\Omega_0=p$ and if the B-field is uniform thereafter, then Eq. (4) becomes

$$\mathbf{v}(z,t)=\hat{\mathbf{e}}_x w_0\cos(pt)+\hat{\mathbf{e}}_x w_0\sin(pt)+\hat{\mathbf{e}}_z u_0 . \tag{5}$$

Equation (5) states that the particle phase is independent of z. Thus, if a stream of particles is continuously ejected from the accelerator, these particles would form an instantaneous line parallel to the z axis, which line remains parallel to, and rotates about, the z axis at the angular frequency $p=\Omega_0$. The phase advance along the trajectory is precisely equal to the phase advance at the accelerator, so that the particles are all in phase. Labeling such a beam "prebunched" fails to convey the character of the beam, which—in this instance—carries no axial density variations. The elementary result given by Eq. (5) suggests that strong coherent radiation might be expected from such a stream of particles. However, this may not necessarily be the case, as shall be demonstrated below, if the particles are to couple to a guided wave with a finite axial wavelength.

Let us now suppose the static magnetic field to be gradually tapered in the drift region, such that the particle motion is adiabatic. In this case, the phase in Eq. (4) becomes

$$pt+\gamma\int_0^z \frac{dz'}{u(z')}[\Omega(z')-p] , \tag{6}$$

where the axial magnetic field variation is embodied in $\Omega(z)$, and where, from the adiabatic conservation of magnetic moment, i.e., $w^2(z)/\gamma^3\Omega(z)=\text{const}$, one has

$$u(z)=\left[V_0^2-(V_0^2-u_0^2)\frac{\Omega(z)}{\Omega_0}\right]^{1/2} \tag{7}$$

with $V_0^2=u^2(z)+w^2(z)=\text{const}$. If the field taper decreases the gyrofrequency from Ω_0 down to a value Ω_1, following which it is uniform, we can write, for the phase in the uniform region,

$$pt+\phi_0-\gamma(p-\Omega_1)z/u_1 , \tag{8}$$

where ϕ_0 is the value of the integral in Eq. (6), integrated from the exit of the accelerator up to the beginning of the uniform field region, where u_1 is the normalized axial momentum in this region and where z is now measured from the beginning of the uniform region. The gyration radius obeys the equation $R(z)=R_0[\Omega_0/\Omega(z)]^{1/2}$. When the phase given by Eq. (8) is inserted in place of that given in Eq. (4), we see how it is that the phase advances with z. A stream of such particles will lie on a helix in space with a pitch number $\gamma(p-\Omega_1)/u_1$, which helix rotates with time at the angular frequency p. Had the B-field taper been increasing with z rather than decreasing, one notes that the pitch number would be negative, corresponding to a rotating helix with a backward advancing spatial phase. [Of course, if the field is increasing, it must not increase so far that $u(z)$ changes sign.] Labeling such a class of equilibria "prebunched" does not convey the rich spatiotemporal qualities outlined here.

A stream of particles with the helical equilibria described above can couple efficiently to a wave with a finite axial wavelength, as shall be shown below. The phase variation given by Eq. (8) is of the same form as one finds, for example, in the first-order perturbations on a beam in many wave-particle interactions, such as in gyrotron traveling-wave tubes and free-electron lasers. But, an important distinction between those cases and what is discussed in this paper is that a stream of particles with normalized momenta given by Eq. (4) and phases given by Eq. (8) is a zeroth-order equilibrium that can couple without additional perturbation to certain radiation fields. As shall be shown, this can lead to rapid rates of energy transfer from the beam to the fields, even in low-impedence waveguide output couplers.

This somewhat pedestrian exposition has been to clarify basic properties of low-density beam equilibria in a static magnetic field that have either been overlooked or imprecisely described in the past. We now have the basis of considering, in the next section of this paper, the radiation fields that can be induced by such a beam.

III. GUIDED WAVE INTERACTION WITH A COLD BEAM

To simplify understanding of the way in which a spatiotemporally modulated gyrating electron beam couples to guided electromagnetic waves, the analysis presented here is for a cold beam and a single-waveguide mode. In view of the discussion in Sec. II of this paper, we can characterize the beam electrons by the following distribution function:

$$\begin{aligned} f_0(w,\phi,u;x,y,z;t)=&\frac{N_0A}{2\pi W}\delta(w-W)\delta(u-U)\\ &\times\delta(\phi-\phi_0-\xi z+pt)\\ &\times\delta(x-R\cos\phi)\delta(y-R\sin\phi) , \end{aligned} \tag{9}$$

