Renormalization-group theory for the modified porous-medium equation

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We analyze the long-time behavior of the modified porous-medium equation $\partial_t u = D\Delta u^{1+n}$ in d dimensions, where n is arbitrary and $D=1$ for $\partial_t u>0$ and $D=1+\epsilon$ for $\partial_t u<0$. This equation describes *inter alia* the height of a groundwater mound during gravity-driven flow in porous media ($d=2$, $n=1$) and the propagation of strong thermal waves following an intense explosion ($d=3$, $n=5$). Using general renormalization-group (RG) arguments, we show that a radially symmetric mound exists of the form $u(r,t)\sim t^{-(d\theta+\alpha)}f(rt^{-(\theta+\beta)},\epsilon)$, where $\theta\equiv 1/(2+nd)$ and α and β are ϵ-dependent anomalous dimensions, obeying the scaling law $n\theta\alpha+(1-nd\theta)\beta=0$. We calculate α and β to $O(\epsilon)$, for general d and n, using a perturbative RG scheme. In the case of groundwater spreading, our results to $O(\epsilon^2)$ are in good agreement with numerical calculations, with a relative error in the anomalous dimension α of about 3% when ϵ is 0.5.

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I. INTRODUCTION

In many physical problems, one is concerned with the limit that one of the dimensionless parameters in the problem, Π_0, tends to zero. The quantity of interest, Π, expressed in dimensionless form as a function, $f(\Pi_0,\Pi_1,\dots,\Pi_n)$, of all of the dimensionless parameters in the problem, $\Pi_0,\Pi_1,\dots,\Pi_n$, is often assumed to be well behaved as $\Pi_0\to 0$:

$$\Pi\sim f(0,\Pi_1,\Pi_2,\dots,\Pi_n) \quad \text{as } \Pi_0\to 0 . \tag{1.1}$$

This being the case, Π_0 may be safely ignored from the outset, and the problem simplified. However, as Barenblatt has emphasized [1], this is often not the case. Instead, with an appropriate choice of exponents $\alpha,\alpha_1,\dots,\alpha_n$, the asymptotic behavior is of the form

$$\Pi\sim\Pi_0^{\alpha}g\left[\frac{\Pi_1}{\Pi_0^{\alpha_1}},\dots,\frac{\Pi_n}{\Pi_0^{\alpha_n}}\right] \quad \text{as } \Pi_0\to 0 , \tag{1.2}$$

where g will be referred to as a scaling function. The exponents are not predicted by dimensional analysis, but must be determined directly from the problem at hand. Such behavior, a consequence of the *incomplete self-similarity* in the governing dimensionless parameters, has become known as *intermediate asymptotics of the second kind* [1]. The use of the term "intermediate" is clearest in nonequilibrium problems, where Π_0 may be taken to be inversely proportional to time, t: here the connotation is of a scaling regime prior to the final, and usually trivial, time-independent state of the system. Barenblatt [1] has provided many diverse examples of this phenomenon, ranging from classical elasticity theory, to shock-wave propagation and fluid flow in porous media. In addition, the problem of velocity selection in dendritic growth appears to be in this category of problems, too [2]. Another set of problems, which may be related to the above, are those in which there is a statistical quantity that exhibits intermediate asymptotics of the second kind. Well-known examples [3] include critical phenomena [4] and spinodal decomposition [5].

Recently, it was shown [6–8] that the exponents $\alpha,\alpha_1,\dots,\alpha_n$ are the anomalous dimensions of the renormalization group (RG), and may thus be computed, *even in cases where there is no statistical aspect to the physical problem*. This RG technique has been applied to the Barenblatt equation, governing the flow of an elastic fluid through an elastoplastic porous medium [1,6–8], convection-diffusion transport with irreversible sorption [9], and two problems in linear continuum mechanics [10]. In all of these problems, the calculation of the anomalous dimensions was accomplished through perturbation theory about a linear problem, using either the Gell-Mann–Low method [6,7,11] or the fixed-point formulation of Wilson [3,8,12].

