Positive P representation

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Although the positive P representation (PPR) is widely used in the literature, there are still many open questions concerning its use as a general tool for solving problems in quantum optics. Recently, there has even been doubt as to whether it gives correct results at all. We present two nonlinear examples in which a comparison with independent methods shows the validity of the PPR. Then we show that, in general, the PPR is not restricted to a lower-dimensional subspace. Finally, we address the problem of initial conditions, which have to be chosen carefully to avoid incorrect results. More specifically, we show that the explicit form of the PPR that was given in the original existence proof leads to unphysical behavior when used as an initial distribution.

I. INTRODUCTION

Since the positive P representation (PPR) was introduced by Drummond and Gardiner[1] in 1980, it has found many applications in quantum optics.[2–33] This is mainly due to the fact that it allows one to avoid the problems of the Glauber-Sudarshan P function, which, in cases where quantum noise dominates, may not exist.[34,35] In contrast to this, the PPR is always a well-behaved positive function that corresponds to a classical stochastic process. This equivalence of a quantum process and a stochastic process is of considerable fundamental interest in itself. Moreover, it could result in a general method of solving nonlinear problems in quantum optics by simulation of the Langevin equations corresponding to the Fokker-Planck equation for the PPR. This would be particularly important, as there are many problems where no method of solution beyond linearization exists. Unfortunately, simulations in the PPR appear to be quite problematic. Every time this method was applied to a nonlinear problem, parameter ranges were found where some trajectories show large excursions from the average which result in "spikes," even after averaging.[2–6] It has been shown that these spikes are not due to numerical errors but are real properties of the stochastic differential equations in the doubled space of the PPR.[3,7] However, no satisfactory explanation of spiking has been given so far. Recently, it has even been shown that simulations in the PPR can produce wrong results.[8,9]

At this point the question arises on how the large number of successful applications of the PPR fit into this picture. The answer is that most authors use the PPR in the framework of linearization.[19–33] Since linearized problems can be solved exactly in any representation, the PPR is here just a convenient way of writing down the equations. This means that most cases where the PPR has been applied successfully are actually cases where one does not need it.

As a matter of fact, there is only one known nonlinear example where the method works perfectly well. This example is the parametric oscillator with the driving field adiabatically eliminated.[10] In that case, the stochastic motion is restricted to a bounded subspace, which eliminates the possibility of spiking. Now one could speculate that this is the generic case, and that it is just difficult to find the appropriate subspace in more complex situations. In Sec. II we show that this is not true by analyzing the above example in the case of nonvanishing detuning of the cavity. In Sec. III, we present two additional nonlinear systems where the PPR gives correct results. For a driven nonlinear absorber as well as for sub-second-harmonic generation with both field modes pumped we demonstrate that the simulation results agree with independent calculations in a regime where linearization fails.

It is important to note that the PRR is not unique.[1] On the other hand, this allows one to choose a positive semidefinite diffusion matrix in the Fokker-Planck equation for the PPR. But on the other hand, this results in ambiguity in the choice of the initial distribution. In Sec. IV, we analyze the laser equations using the PPR and show that the transient solution depends on that choice. Whereas an initial δ function results in the correct behavior, the use of the explicit form of the PPR given by Drummond and Gardiner in their original paper[1] leads to unphysical results.

II. IS THE PPR RESTRICTED TO A SUBMANIFOLD?

In a recent Letter,[10] Wolinsky and Carmichael use the PPR to describe the degenerate parametric oscillator. After adiabatic elimination of the second-harmonic field mode, they end up with the following set of Ito stochastic differential equations (the scaling has been changed slightly from that used in Ref. 10):

$$\begin{aligned}\dot{\alpha}&=(-\gamma-i\delta)\alpha+\beta(g-\alpha^2)+(g-\alpha^2)^{1/2}\xi_1(t)\ ,\\ \dot{\beta}&=(-\gamma+i\delta)\beta+\alpha(g-\beta^2)+(g-\beta^2)^{1/2}\xi_2(t)\ .\end{aligned}\tag{1}$$

Here, γ is the cavity damping, δ is the detuning which is zero in Ref. 10, and g is proportional to both the strength of the driving field and the coupling between the modes.