where the cylindrical normalized momentum coordinates (w,ϕ,u) are defined by $\mathbf{v}\cdot\mathbf{v}=c^2(\gamma^2-1)$ $u=\hat{\mathbf{e}}_z\cdot\mathbf{v}$, $w^2=\mathbf{v}\cdot\mathbf{v}-u^2$, and $\cos\phi=\hat{\mathbf{e}}_x\cdot\mathbf{v}/w$. The imposed uniform static magnetic field is along $\hat{\mathbf{e}}_z$, and the accelerator radian pump frequency is p. In Eq. (9), the electron gyration radius $R=w/\gamma\Omega$, the average density is N_0, the effective beam cross-sectional area is A, and the specific axial and transverse normalized momenta are U and W. Equation (9) describes a beam of particles following identical helical axis-encircling orbits. The combination of both spatial and temporal coherence in the zeroth-order electron distribution function leads, as shall be shown, to novel possibilities for wave growth. By developing the formalism from the distribution function, we have laid the basis for the theory to be extended to include guiding center, energy, and momentum spreads, although these are not specifically dealt with in this paper.

From Eq. (9), one has the following expressions, respectively, for the beam density $N(x,y,z;t)$, the axial current density $J_{0z}(x,y,z;t)$, the dc axial current I_{0z}, and the x component of the transverse current density $J_{0x}(x,y,z;t)$:

$$\begin{aligned} N(x,y,z;t)&=\int du\int dw\, w\int d\phi\, f_0\\ &=AN_0\delta(x-R\cos\phi')\delta(y-R\sin\phi') , \end{aligned} \tag{10}$$

$$\begin{aligned} J_{0z}(x,y,z;t)&=-e\int du\int dw\, w\int d\phi\frac{u}{\gamma}f_0\\ &=-eN(x,y,z;t)\frac{U}{\gamma} , \end{aligned} \tag{11}$$

$$I_{0z}=\int dx\int dy\, J_{0z}=-eAN_0\frac{U}{\gamma} , \tag{12}$$

and

$$\begin{aligned} J_{0x}(x,y,z;t)&=-e\int du\int dw\, w\int d\phi\frac{w}{\gamma}\cos\phi f_0\\ &=-eN(x,y,z;t)\frac{W}{\gamma}\cos\phi' , \end{aligned} \tag{13}$$

where $\phi'(z;t)=\phi_0+\xi z-pt$. We seek an expression for the power transfer between the electron beam and a guided electromagnetic wave whose electric field is given by

$$\mathbf{E}(x,y,z;t)=\hat{\mathbf{e}}_x E_x(x,y,z;t) =\hat{\mathbf{e}}_x[E_r(z)\cos k_\perp y+E_s(z)\sin k_\perp y] \times\sin(k_\parallel z-\omega t)\ . \tag{14}$$

In Eq. (14), symmetric modes (i.e., the TE_{r0} modes with odd values of r) are identified with the amplitudes $E_r(z)$, while antisymmetric modes (i.e., the TE_{s0} modes with even values of s) are identified with the amplitudes $E_s(z)$. The amplitudes $E_r(z)$ and $E_s(z)$ are assumed to vary slowly in space such that

$$k_\parallel^{-1}\frac{d}{dz}[\ln E_q(z)]\ll 1\ ,$$

where q designates either r or s. In Eq. (14), the frequency and wave numbers are related by a guided wave dispersion relation

$$(k_\perp^2+k_\parallel^2)c^2=\omega^2\ , \tag{15}$$

where c is the speed of light. For a rectangular guide of width a and height b, $k_\perp=q\pi/a$ for the TE_{q0} mode. For a standing-wave cavity field for the TE_{q0l} mode, where $k_\parallel=l\pi/L$, one can find the power transfer using results for a traveling wave by taking one-half the sum of the power transfer found for $\omega>0$ and $\omega<0$. This is valid since here power transfer is, to lowest order, linear in $\mathbf{E}(x,y,z;t)$.