The purpose of this paper is to extend the RG technique to a modification of the so-called porous-medium equation in d dimensions,

$$\partial_t u(\mathbf{x},t)=D\Delta_d u^{1+n} , \tag{1.3}$$

which describes a variety of physical phenomena, ranging from groundwater flow under gravity to radiative heat transfer following intense explosions [1,13]. The notation Δ_d denotes the Laplacian operator in d dimensions. Equation (1.3) is intrinsically nonlinear, yet it is possible to make progress in situations of sufficiently high symmetry that the similarity solution due to Zel'dovich and Kompaneets [14] may be used as a starting point for perturbation theory.

As in our earlier studies, the transport coefficient D is assumed to be a discontinuous function of $\partial_t u$:

$$D=\begin{cases}1 & \text{for } (\partial_t u>0)\\ 1+\epsilon & \text{for } (\partial_t u<0).\end{cases} \tag{1.4}$$

How this form for D arises is discussed in detail by Barenblatt [15]: in the case of the gravity-driven flow of a groundwater mound in a porous medium, the behavior of D reflects the fact that there is a fundamental asymmetry between flow of water into and out of a pore. When water flows into a pore, it occupies a certain fraction of the pore, whereas when water flows out, a thin wetting layer is left behind. Thus, the dynamics in the part of the mound where water is draining out of previously filled pores is different from that in those parts where water is filling previously empty pores.

In general, such discontinuous behavior of the transport coefficient in linear or nonlinear diffusion equations reflects the presence of a dissipative or loss mechanism, which breaks the conservation law usually associated with these equations. Nevertheless, u^{1+n} is continuous and even twice differentiable. Its first derivative is continuous because the flux must be continuous for physical reasons. The second derivative is trivially continuous everywhere, except possible at the point in space where D is discontinuous; but this occurs when $\Delta_d u^{1+n}=0$. Hence, the second derivative must be continuous everywhere, although the third spatial derivative of u^{1+n} may exhibit a discontinuity where $\Delta_d u^{1+n}=0$. The existence and uniqueness of the solutions that we seek has been rigorously proved in the case $d=1$, $n=0$ (the so-called Barenblatt equation) by Kamenomostskaya [16] and in the case $n=0$, $d\geq 1$ by Kamin, Peletier, and Vazquez [17]. The latter authors have also proved the existence, uniqueness, and some other analytic properties of the anomalous dimension in the case that they consider. To our knowledge, no such proofs are available for the case $n>0$: these would be useful, being essentially proofs of renormalizability, a property that we have had to *assume* in the present paper. We emphasize that such an assumption is perfectly natural, and corresponds to the assumption that a phenomenological description of the dynamics is possible [6,10].

The long-time behavior of the porous-medium equation, starting with a radially symmetrical initial condition for u that decays sufficiently fast at infinity, is of the form

$$u(\mathbf{x},t)\sim\frac{1}{t^{d\theta}}f\left[\frac{r}{t^{\theta}}\right], \tag{1.5}$$

where $r\equiv|\mathbf{x}|$,

$$\theta\equiv\frac{1}{2+nd}, \tag{1.6}$$

and f is some scaling function that may be determined. In Sec. II, we use the RG to consider the long-time behavior of the modified porous-medium equation, where $\epsilon\neq 0$. We find that the long-time behavior is renormalized by the nonconserving terms in the dynamics, with the result that

$$u(r,t)\sim\frac{1}{t^{d\theta+\alpha}}f\left[\frac{r}{t^{\theta+\beta}}\right], \tag{1.7}$$

where the so-called *anomalous dimensions* α and β are related by the scaling law

$$\alpha+\frac{1-nd\theta}{n\theta}\beta=0. \tag{1.8}$$

The actual values of the exponents and scaling functions are determined in Sec. III by using the RG in conjunction with perturbation theory in the parameter ϵ. For ease of presentation, the calculation is presented in detail for the groundwater problem ($n=1$, $d=2$). In Sec. IV, we give the key results for the general calculation with arbitrary n and d.

A remarkable property of the porous-medium equation is that the limit $n\to 0$ is smooth: even though the solutions for $n>0$ possess well-defined propagating fronts, these crossover into the diffusion equation tails as $n\to 0$. This property is preserved when $\epsilon\neq 0$, and by setting $n\to 0$ in our general formula for the anomalous dimension α, we are able to recover the anomalous dimension for the Barenblatt equation ($n=0$).