The real stochastic forces ξ_1 and ξ_2 are assumed to obey $\langle \xi_i(t)\xi_j(0)\rangle = \delta_{ij}\delta(t)$. The fundamental field mode is described by the Bose operators a and $a^\dagger$, which are linked to the complex amplitudes α and β via their normally ordered moments

$$\langle a^{\dagger m} a^n \rangle = \langle \beta^m \alpha^n \rangle , \tag{2}$$

where the right-hand side can be obtained by averaging over many trajectories of the Langevin dynamics (1). The deterministic part of Eq. (1) has a fixed point at $\alpha=\beta=0$ characterized by eigenvalues λ_i and eigenvectors $e_i=(x_1,x_2,x_3,x_4)$, where $\alpha=x_1+ix_2$ and $\beta=x_3+ix_4$. They depend on the detuning δ and are given by

$$\begin{aligned}
\lambda_1(\delta) &= -\gamma + (g^2-\delta^2)^{1/2}, \quad e_1(\delta)=(1,\eta,1,-\eta) ,\\
\lambda_2(\delta) &= -\gamma - (g^2-\delta^2)^{1/2}, \quad e_2(\delta)=(\eta,1,\eta,-1) ,\\
\lambda_3(\delta) &= -\gamma + (g^2-\delta^2)^{1/2}, \quad e_3(\delta)=(-\eta,1,\eta,1) ,\\
\lambda_4(\delta) &= -\gamma - (g^2-\delta^2)^{1/2}, \quad e_4(\delta)=(-1,\eta,1,\eta) ,
\end{aligned} \tag{3}$$

where

$$\eta = g/\delta - [(g/\delta)^2-1]^{1/2} \simeq \delta/(2g) \quad (\delta \to 0) .$$

For $\delta=0$, the plane which is subtended by the vectors $e_1(0)$ and $e_4(0)$ is invariant under the deterministic flow of Eq. (1). The rectangular portion of that plane defined by $|\alpha|,|\beta| \le \sqrt{g}$ is also invariant under the stochastic part of Eq. (1). Since Eq. (1) has to be interpreted as an Ito equation, it can easily be seen that a trajectory starting inside that rectangular will remain there forever, which means that no spiking will occur in this system. Indeed, Wolinsky and Carmichael found a completely well-behaved analytical solution of Eq. (1) on the rectangular subspace defined above. However, this success depends very much on the fact that they were able to find the appropriate invariant manifold.[36] Since there exists no practical method of finding invariant manifolds of a nonlinear dynamical system, an intriguing question arises. Does spiking occur just because—due to numerical errors in the simulation—the trajectories do not remain in the correct subspace? Such a subspace can have a complicated structure and may be very difficult to discover, which would explain why only one such case has been found so far.

To answer this question, we investigate whether there exists an invariant manifold for Eq. (1) when one allows for some detuning δ. If δ is very small, one expects to obtain that manifold just by a slight deformation of the rectangle which is invariant for $\delta=0$. At the fixed point $\alpha=\beta=0$, the manifold would then be tangent to the plane subtended by the vectors $e_i(\delta)$ and $e_4(\delta)$. However, for $\delta\neq 0$, the stochastic forces at the origin do not fall into that plane. From this it follows that close to the origin, trajectories will not be restricted to a two-dimensional manifold but will make use of the entire four-dimensional space.