The spatial growth or decay of the field is determined by the rate of energy transfer $P(z,t)$ from the beam to the fields where, to lowest order,

$$dP(z,t)/dz=-\int dx\int dy\ J_{0x}(x,y,z;t)E_x(x,y,z;t) =I_{0z}E_q(z)\frac{W}{2U}\sum_{n=0}^{\infty}\begin{Bmatrix}\epsilon_n[\cos(2n+1)\phi'+\cos(2n-1)\phi']J_{2n}(k_\perp R)\\ 2[\sin(2n+2)\phi'+\sin(2n)\phi']J_{2n+1}(k_\perp R)\end{Bmatrix}\sin(k_\parallel z-\omega t)\ , \tag{16}$$

where the upper quantity in large curly brackets applies to symmetric modes and the lower quantity in large curly brackets applies to antisymmetric modes. In finding Eq. (16) we have used the relationship

$$\exp(i\zeta\sin\theta)=\sum_{n=0}^{\infty}[\epsilon_n J_{2n}(\zeta)\cos 2n\theta+2iJ_{2n+1}(\zeta)\sin(2n+1)\theta]\ ,$$

where $J_{2n}(\zeta)$ is the Bessel function of the first kind and ϵ_n is the Neumann symbol, equal to 1 for $n=0$ and otherwise equal to 2. The sum in Eq. (16) may be simplified by use of the identity $J_{n-1}(\zeta)+J_{n+1}(\zeta)=(2n/\zeta)J_n(\zeta)$, to yield

$$\frac{dP(z,t)}{dz}=2I_{0z}E_q(z)\frac{W}{U}\sum_{m=0}^{\infty}\left[\frac{m}{k_\perp R}\right]J_m(k_\perp R)\times\begin{Bmatrix}\cos m\phi'\\ \sin m\phi'\end{Bmatrix}\times\sin(k_\parallel z-\omega t)\ , \tag{17}$$

where $\cos m\phi'$ takes the odd values of m and goes with the symmetric waveguide modes, while $\sin m\phi'$ takes the even values of m and goes with the antisymmetric waveguide modes.

For the electric field given by Eq. (14) we have, in general,

$$P(z,t)=\int dx\int dy\ E_x(x,y,z;t)H_y(x,y,z;t) =\frac{1}{\omega\mu_0}\int dx\int dy\begin{Bmatrix}\cos^2 k_\perp y\\ \sin^2 k_\perp y\end{Bmatrix}\times\left[k_\parallel E_q^2(z)\sin^2(k_\parallel z-\omega t)+\frac{1}{2}E_q(z)\frac{dE_q(z)}{dz}\sin(2k_\parallel z-2\omega t)\right]\ , \tag{18}$$

where H_y is the rf magnetic field and μ_0 is the permeability of free space. Omitting the term without a time average and performing the integrals over x and y, we find

$$\frac{dP(z,t)}{dz}=ab\epsilon_0 c(k_\parallel c/\omega)E_q(z)\frac{dE_q(z)}{dz}\sin^2(k_\parallel z-\omega t)\ , \tag{19}$$

where $\epsilon_0=1/c^2\mu_0$ is the permittivity of free space. Equation (19) may be equated to Eq. (17), eliminating $E_q(z)$ on both sides, to yield

$$\sin^2(k_\parallel z-\omega t)dE_q(z)/dz=\frac{2I_{0z}}{ab\epsilon_0 c}\left[\frac{\omega}{k_\parallel c}\right]\frac{W}{U}\sum_m K_m\times\begin{Bmatrix}\cos m\phi'\\ \sin m\phi'\end{Bmatrix}\times\sin(k_\parallel z-\omega t)\ , \tag{20}$$

where $K_m(k_\perp R)=(m/k_\perp R)J_m(k_\perp R)$. Integration over z should follow averaging over time, since the time of interaction $T(z)=z\gamma/U$ is a function of z. On the left-hand side of Eq. (20), we replace the time average by $\frac{1}{2}$, anticipating $\omega T\gg 1$. On the right-hand side, we require the integrals

$$g_m(z)=\frac{1}{T}\int_0^T dt\ \cos(m\phi_0+m\xi z-mpt)\sin(k_\parallel z-\omega t) =\frac{\cos[m\phi_0+(m\xi-k_\parallel)z]-\cos[(\omega-mp)T+m\phi_0+(m\xi-k_\parallel)z]}{2(\omega-mp)T} \tag{21}$$

and

$$h_m(z)=\frac{1}{T}\int_0^T dt\,\sin(m\phi_0+m\xi z-mpt)\sin(k_\parallel z-\omega t)$$
$$=\frac{\sin[(\omega-mp)T+m\phi_0+(m\xi-k_\parallel)z]-\sin[m\phi_0+(m\xi-k_\parallel)z]}{2(\omega-mp)T}\,, \tag{22}$$

where rapidly varying terms which average to zero have been dropped.