Finally, we remark on the stability of the similarity solutions that are the topic of this paper: the similarity solution is not physically relevant if it is unstable. The appropriate definition of stability is the following [18]. Let us suppose that for a given problem, the similarity solution is of the form

$$u_s(x,t)=At^{-\alpha}f(\xi),\quad \xi\equiv xt^{-\beta}. \tag{1.9}$$

An infinitesimal perturbation, δu, present at time t_0 may grow or shrink as $t\to\infty$; if the perturbation shrinks, then the solution is linearly stable. Without loss of generality, we may write the perturbation in the form

$$\delta u(x,t)\sim t^{-\alpha}[\delta AF(\xi)+\eta(\xi,t)]. \tag{1.10}$$

The first term in the square brackets simply represents a change in the amplitude of the similarity solution. The last term represents the true deviation from the similarity form. Thus, the correct criterion for stability is that $\eta/u_s\to 0$ as $t\to\infty$.

In the present case, Barenblatt [1] has performed the linear stability analysis for the Barenblatt equation, with the result that the similarity solution is linearly stable. It is straightforward to verify that a similar analysis goes through for the modified porous-medium equation with arbitrary n and d. There is nothing specific to the renormalization group in this analysis, and so we do not present this calculation here.

II. DIMENSIONAL ANALYSIS AND THE RENORMALIZATION GROUP

In this section, we investigate the modified porous-medium equation [Eqs. (1.3) and (1.4)] using dimensional analysis. We consider the initial value problem, with initial conditions of a bell-shaped distribution for $u(r,0)$, with width l and volume of fluid

$$Q_l\equiv\int u(r,0)S_d r^{d-1}dr, \tag{2.1}$$

where S_d is the surface area of a unit sphere in d dimensions. The solution u depends upon the governing parameters as follows: $u=u(r,t,Q_l,l,\epsilon)$. By choosing as the two independent dimensional quantities u and r, we find that $[u]\equiv U$, $[r]\equiv L$, $[t]=L^2U^{-n}$, $[Q_l]=UL^d$, $[l]=L$, and $[\epsilon]=1$. The solution to the initial value problem may then be written in the form

$$\Pi=f(\Pi_1,\Pi_2,\Pi_3)\ , \tag{2.2}$$

with

$$\Pi\equiv\frac{u(Q_l^n t)^{d\theta}}{Q_l},\quad \Pi_1\equiv\frac{r}{(Q_l^n t)^{\theta}},\quad \Pi_2\equiv\frac{l}{(Q_l^n t)^{\theta}},\quad \Pi_3\equiv\epsilon\ . \tag{2.3}$$

For long times, the asymptotic behavior of the initial value problem may be found from the limit $\Pi_2\rightarrow 0$. In the case that $\epsilon=0$, it can be shown that this limit is regular; a similarity solution is obtained by simply setting $\Pi_2=0$ in Eq. (2.2), and solving the ordinary differential equation for f, obtained by substituting Eq. (2.2) into the original problem [14].

In the case $\epsilon\neq 0$, there is no similarity solution of the form $\Pi=f(\Pi_1,0,\Pi_3)$ [15]. The integral $\int u(r,t)S_d r^{d-1}dr$ is not a constant of the motion, and so Q_l is not observable at late times. To elaborate further on this point, consider the system at some late time $\tilde{t}$ with volume of fluid $\tilde{Q}$. What was the initial condition that gave rise to this state? There is no unique answer: for an initial condition with a width l, there is a corresponding initial volume of fluid Q_l that can give rise to the observable late time state. In particular, we can consider the limit $l\rightarrow 0$: even in this limit, the observed volume of fluid $\tilde{Q}$ at time $\tilde{t}$ must be reproduced. The observed quantity $\tilde{Q}$ is related to the unobservable quantity Q_l: by dimensional analysis, they must be proportional to one another. Thus [19]