The conjecture that there exists no invariant manifold for $\delta\neq0$ is further substantiated by the results of a linearized treatment of Eq. (1). Close to the origin $\alpha=\beta=0$, the linearized equations are

$$\begin{aligned}
\dot\alpha &= (-\gamma - i\delta)\alpha + g\beta + \sqrt{g}\,\xi_1(t) ,\\
\dot\beta &= (-\gamma + i\delta)\beta + g\alpha + \sqrt{g}\,\xi_2(t) .
\end{aligned} \tag{4}$$

The stationary solution of these equations can be given in the form of a probability distribution $P(\alpha,\beta)$ which is just the PPR. Below threshold, i.e., for $g<\gamma$, one finds for vanishing detuning $\delta=0$:

$$\begin{aligned}
P(\alpha,\beta) &= P(x_1,x_2,x_3,x_4)\\
&= N \exp[(-\gamma/g)(x_1^2+x_3^2)+2x_1x_3]\delta(x_2)\delta(x_4) ,
\end{aligned} \tag{5}$$

which corresponds to the result of Ref. 10, N being a normalization factor. As expected, this distribution is nonzero only on the two-dimensional subspace defined by $x_2=x_4=0$. However, if one allows for detuning $\delta\neq0$, one gets the following solution which extends over the entire four-dimensional space:

$$P(\alpha,\beta)=P(x_1,x_2,x_3,x_4)=N\exp[-\tfrac{1}{2}x_i(\sigma^{-1})_{ij}x_j] . \tag{6}$$

The matrix σ^{-1} is given by

$$\sigma^{-1} = \begin{pmatrix}
4\gamma/g & 4\gamma^2/(\delta g) & 0 & -4\gamma/\delta\\
4\gamma^2/(\delta g) & 4\gamma(2\gamma^2+\delta^2)/(\delta^2 g) & 4\gamma/\delta & -8\gamma^2/\delta^2\\
0 & 4\gamma/\delta & 4\gamma/g & -4\gamma^2/(\delta g)\\
-4\gamma/\delta & -8\gamma^2/\delta^2 & -4\gamma^2/(\delta g) & 4\gamma(2\gamma^2+\delta^2)/(\delta^2 g)
\end{pmatrix} . \tag{7}$$

We conclude that the solution of the problem of spiking seems not to lie in the existence of invariant submanifolds.

III. TWO NONLINEAR EXAMPLES WHERE THE PPR DOES WORK

As was mentioned in the Introduction, most authors use the PPR in the framework of linearization. With the exception of the example discussed in Sec. II, in all known applications to the PPR to fully nonlinear problems serious difficulties arise, such as spiking or wrong behavior in some parameter regime. In this section, we present two examples for which the PPR results can be checked by independent calculations in a regime where linearization fails.

The first example is a coherently driven single-mode interferometer with a nonlinear absorber as it was used by Gardiner[37] to illustrate the generalized P representations.

For this system, both complex P representation (CPR) and the PPR obey the same quasi-Fokker-Planck equation,

$$\frac{\partial P}{\partial t}=\left[-\frac{\partial}{\partial\alpha}(\epsilon-K\alpha^2\beta)-\frac{\partial}{\partial\beta}(\epsilon-K\alpha\beta^2)-\tfrac{1}{2}K\frac{\partial^2}{\partial\alpha^2}\alpha^2 -\tfrac{1}{2}K\frac{\partial^2}{\partial\beta^2}\beta^2\right]P\ , \quad (8)$$

where $P=P(\alpha,\beta)$ stands either for the CPR or for the PPR and where ϵ and K are the classical driving field and the strength of the nonlinear absorber, respectively. Since Eq. (8) obeys potential conditions, an exact stationary solution can be found in the CPR. This leads to the following expression for the stationary moments (in this formula, some minor errors in Ref. 37, p. 419 have been corrected):

$$\langle (a^\dagger)^m a^n\rangle = M_{mn}/M_{00}$$

where

$$M_{mn}=\sum_{r=0}^{\infty}\frac{(2\epsilon/K)^{n+m+2r-2}2^r}{r!(n+r-1)!(m+r-1)!} \quad (9)$$

and where the $r=0$ term has to be omitted when $n=0$ and $m=0$ or both. The classical dynamics corresponding to the quasi-Fokker-Planck equation (8) has the physical fixed point $\alpha=\beta=(\epsilon/K)^{1/3}$. Linearization at this fixed point leads to the stationary photon number