As a result, we can integrate Eq. (20) to find

$$E_r(L)-E_r(0)=\frac{4I_{0z}}{ab\epsilon_0 c}\left[\frac{\omega}{k_\parallel c}\right]\frac{W}{U}\sum_{m=1}^{\infty}K_m G_m(L) \tag{23}$$

for odd values of m corresponding to the symmetric modes, and

$$E_s(L)-E_s(0)=\frac{4I_{0z}}{ab\epsilon_0 c}\left[\frac{\omega}{k_\parallel c}\right]\frac{W}{U}\sum_{m=1}^{\infty}K_m H_m(L) \tag{24}$$

for even values of m corresponding to the antisymmetric modes. In Eqs. (23) and (24), we have the integrals

$$G_m(L)=\int_0^L dz\, g_m(z)$$

and

$$H_m(L)=\int_0^L dz\, h_m(z)\,.$$

The functions $G_m(L)$ and $H_m(L)$ may be expressed in terms of the tabulated integrals [7] Si(L) and Ci(L). Examination of the properties of these integrals shows that both $G_m(L)$ and $H_m(L)$ tend toward zero as L increases, unless both $(\omega-mp)$ and $(k_\parallel-m\xi)$ are themselves zero. The vanishing of these two quantities will be referred to hereafter as conforming to matching conditions. These matching conditions can be understood mathematically, since the integrands contain a $1/z$ factor multiplied by terms oscillatory in z that are proportional to z for small z. For purposes of what is discussed in most of the balance of this paper, we shall assume that both of the matching conditions have been met. If this is so then $G_m(L)$ and $H_m(L)$ can both be shown to be equal to $L/2$, provided $\sin m\phi_0=1$ for odd m and $\cos m\phi_0=1$ for even m.

The requirement for the two matching conditions, i.e., $\omega=mp$ and $k_\parallel=m\xi$, can be understood on elementary physical grounds. Unless the wave frequency matches one of the frequencies present in the temporal spectrum of the electron current, phase interference will prevent any significant power transfer from the current to the fields. Likewise, unless the wave's spatial variation matches one of the space harmonics on the beam, spatial phase interference will prevent significant power transfer as well. However, if the system is short, such that both $(\omega-mp)\gamma L/U$ and $(k_\parallel-m\xi)L$ are much less than π, then $G_m(L)$ and $H_m(L)$ may not have insignificant values, and power transfer can occur. This was evidently the situation for the experiments reported in Ref. [4].

We shall henceforth assume that both frequency and wave number matching do occur, so that $G_m(L)=H_m(L)=L/2$. The magnitude of the time-average power transfer $P_m(L)$ from the beam to the fields at the mth harmonic can be found from Eqs. (23) and (24) using the general expression

$$P_m(L)=ab\epsilon_0 c(k_\parallel c/\omega)E_m^2(L)/4\,,$$

where E_m is the field amplitude at the mth harmonic of the pump frequency p for either the symmetric (odd m) or antisymmetric (even m) waveguide modes, and where $k_\parallel=m\xi$ satisfies the waveguide dispersion relation at $\omega=mp$. Equations (23) and (24) give $E_m(L)$ so that, in W,

$$P_m(L)=P_m(0)+I_{0z}LE_m(0)\frac{W}{U}K_m+120\pi\left[\frac{L^2}{ab}\right]\left[\frac{W}{U}\right]^2\left[\frac{\omega}{k_\parallel c}\right]I_{0z}^2K_m^2\,, \tag{25}$$

for I_{0z} in A.

Equation (25) indicates that power flow into the waveguide will occur whether or not an input amplitude $E_m(0)$ is present. If indeed $E_m(0)=0$, the power is seen to increase quadratically with the interaction length L. This occurs since an exact impedance match is achieved, and power will flow from the beam into the circuit so long as the match is maintained, i.e., so long as frequency and wave-number matching is preserved. This type of power transfer cannot occur without initial spatiotemporal modulation on the beam.

Failure to achieve an exact match in either frequency or wave number will result in values of $G_m(L)$ and $H_m(L)$ less than $L/2$, with a concomitant reduction in the growth rate for radiation at the mth harmonic. This can be shown quantitatively by taking $\omega=mp$, but with $k_\parallel\neq m\xi$. Equations (21) and (22) lead to

$$|G_m(L)|=|H_m(L)|=\frac{L}{2}\left|\frac{\sin(k_\parallel-m\xi)L}{(k_\parallel-m\xi)L}\right|\,, \tag{26}$$

where, to maximize the results, $m\phi_0$ has been taken to be an odd integer multiple of $\pi/2$ in evaluating $G_m(L)$, and an integer multiple of π in evaluating $H_m(L)$. Since the harmonic power is proportional to the square of either $G_m(L)$ or $H_m(L)$, we see that it drops to less than half that for perfect wave-number matching when $(k_\parallel-m\xi)L$ exceeds 1.39. On the other hand, a competing mode can probably be neglected if its wave number $k'_\parallel$ is such that $(k'_\parallel-m\xi)L$ exceeds $3\pi/2$, where Eq. (26) gives $G_m^2(L)=0.045(L/2)$. Similar considerations would apply to competition from modes at harmonics other than the design harmonic.