$$\tilde{Q}=Z^{-1}Q_l\ . \tag{2.4}$$

The dimensionless constant or proportionality, Z, must depend upon l so that $\tilde{Q}$ is independent of l. However, being dimensionless, Z must have the functional dependence $Z=Z(l/\mu)$, where μ is an arbitrary length. In addition, Z may depend upon the dimensionless parameter ϵ. The introduction of Z—an example of a renormalization constant [20]—has required that an additional length scale μ enter the problem and the dimensional analysis. Thus, we can rewrite Eq. (2.2) as

$$u(r,t)=\frac{(Z\tilde{Q})^{1-nd\theta}}{t^{d\theta}}f(\xi,\eta,\epsilon)\ , \tag{2.5}$$

where we have defined

$$\xi\equiv\frac{r}{t^{\theta}(Z\tilde{Q})^{n\theta}}\ , \tag{2.6}$$

$$\eta\equiv\frac{\mu}{t^{\theta}(Z\tilde{Q})^{n\theta}}\ . \tag{2.7}$$

The arbitrary length μ was not present in the original formulation of the initial value problem, and so it is not possible that u can depend upon it. This is expressed by the renormalization-group equation

$$\mu\frac{du}{d\mu}=0\ , \tag{2.8}$$

where the differentiation is performed at fixed values of Q_l, l, r, t, and ϵ. Performing the differentiation yields

$$\gamma n\theta\xi\frac{\partial f}{\partial\xi}+(1+n\gamma\theta)\eta\frac{\partial f}{\partial\eta}-\gamma(1-nd\theta)f=0\ , \tag{2.9}$$

where we have defined the constant

$$\gamma\equiv-\frac{d\ln Z}{d\ln\mu} \tag{2.10}$$

as $l\rightarrow 0$. The general solution is found, from the characteristic equations

$$\frac{d\xi}{\gamma n\theta\xi}=\frac{d\eta}{(1+n\gamma\theta)\eta}=\frac{df}{\gamma(1-nd\theta)f}\ , \tag{2.11}$$

to be of the form

$$u(r,t)=t^{-(d\theta+\alpha)}F\left[\frac{r}{t^{\theta+\beta}}\right]\ , \tag{2.12}$$

with F a function to be determined and anomalous dimensions

$$\alpha=\frac{\theta\gamma(1-nd\theta)}{1+n\gamma\theta} \tag{2.13}$$

and

$$\beta=\frac{\theta\gamma n\theta}{1+n\gamma\theta}\ . \tag{2.14}$$

The anomalous dimensions are not independent, but satisfy the scaling law

$$\alpha+\left[\frac{1-nd\theta}{n\theta}\right]\beta=0 \tag{2.15}$$

as guessed for the groundwater problem case ($d=2$, $n=1$ [15]. Equation (2.12) is the form of the similarity solution governing the behavior of the initial value problem at long times.

III. THE SPREADING OF GROUNDWATER

Consider a horizontal stratum of porous rock containing a groundwater mound [15]. Under the influence of gravity, the mound spreads out and flows along an underlying impermeable bed. In discussing this problem, we shall use three simple assumptions.

(i) The mound is axially symmetric.

(ii) The height of the mound, $h(r,t)$, decreases with increasing radius, with boundary conditions at infinity: $h(r,t)\rightarrow 0$ as $r\rightarrow\infty$.

(iii) The dynamics depends upon whether or not a given pore of the rock was previously occupied.

The resulting equation for the mound height h may be written as

$$\frac{\partial h}{\partial t}=D\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}h^2\right]\ . \tag{3.1}$$

with $D=\kappa$ for $\partial_t h\leq 0$ and $D=\kappa(1-\epsilon)$ for $\partial_t h\geq 0$, where κ is a diffusion coefficient determined by the porous medium itself. The positive constant ϵ is equal to the ratio of the volume fraction of an empty pore, which is occupied by the residual wetting layer of fluid, to the volume fraction of the pore, which may be filled by fluid during saturation. Using the available data [21], we estimate that ϵ is typically about 0.25 in situations of practical interest.