$$\langle a^\dagger a\rangle=\tfrac{1}{6}+(\epsilon/K)^{2/3}\ . \quad (10)$$

For small nonlinear coupling, i.e., for $\epsilon/K \gg 1$, it can be shown that Eq. (10) agrees with Eq. (9). However, for $\epsilon/K \ll 1$, Eqs. (9) and (10) lead to different results, which means that linearization fails in this regime. Now what are the predictions of the positive P representation? The Langevin equations derived from Eq. (8) in the PPR are

$$\dot\alpha=\epsilon-K\alpha^2\beta+i\sqrt{K}\,\alpha\xi_1(t)\ ,$$
$$\dot\beta=\epsilon-K\alpha\beta^2+i\sqrt{K}\,\beta\xi_2(t)\ . \quad (11)$$

The real stochastic forces ξ_1 and ξ_2 are defined as in Sec. II. Figure 1 compares the result of a simulation of Eqs. (11) with the results of Eqs. (9) and (10) for the parameter values $K=2$ and $\epsilon=0.1$, i.e., for a large nonlinearity and a small driving field. All trajectories start at $\alpha=\beta=0$. Regardless of some spiking, the stationary value obtained from the PPR is in perfect agreement with the CPR, whereas the linear result is too small. To check the transient regime, a small-time expansion for the moments has been carried out. For the photon number, one obtains up to fifth order

$$\langle a^\dagger a\rangle(t)=\epsilon^2t^2-\tfrac{1}{2}K\epsilon^4t^5+O(t^6)\ . \quad (12)$$

The second-order term has been plotted in Fig. 1. One sees that the small-time behavior of the simulation is identical to Eq. (12). We conclude that in this case the PPR is reliable—even far in the nonlinear regime.

The second example we want to discuss is sub-second-harmonic generation in an externally driven cavity which contains a $\chi^{(2)}$ nonlinear element. For this system, the

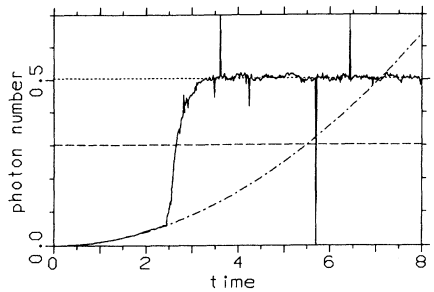

FIG. 1. Photon number $\langle a^\dagger a\rangle=\langle\beta\alpha\rangle$ vs time in dimensionless units for the driven nonlinear absorber Eq. (8). The parameters are $\epsilon=0.1$ and $K=2$. Solid line: simulation of Eqs. (11), 10 000 trajectories, time step $\Delta t=10^{-3}$. Dotted line: stationary value, Eq. (9), derived in the CPR. Dashed line: stationary value, Eq. (10), derived from the linearized theory. Dashed-dotted line: small-time expansion, Eq. (12), up to second order.

Langevin equations in the PPR are[2,19]

$$\dot\alpha_1=-\gamma_1\alpha_1+\chi\beta_1\alpha_2+F_1+\sqrt{\chi\alpha_2}\xi_1(t)\ ,$$
$$\dot\alpha_2=-\gamma_2\alpha_2-\tfrac{1}{2}\chi\alpha_1^2+F_2\ ,$$
$$\dot\beta_1=-\gamma_1\beta_1+\chi\alpha_1\beta_2+F_1+\sqrt{\chi\beta_2}\xi_2(t)\ ,$$
$$\dot\beta_2=-\gamma_2\beta_2-\tfrac{1}{2}\chi\beta_1^2+F_2\ . \quad (13)$$