The degradation in growth due to energy or axial momentum spread can be calculated in the same way. To illustrate, we consider a monoenergetic beam with a spread in pitch-angle caused by a spread in parallel momentum from $\bar{u}-\Delta u/2$ to $\bar{u}+\Delta u/2$. If the distribution in u is constant within this interval, we find the average value of $G_m(L)$ or $H_m(L)$ to be

$$\bar{G}_m(L)=\bar{H}_m(L)=\frac{L}{2}\frac{1}{z}\text{Si}(z) , \tag{27}$$

where $\text{Si}(z)$ is the sine-integral function, and $z=k_{\parallel}L(\Delta u/2\bar{u})$. From the tabulated values [7], one can determine that $\text{Si}(z)/z$ falls to less than 0.707 for z greater than about 2.5. As a result, one can expect power growth to decrease from the cold-beam value by a factor of 2 or more once the fractional parallel momentum spread $\Delta u/\bar{u}$ exceeds about $5/k_{\parallel}L$. This result is similar to what is found for traveling wave amplifiers and free-electron lasers.

If a cavity rather than a waveguide surrounds the beam, the power flow is calculated as in Ref. [4]. The field amplitude assumes a value such that the power transfer from the beam balances the cavity losses. Thus,

$$abL\epsilon_0 E_m^2\omega/Q=-\left\langle \int dx \int dy \int dz\, J_{0x}E_x \right\rangle , \tag{28}$$

where Q is the quality factor of the cavity resonator, and where the brackets indicate a time average. The amplitude E_m which satisfies Eq. (28) is

$$E_m=8QI_{0z}\frac{W}{ab\epsilon_0\omega U}K_m , \tag{29}$$

provided $L=l\pi/k_{\parallel}=l\pi U/\gamma m(p-\Omega)$, where l is the axial eigenvalue for the TE_{q0l} mode. This gives, for the steady-state power level (in W) into the cavity,

$$P_m=960\pi f Q I_{0z}^2(W/U)^2K_m^2 \tag{30}$$

for I_{0z} in A, and where the geometric factor $f=(c/\omega L)(L^2/ab)$. Equation (30) agrees essentially with that found in Ref. [4], after one takes into account geometric differences and the distinction arising from our use of both wave-number and frequency matching. What is seen from Eq. (30) is that power transfer increases as the square of the dc beam current and linearly with the cavity Q.

There are several reasons why a waveguide coupler is preferred to a cavity. In order to effect both frequency and wave-number matching, except at the lower harmonic numbers, the axial mode number for a cavity long enough to allow significant power extraction from the beam could be quite high. This could place the operating mode within a highly crowded spectrum of adjacent modes, and perhaps exacerbate mode competition. Use of a cavity will, in most instances, consume a greater fraction of the generated rf power in wall losses than will a waveguide coupler. Finally, with a waveguide coupler, extraction of a large portion of the beam powr will require a precisely tapered axial magnetic field; implementing this tapered-field concept within a short resonant cavity could be a difficult task.

IV. EXAMPLES

In this section we shall examine the consequences of requiring both frequency and wave-number matching between the spatiotemporally modulated electron beam and a single propagating waveguide mode. We shall also provide some examples of millimeter and submillimeter harmonic conversion in a traveling fast wave output coupler, as described by Eq. (25).

The waveguide dispersion relation, Eq. (15), upon substituting the matching conditions

$$\omega - mp$$

and

$$k_{\parallel}=m\xi=m\gamma(p-\Omega)/U ,$$

becomes

$$1=\frac{1}{\beta_\perp^2}\left[\frac{k_\perp R}{m}\right]^2\left[\frac{\Omega}{p}\right]^2+\frac{1}{\beta_z^2}\left[1-\frac{\Omega}{p}\right]^2 , \tag{31}$$

where $\beta_\perp=W/\gamma c$ and $\beta_z=U/\gamma c$. For maximizing harmonic conversion, it is necessary to maximize the harmonic coupling coefficient $K_m(k_\perp R)=(m/k_\perp R)J_m(k_\perp R)$. This requires $k_\perp R$ to be as close to m as possible. We can find the largest allowed value of $k_\perp R$ by rearranging Eq. (31) into the following form:

$$\frac{1}{\beta_\perp^2}\left[\frac{k_\perp R}{m}\right]^2=f(x)=\frac{\beta_z^2-(1-x)^2}{\beta_z^2x^2} \tag{32}$$

and differentiating $f(x)$ with respect to $x=\Omega/p$ to find the maximum. The result is

$$\max\left[\frac{k_\perp R}{m\beta_\perp}\right]^2=(1-\beta_z^2)^{-1} \text{ at } x=x^*=1-\beta_z^2 . \tag{33}$$