We use the initial condition

$$h(r,0)=\frac{Q_l}{16l^2}\left[8-\frac{r^2}{l^2}\right]\Theta\left[\sqrt{8}l-r\right], \tag{3.2}$$

where Θ is the Heaviside step function, with normalization

$$\int_{-\infty}^{\infty} 2\pi r\, dr\, h(r,0)=2\pi Q_l , \tag{3.3}$$

satisfying the localization condition that $h(r,0)\to 0$ as $r\to\infty$. The initial condition was obtained by the device of setting $\epsilon=0$, and evolving h forward in time from a δ function initial condition until the front was at a radial distance l from the origin. Note that the width of the initial distribution is the crucial regularizing parameter in what follows.

To proceed, we posit a naive expansion of h:

$$h(r,t)=h_0(r,t)+\epsilon h_1(r,t)+\cdots . \tag{3.4}$$

The position of the propagating front of h is given by

$$r_\epsilon(t)=r_0(t)+\epsilon r_1(t)+\cdots \tag{3.5}$$

and is defined by the smallest r satisfying $h(r,t)=0$. The functions h_0, h_1, etc. are determined by matching powers of ϵ. As in our earlier work on the Barenblatt equation [6,7], we anticipate that the naive expansion (3.4) will be divergent.

The zeroth-order equation is

$$\frac{\partial h_0}{\partial t}=\frac{\kappa}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}h_0^2\right], \tag{3.6}$$

with solution

$$h_0(r,t)=\frac{Q_l}{16(Q_l\kappa t+l^4)^{1/2}}\left[8-\frac{r^2}{(Q_l\kappa t+l^4)^{1/2}}\right]\times\Theta\left[8-\frac{r^2}{(Q_l\kappa t+l^4)^{1/2}}\right]. \tag{3.7}$$

At large time $t\gg l^4/Q_l\kappa$, this solution tends toward the self-similar solution

$$\lim_{t\to\infty} h_0(r,t)\sim\frac{1}{t^{1/2}}f\left[\frac{r}{t^{1/4}}\right], \tag{3.8}$$

where f is a scaling function, which may be read off from Eq. (3.7). Note that we can also achieve this limit by keeping t fixed, and taking the limit $l\to 0$, ensuring that the condition $t\gg l^4/Q_l\kappa$ holds.

The first-order equation is

$$\frac{\partial h_1}{\partial t}=\frac{\kappa}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(2h_0h_1)\right]-\frac{\kappa}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}h_0^2\right]\Theta\left[\frac{\partial h_0}{\partial t}\right]. \tag{3.9}$$

Using the solution for h_0, we may write this as

$$\frac{\partial h_1}{\partial t}=\frac{4\kappa Q_l}{\hat{r}_0^2}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\rho}{\partial r}\Theta(\rho)h_1\right]-\frac{16Q_l^2\kappa\rho}{\hat{r}_0^6}\Theta[r-\hat{r}_0(t)], \tag{3.10}$$

where $\hat{r}_0(t)\equiv 2(Q_l\kappa t+l^4)^{1/4}$ is the point where $\partial_t h_0=0$ and $\rho\equiv 1-r^2/\hat{r}_0^2$. The zeroth-order solution h_0 vanishes for $r\geq r_0=\sqrt{8}(Q_l\kappa t+l^4)^{1/4}$. The initial condition is $h_1(r,0)\equiv 0$. Since $r=r_0(t)$ is a boundary where $h_0(r,t)$ is not differentiable, in the above equation, there exists a δ function at $r=r_0$. However, it will turn out that for *positive* ϵ, $h_0+\epsilon h_1$ first becomes zero at $r=r_0+\epsilon r_1$, where r_1 is *negative*. Thus, in solving for h_1, the domain of r is always such that the Θ function has the value unit. The equation for h_1 can be simplified by the substitutions

$$s=8(Q_l\kappa t+l^4)^{1/2} \tag{3.11}$$

and

$$y=\frac{2r^2}{s}-1,\quad -1\leq y . \tag{3.12}$$

The variable s is bounded below by $\delta\equiv 8l^2$ and is unbounded above. These substitutions lead to the equation

$$s\frac{\partial h_1}{\partial s}-\left[(1-y^2)\frac{\partial^2 h_1}{\partial y^2}-2y\frac{\partial h_1}{\partial y}-h_1\right]=-\frac{4Q_l}{s}y\Theta(y) . \tag{3.13}$$