The stochastic forces ξ_1 and ξ_2 are defined as above; χ is the nonlinear coupling constant; γ_1 and γ_2 are the cavity damping constants in the fundamental and second-harmonic modes, respectively; and F_1 and F_2 are the corresponding classical driving fields for which we assume zero detuning. The quadrature variances in the fundamental mode, which are shown in Fig. 2, are obtained from the second-order moments of Eqs. (13):

$$\langle\Delta^2x_1^\pm\rangle=1-2\langle\alpha_1\beta_1\rangle\pm\langle\alpha_1^2\rangle\pm\langle\beta_1^2\rangle+2\langle\alpha_1\rangle\langle\beta_1\rangle \mp\langle\alpha_1\rangle^2\mp\langle\beta_1\rangle^2\ . \quad (14)$$

In Fig. 2, the results of a simulation[2] of the Langevin equations (13) are compared with a linearized treatment[19]

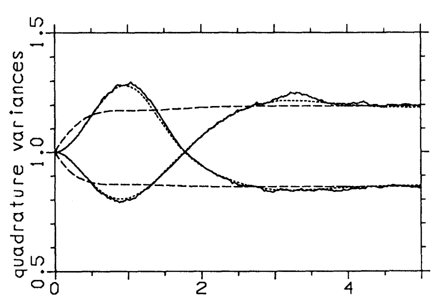

FIG. 2. Sub-second-harmonic generation. Quadrature variances $\langle\Delta^2x_1^+\rangle$ (stationary value is greater than 1) and $\langle\Delta^2x_1^-\rangle$ (stationary value is less than 1) in the fundamental mode vs time in dimensionless units. The parameters are $\gamma_1=\gamma_2=1$, $\chi=0.15$, $F_1=\frac{40}{3}$, $F_2=\frac{20}{3}$. Solid line: simulation of Eqs. (13), 1000 trajectories, time step $\Delta t=10^{-2}$. Dotted line: solution of the moment hierarchy. Dashed line: linearized theory.

and with the solution of the moment hierarchy of the problem.[38] While all of these methods lead to the same stationary values, linearization gives an incorrect transient behavior. Especially the characteristic crossing of the variances in the two different quadrature components is not predicted by the linearized equations.

The moment hierarchy has been solved by truncating it under the assumption that all cumulants of order higher than n vanish. In this case, the three approximation orders corresponding to $n=3$, 4, and 5, respectively, give identical results which are shown as dotted lines in Fig. 2. The slight discrepancy between these lines and the simulation curves are due to the fact that the number of summed trajectories is fairly low. Perfect agreement is easily obtained by averaging over a higher number of trajectories. This case is not shown because the dotted lines in Fig. 2 would then be completely covered by the solid lines. Thus, for the example of sub-second-harmonic generation, one finds again that the PPR treatment is correct where linearization fails.

IV. INITIAL CONDITIONS

In their original paper[1] Drummond and Gardiner give an explicit expression for the PPR in order to prove the theorem that a positive P representation exists for any quantum density operator ρ. The expression is

$$P_{\rm expl}(\alpha,\beta)=\frac{1}{4\pi^2}\exp(-\tfrac{1}{4}|\alpha-\beta^*|^2)\times\langle\tfrac{1}{2}(\alpha+\beta^*)|\rho|\tfrac{1}{2}(\alpha+\beta^*)\rangle\ . \qquad (15)$$

As is shown in the Appendix for a particular example, the distribution $P_{\rm expl}(\alpha,\beta)$ in general does not belong to the class of positive P functions which obey a Fokker-Planck equation. If one wants to derive a Fokker-Planck equation in the PPR, one has to make use of the ambiguity in the definition of the PPR in order to obtain a positive-semidefinite diffusion matrix. This ambiguity is therefore a necessary feature of the method.

Unfortunately, the same ambiguity exists for the initial conditions which must be specified for the Fokker-Planck equation. For instance, it can easily be shown that the distribution

$$P(\alpha,\beta)=(pq/\pi)^2\exp(-p^2|\alpha|^2-q^2|\beta|^2) \qquad (16)$$

is a PPR for the vacuum state $\rho=|0\rangle\langle 0|$ for any real p and q. Now Drummond and Gardiner suggest the use of $P_{\rm expl}$ as the initial distribution, i.e., to set $P(\alpha,\beta,t=0)=P_{\rm expl}(\alpha,\beta,\rho(t=0))$ for an arbitrary initial state $\rho(t=0)$. Since they claim that different positive P functions lead to the same observable moments, they conclude that this should be a correct procedure.