Under these conditions

$$k_{\parallel}c=\omega\beta_z$$

and

$$k_\perp R=\frac{m\beta_\perp}{(1-\beta_z^2)^{1/2}}=m\left[\frac{\gamma^2\beta_\perp^2}{1+\gamma^2\beta_\perp^2}\right]^{1/2} . \tag{34}$$

which, in the limit $\gamma\beta_\perp \gg 1$, allows $K_m(k_\perp R)$ to go over to $J_m(m)$. This approximation is seen to be justified even if $\beta_\perp$ is not particularly close to unity. Without wave-number matching, $k_\perp R\approx m\beta_\perp$, and the values of $J_m(k_\perp R)$ can be smaller than those resulting from the use of Eq. (34). This shows that the high-harmonic coupling can be larger when wave-number matching is observed than otherwise. The condition $\Omega/p=x^*=1-\beta_z^2$ shows that the magnetic field strength required for both temporal and spatial phase matching is lower than that required when only temporal phase matching is considered.

The values of $J_m(m)$ are given in Table I for m from 1 to 10. The asymptotic form [8] is $J_m(m)\sim 0.447\,31m^{-1/3}$, which is seen to be accurate to within 0.1% for m greater than 9. This approximate form is valid for m values much less than the critical

TABLE I. Values of $J_m(m)$ for m from 1 through 10. The asymptotic form $0.447\,31m^{-1/3}$ is accurate to better than 0.1% for m greater than 9.

m	$J_m(m)$	m	$J_m(m)$
1	0.440 05	6	0.245 84
2	0.352 83	7	0.233 58
3	0.309 06	8	0.223 45
4	0.281 13	9	0.214 48
5	0.261 14	10	0.207 49

value $m_{\rm cr}=3(\gamma\beta_\perp)^3$, a fact which emerges upon expressing $J_m(k_\perp R)$ in a Nicholson asymptotic expansion [8], valid when $k_\perp R$ is slightly less than m. This expansion gives

$$J_m(k_\perp R)\sim\frac{1}{\pi\sqrt{3}\gamma\beta_\perp}K_{1/3}\left[\frac{m}{m_{\rm cr}}\right], \tag{35}$$

where $K_{1/3}(s)$ is the Bessel function of the third kind. For $m \ll m_{\rm cr}$, Eq. (35) goes over to the asymptotic form used above, namely, $J_m(m)\sim 0.447\,31m^{-1/3}$, while for $m \gg m_{\rm cr}$, one has

$$J_m(k_\perp R)\sim\left[\frac{\gamma\beta_\perp}{2\pi m}\right]^{1/2}\exp\left[-\frac{m}{m_{\rm cr}}\right]. \tag{36}$$

It can be noted that these forms are not identical to those which emerge in treating either spontaneous [9] or stimulated [10] synchrotron radiation, which are derived assuming $\beta_z=0$, in which case $\beta_\perp$ in the above formulas is set equal to unity. For the problem under discussion here, β_z is not zero, and the above forms apply.

For weakly relativistic beams i.e., where $(\gamma\beta_\perp)^2$ is not very much greater than unity, one must use the exact form given by Eq. (34) for $k_\perp R$. Examples using electron energies less than about 2 MeV fall into this category. Two such examples are given below showing the parameters for devices that would produce power at a frequency of 94 GHz, corresponding to a wavelength of 3.19 mm. Equation (25), with $E_m(0)=0$, has been used to find the ratio $P_m(L)/L^2$ which, together with the other parameters, is listed in Table II for two beam energies. The dc beam power for both examples is 2 MW. It is tempting to use the final entry in Table II to find the device lengths needed for an output power of, say, 500 kW, namely 6.1 and 22.5 cm. However, use of results of the linearized theory developed here out to these lengths would not be justified, since energy depletion would probably terminate the growth process well before the efficiency reaches 25%. One can speculate that tapering of either the B field or the waveguide dimensions can overcome the limitations of energy depletion, a point which is discussed below. Nevertheless, the relatively short growth lengths found here do serve to give a measure of the strength of harmonic coupling under conditions described.