Here we show that this is not true. Different positive P functions can give different observable properties, and the use of $P_{\rm expl}$ as initial distribution actually can lead to invalid results. The example we will use is the ordinary laser which can be described by the following Fokker-Planck equation[39] for the Glauber-Sudarshan P function:

$$\frac{\partial}{\partial t}P(\alpha,\alpha^*)=\left[-\frac{\partial}{\partial\alpha}(d-\alpha\alpha^*)\alpha-\frac{\partial}{\partial\alpha^*}(d-\alpha\alpha^*)\alpha^*+2Q\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P(\alpha,\alpha^*)\ . \qquad (17)$$

The equation is scaled such that it depends only the pump parameter d and the noise parameter Q. As the laser equation has a positive-semidefinite diffusion matrix, it is equivalent to the Langevin equations

$$\dot\alpha=(d-\alpha\alpha^*)\alpha+\sqrt{Q}\,\xi(t)\ ,$$
$$\dot\alpha^*=(d-\alpha\alpha^*)\alpha^*+\sqrt{Q}\,\xi^*(t)\ . \qquad (18)$$

The complex stochastic force $\xi(t)$ obeys $\langle\xi(t)\xi^*(0)\rangle=2\delta(t)$ and $\langle\xi(t)\xi(0)\rangle=0$. These equations, which have been solved in many ways, can also be solved by simulation. For the initial state we choose the vacuum which is described by the Glauber-Sudarshan P function $P(\alpha,\alpha^*,t=0)=\delta^2(\alpha)$. Therefore, all trajectories start at $\alpha(t=0)=0$. The result of such a simulation is shown by the dashed line in Fig. 3.

Although there is no need to double the dimensions of the phase space, we will do it nevertheless, replacing in Eq. (17) α^* by β in order to obtain the Fokker-Planck equation for the PPR. The equivalent Langevin equations in the PPR are then

$$\dot\alpha=(d-\alpha\beta)\alpha+\sqrt{Q}\,\xi(t)\ ,$$
$$\dot\beta=(d-\alpha\beta)\beta+\sqrt{Q}\,\xi^*(t)\ . \qquad (19)$$

The most natural choice of a PPR describing the initial vacuum is

$$P(\alpha,\beta,t=0)=\delta(\alpha)\delta(\beta)\ . \qquad (20)$$

in analogy to the Glauber-Sudarshan case above. This corresponds to trajectories starting at $\alpha(t=0)=\beta(t=0)=0$. Since the stochastic forces in Eqs. (19) are the complex conjugate to each other, trajectories that initially obey the condition $\alpha^*=\beta$ will always stay in the subspace defined by this condition. This means that for our choice Eq. (20) of an initial distribution, Eqs. (18) and

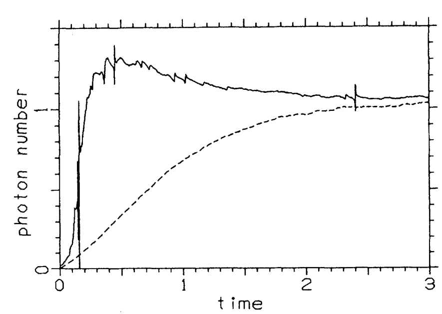

FIG. 3. Photon number vs time in dimensionless units for the laser equation (17). The parameters are $\gamma=1$ and $Q=0.25$. Dashed line: correct result, obtained via simulation of Eq. (18). Solid line: unphysical result obtained via simulation of Eq. (19), using Eq. (21) as initial distribution. The number of trajectories is 3000 and the time step $\Delta t=5\times10^{-4}$ for both curves.