The attenuation values for the copper waveguide have been listed in Table II to show that the total power loss in the device walls is probably less than 0.5 db. This follows from adding attenuation to the wave growth as described by the differential equation

$$\frac{dP_m(z)}{dz}=2Az-\alpha P_m(z)\ , \tag{37}$$

where A is the last term in Eq. (25), divided by L^2, and α is the attenuation factor in Nps/cm (equal to the values in Table II divided by 8.686). The solution of Eq. (37), divided by its solution when $\alpha=0$, is

$$\frac{2}{\alpha^2L^2}(e^{-\alpha L}+\alpha L-1)\ , \tag{38}$$

representing the ratio of output power with attenuation to that in the absence of attenuation. For the two cases

TABLE II. Parameters for two examples of harmonic converters providing output power at 94 GHz.

Beam voltage (kV)	300	500
Output wavelength (mm)	3.19	3.19
Harmonic number m	5	9
Pump frequency (GHz)	18.8	10.4
Beam current (A)	6.7	4.0
Momentum ratio W/U	2.0	2.0
Magnetic field (kG)	9.37	6.25
Axial wave number $k_\parallel$ ($\rm cm^{-1}$)	6.84	7.57
Perp. wave number $k_\perp$ ($\rm cm^{-1}$)	18.46	18.17
Coupling constant K_m	0.1347	0.1089
Waveguide mode	TE_{30}	TE_{50}
Waveguide width a (cm)	0.51	0.87
Gyration radius (cm)	0.201	0.416
Frequency/cutoff frequency	1.066	1.084
Output frequency/gyration frequency	5.69	10.62
Waveguide loss (Cu, db/cm)	0.040	0.017
$P_m(L)/L^2$ ($\rm kW/cm^2$)	13.6	0.99

above, one has $\alpha L = 0.0279$ and 0.0440, respectively, for which Eq. (28) gives ratios of 0.991 and 0.986. If applied to an output power of 500 kW, these would represent a (nonuniform) heat deposition of only 4.6 and 7.3 kW. These values are much smaller than heat load values for 94-GHz, 500-kW cavity gyrotrons. Furthermore, cavity gyrotrons have much smaller surface areas upon which this heat must be deposited. It must be stressed that the device parameters for high-efficiency power extraction, as well as the actual heat loading, cannot be known accurately until a nonlinear analysis is carred out. The estimates given here are not intended for more than order-of-magnitude purposes.

Let us now consider a relativistic beam such that the asymptotic form $K_m(k_\perp R) \sim 0.44731 m^{-1/3}$ is appropriate. This would be so, as an example, for a beam energy of 4.60 MeV with $W/U = 10$, so that $\gamma\beta_\perp = 9.90$. Significant power can be extracted from this beam at a high harmonic using fairly modest magnetic field strengths. To show this, we provide an illustrative example where 359th harmonic power is extracted from the beam at 1000 GHz. Table III shows a set of parameters for such a device.

As is seen, this device requires a magnetic field of only 9.6 kG, which can be furnished from either permanent magnets or nonsuperconducting coils. To resist mode competition, both the harmonic number (359) and the perpendicular mode number (233) are prime. This helps to guard against coincidental solutions of the waveguide dispersion relation at lower frequencies. At higher frequencies, such as at twice the design frequency, the dispersion relation would be satisfied, but here the harmonic coupling coefficient $K_m(k_\perp R)$ is much smaller than at the design frequency. The linear efficiency for the device is about 2.4%, which is comparable to or higher than that of many molecular gas lasers in the same wavelength range.

However, there is a limiting efficiency for these traveling-wave harmonic converters. This arises since the harmonic coupling is affected by a spatial phase slip between the wave and the modulation on the beam. (The relative temporal phase is not affected by electron energy depletion.) The variation in relative spatial phase brought on by a depletion in electron energy $\Delta\gamma$ is

$$\Delta\left[k_\parallel - m\frac{\gamma}{u}(p-\Omega)\right]L = -\frac{mpL}{u}\Delta\gamma = -\frac{\omega L}{c\beta_z}\frac{\Delta\gamma}{\gamma} ,$$

since $U = \text{const}$ to first order. Once this phase variation approaches $\pi/2$, energy transfer to the wave can cease, and can even reverse sign. This condition amounts to limiting the energy loss fraction to $\Delta\gamma/\gamma < \beta_z\lambda/4L$, where λ is the radiation wavelength. This limit is similar to that encountered in free-electron lasers in the absence of tapered wigglers. Similar considerations show that wave growth would be limited by an initial spread on the beam in energy, or in axial momentum. The linearized efficiency values given above are seen to exceed this limit.