(19) are completely equivalent and will lead to identical results.

However, if $\rho=|0\rangle\langle 0|$ is inserted in the expression for the explicit form P_{expl} of the PPR equation (15), one finds the initial distribution

$$P(\alpha,\beta,t=0)=\frac{1}{4\pi^2}\exp[-\tfrac{1}{2}(|\alpha|^2+|\beta|^2)] , \tag{21}$$

which is a special case of Eq. (16). If one chooses the starting points $\alpha(t=0)$ and $\beta(t=0)$ of the trajectories at random according to the probability distribution equation (21), one obtains a result different from the previous one. The solid curve in Fig. 3 shows a typical simulation. Thus, the choice of P_{expl} for the initial distribution leads to spiking and incorrect transient behavior. For the present example of the laser, the mechanism leading to those difficulties is particularly easy to understand. The four-dimensional deterministic flow of Eqs. (19) has an unstable direction for $\alpha^*=-\beta$. Since the initial distribution equation (21) has a nonvanishing probability close to this instability, some trajectories will make large excursions in the unstable direction. These trajectories are responsible for the spikes and contribute to the unphysical transient behavior.

V. CONCLUSION

Difficulties with the use of the positive P representation arise only in fully nonlinear problems. Only nonlinear deterministic equations develop additional instabilities when they are transformed into the higher-dimensional space of the PPR. Unphysical results and spiking are caused just by those instabilities. We have shown that, nevertheless, simulation of the Langevin dynamics derived in the PPR can be a valuable method beyond linearization for solving problems in nonlinear quantum optics. It would be very important, both from a fundamental and from a practical point of view, to overcome the remaining difficulties of the method. We have demonstrated that this is most probably not achieved by searching for subspaces which are invariant under the stochastic flow. No solution to the problem of the positive P representation would be complete without a clarification of the role of the explicit form of the PPR and without a prescription for choosing the initial distribution.

APPENDIX

Here we show that, in the case of the driven nonlinear absorber, the explicit form of the PPR equation (15) obeys no Fokker-Planck equation. The master equation for this problem is[37]

$$\dot{\rho}=\epsilon[a^\dagger-a,\rho]+\tfrac{1}{2}K(2a^2\rho a^{\dagger 2}-\rho a^{\dagger 2}a^2-a^{\dagger 2}a^2\rho) , \tag{A1}$$

which is equivalent to the Fokker-Planck equation (8). In order to find the equation for $P_{\text{expl}}(\alpha,\beta)$, it is useful to derive operator correspondences in a way analogous to Ref. 37. One obtains

$$\begin{aligned} a\rho &\Longrightarrow \left[\frac{\partial}{\partial\alpha^*}+\frac{\partial}{\partial\beta}+\frac{1}{2}(\alpha+\beta^*)\right]P_{\text{expl}} , \\ \rho a &\Longrightarrow \tfrac{1}{2}(\alpha+\beta^*)P_{\text{expl}} , \\ a^\dagger\rho &\Longrightarrow \tfrac{1}{2}(\alpha^*+\beta)P_{\text{expl}} , \\ \rho a^\dagger &\Longrightarrow \left[\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta^*}+\frac{1}{2}(\alpha^*+\beta)\right]P_{\text{expl}} . \end{aligned} \tag{A2}$$

If one inserts these relations into the term $a^2\rho a^{\dagger 2}$ in Eq. (A1), one sees immediately that the partial differential equation for P_{expl} contains fourth-order derivatives. Therefore, this example shows that P_{expl} does not necessarily obey a Fokker-Planck equation when a Fokker-Planck equation exists for the PPR.