In principle, it should be possible to avoid spatial phase slip due to nonlinear energy depletion by tapering the magnetic field or the waveguide dimension. The taper would have to be designed so as to hold constant the quantity $k_\parallel - m\gamma(p-\Omega)/U$ over the length of the interaction. Since all the particles would have identical phases at each point along the interaction region, it should be possible to maintain energy flow continuously from the particles to the fields by proper choice of the field or guide taper profile. From these arguments one can speculate that the device efficiency can be relatively high.

TABLE III. Parameters for a harmonic converter providing output power at 1000 GHz.

Output wavelength (cm)	0.03
Beam voltage (kV)	4600
Beam current (A)	1.0
Magnetic field (kG)	9.851
Pump frequency (GHz)	2.786
Harmonic number m	359.
Momentum ratio W/U	10
Axial wave number (cm^{-1})	29.4
Perp. wave number (cm^{-1})	208.4
Waveguide width (cm)	3.51
Perp. mode number	233
Coupling constant K_m	0.0629
Gyration radius (cm)	1.71
Waveguide length (cm)	30
Output power (kW)	110

V. CONCLUSIONS

An analysis has been made of the orbits of axis-encircling electrons in a down-tapered axisymmetric static magnetic field after acceleration using cyclotron autoresonance. It was shown that a beam with spatiotemporal modulation can be created which can drive guided fast waves in a first-order harmonic conversion process. As a result, wave growth in a waveguide can arise without an input signal, and cavity oscillations can occur without a threshold starting current.

The harmonic conversion process described allows significant power flow between a spatiotemporally modulated beam and radiation fields at harmonics of the fundamental temporal modulation frequency. Power transfer will only occur cumulatively if *both* frequency and wavenumber matching occur between the beam and the wave. The harmonic range where power transfer can occur increases as the beam energy increases. A harmonic coupling coefficient is identified whose magnitude falls off as $m^{-1/3}$, where m is the harmonic number, so long as m is much less than the critical harmonic number $3(\gamma\beta_\perp)^3$, where $\beta_\perp$ is the electron velocity component perpendicular to the magnetic field divided by c, and where γ is the relativistic energy factor. For $m \gg m_{\text{cr}}$, the coupling falls off as $\exp(-m/m_{\text{cr}})$. Wave growth in a waveguide is shown to increase quadratically with the product of guide length and dc beam current. Use of a cavity as an output coupler at high harmonics can be less effective than a waveguide because of mode competition.

Examples for providing power at 94 GHz are given using 300- and 500-kV beams. Operation at the fifth and ninth harmonics, respectively, is shown to require magnetic fields of only 9.4 and 6.3 kG. Other advantages in this wavelength region, in comparison with cavity gyrotrons, include phase locking to the fundamental pump source frequency to combat mode competition, lower wall heat loading, and potentially simpler beam collector design, provided the anticipated high efficiency is achieved.

An example for providing submillimeter wave power at 1000 GHz using a magnetic field of less than 10 kG is also given. This device would operate at the 359th harmonic with the waveguide in the 233rd transverse mode. These high values do not necessarily invite problems from mode competition since, as they are prime numbers, satisfaction of the waveguide dispersion relation cannot occur at lower frequencies. At higher frequencies, the harmonic couplng coefficients are smaller than at the design frequency. For the example given, 110 kw of 0.3-mm radiation would be generated in a 30-cm-long waveguide coupler.

The main outstanding theoretical issue has been shown to be the efficiency limit for such a harmonic coupler when the device parameters are uniform along the interaction region. Spatial phase slip between the wave and the beam modulations can be expected once $\Delta\gamma/\gamma$ exceeds $\lambda/4L\beta_z$. This phase slip due to energy depletion can be avoided by employing a gradual taper in the magnetic field or the waveguide dimension, so as to hold close to zero the spatial phase difference. Initial energy and momentum spread on the beam is also expected to lead to a degradation in the power transfer rates. However, it appears possible to operate a cyclotron autoresonance accelerator with strong phase focusing so that the beam could be initially formed with a small spread.

Extension of these concepts to the shorter-wavelength portions of the electromagnetic spectrum would appear to offer an alternative to either molecular gas lasers or free-electron lasers, provided beams of sufficient quality can be produced.

ACKNOWLEGMENTS

Useful discussions were held with C. M. Armstrong, I. B. Bernstein, A. K. Ganguly, and G. S. Park. This research was supported by the U.S. Office of Naval Technology in collaboration with the U.S. Naval Research Laboratory.

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