[1]P. D. Drummond and C. W. Gardiner, J. Phys. A **13**, 2353 (1980).
[2]M. Dörfle and A. Schenzle, Z. Phys. B **65**, 113 (1986).
[3]H. J. Carmichael, J. S. Satchell, and S. Sarkar, Phys. Rev. A **34**, 3166 (1986).
[4]S. Sarkar, J. S. Satchell, and H. J. Carmichael, J. Phys. A **19**, 2765 (1986).
[5]S. Sarkar and J. S. Satchell, J. Phys. A **20**, 2147 (1987).
[6]J. S. Satchell and S. Sarkar, J. Phys. A **19**, 2737 (1986).
[7]I. J. D. Craig and K. J. McNeil, Phys. Rev. A **39**, 6267 (1989).
[8]A. M. Smith and C. W. Gardiner, Phys. Rev. A **39**, 3511 (1989).
[9]K. J. McNeil and I. J. D. Craig, Phys. Rev. A **41**, 4009 (1990).
[10]M. Wolinsky and H. J. Carmichael, Phys. Rev. Lett. **60**, 1836 (1988).
[11]M. S. Kim, F. A. M. de Oliveira, and P. L. Knight, Phys. Rev. A **40**, 2494 (1989).
[12]S.-Y. Zhu and N. Lu, Phys. Lett. A **137**, 191 (1989).
[13]N. Lu, S.-Y. Zhu, and G. S. Agarwal, Phys. Rev. A **40**, 258 (1989).
[14]M. L. Steyn-Ross and D. F. Walls, Opt. Acta **28**, 201 (1981).
[15]A. M. Smith and C. W. Gardiner, Phys. Rev. A **38**, 4073 (1988).
[16]T. A. B. Kennedy and E. M. Wright, Phys. Rev. A **38**, 212 (1988).
[17]T. A. B. Kennedy and P. D. Drummond, Phys. Rev. A **38**, 1319 (1988).
[18]V. Perinova, J. Perina, A. Luks, C. Sibilia, and M. Bertolotti, Opt. Acta **31**, 735 (1984).
[19]P. D. Drummond, K. J. McNeil, and D. F. Walls, Opt. Acta **28**, 211 (1981).
[20]J. R. Klauder, S. L. McCall, and B. Yurke, Phys. Rev. A **33**, 3204 (1986).
[21]F. A. M. de Oliveira and P. L. Knight, Phys. Rev. A **39**, 3417 (1989).
[22]V. Perinova, J. Krepelka, and J. Perina, Opt. Acta **33**, 1263 (1986).
[23]W. Kaige, Phys. Rev. A **37**, 4785 (1988).

[24]H. J. Carmichael, D. F. Walls, P. D. Drummond, and S. S. Hassan, Phys. Rev. A **27**, 3112 (1983).
[25]M. Xiao, H. J. Kimble, and H. J. Carmichael, Phys. Rev. A **35**, 3832 (1987).
[26]M. D. Reid and D. F. Walls, Phys. Rev. A **34**, 4929 (1986).
[27]H. J. Carmichael, Phys. Rev. A **33**, 3262 (1986).
[28]M. J. Collett and D. F. Walls, Phys. Rev. Lett. **61**, 2442 (1988).
[29]T. A. B. Kennedy and D. F. Walls, Phys. Rev. A **37**, 152 (1988).
[30]B. Huttner and Y. Ben-Aryeh, Phys. Rev. A **40**, 2479 (1989).
[31]P. D. Drummond and M. D. Reid, Phys. Rev. A **37**, 1806 (1988).
[32]P. D. Drummond and S. J. Carter, J. Opt. Soc. Am. B **4**, 1565 (1987).
[33]P. D. Drummond and D. F. Walls, Phys. Rev. A **23**, 2563 (1981).
[34]M. D. Reid and D. F. Walls, Phys. Rev. Lett. **53**, 955 (1984).
[35]R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).
[36]H. J. Carmichael (private communication).
[37]C. W. Gardiner, *Handbook of Stochastic Methods,* 2nd ed. (Springer, Berlin, 1985).
[38]R. Schack and A. Schenzle, Phys. Rev. A **41**, 3847 (1990).
[39]H. Haken, *Laser Theory,* Vol. XXV of *Handbuch der Physik* (Springer, Berlin, 1970).