The formal solution to Eq. (3.13) is given by

$$h_1(y,s)=\int_\delta^s ds'\int_{-1}^{1} dy' G(s,y;s',y')\times\left[-\frac{4Q_l}{s'}y'\right]\Theta(y') \tag{3.14}$$

with the initial condition $h_1(s=\delta,y)=0$, where G is the bounded Green function satisfying

$$s\frac{\partial G}{\partial s}-\left[(1-y^2)\frac{\partial^2 G}{\partial y^2}-2y\frac{\partial G}{\partial y}-G\right]=\delta(y-y')\delta(s-s') . \tag{3.15}$$

The solution is

$$G(y,s;y',s')=\sum_{n=0}^{\infty}\frac{2n+1}{2}P_n(y)P_n(y')\frac{1}{s'}\times\left[\frac{s'}{s}\right]^{n(n+1)+1}\Theta(s-s') , \tag{3.16}$$

where $P_n(y)$ is the Legendre polynomial of degree n. Then, for $s\gg\delta$, we find that

$$h_1(y,s)=-\frac{Q_l}{s}\ln\frac{s}{\delta}-\frac{2Q_l}{s}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}B_nP_n(y) , \tag{3.17}$$

where

$$B_n \equiv \int_0^1 dy\, y P_n(y) \ . \tag{3.18}$$

Evaluating the integral gives $B_1=\frac{1}{3}$, $B_{2k+1}=0$, and

$$B_{2k}=(-1)^{k+1}\frac{(2k-3)!!}{(2k+2)!!}, \quad k=1,2,\ldots \ . \tag{3.19}$$

The bare-perturbation result is

$$h(r,t)=\frac{2Q_l}{\hat{r}_0^2}\left[1-\frac{r^2}{2\hat{r}_0^2}\right]-\epsilon\frac{Q_l}{4\hat{r}_0^2}\ln\left[\frac{Q_l\kappa t}{l^4}\right] -\epsilon\frac{Q_l}{\hat{r}_0^2}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}B_nP_n\left[\frac{r^2}{\hat{r}_0^2}-1\right]+O(\epsilon^2) \ . \tag{3.20}$$

As anticipated, $h(r,t)$ exhibits a leading singularity, $\ln(Q_l\kappa t/l^4)$, in the limit $t/l^4\to\infty$. Lastly, we rewrite $h(r,t)$ in the form

$$h(r,t)=\frac{Q_l}{2(Q_l\kappa t+l^4)^{1/2}}\left[1-\frac{\epsilon}{8}\ln\left[\frac{Q_l\kappa t}{l^4}\right]+O(\epsilon^2)\right] -\frac{Q_l r^2}{16(Q_l\kappa t+l^4)}[1-a\epsilon+O(\epsilon^2)] \ . \tag{3.21}$$

Here, a is a finite function, which may be read off from Eq. (3.20); it is not important for the purpose of determining the anomalous dimensions, but leads to *finite* $O(\epsilon)$ corrections to the scaling function. It is these terms to which we refer when we write $O(\epsilon)$ corrections in the following analysis.

Just as in quantum field theory, we treat the divergence of the bare-perturbation result by regarding l as a regularization parameter. The singularity can be removed by introducing one renormalization constant $Z=Z(l/\mu,\epsilon)$, which absorbs the divergence in the limit $l\to 0$, order by order in ϵ. In this way, the renormalized $h(r,t)$, which we shall denote by h_R, remains finite even in the limit $l\to 0$. We replace Q_l by $Z(l/\mu,\epsilon)\tilde{Q}$, and assume a Taylor expansion for Z:

$$Z=\sum_n^{\infty}a_n\left[\frac{l}{\mu}\right]\epsilon^n \tag{3.22}$$

with $a_0=1$. The coefficients a_n $(n\geq 1)$ are determined order by order in ϵ in such a way that all the divergences in h are canceled out. Using

$$\ln Z=\ln[1+a_1\epsilon+O(\epsilon^2)]\approx a_1\epsilon+O(\epsilon^2) \ , \tag{3.23}$$

the first term in Eq. (3.21) becomes

$$h_R^{(1)}=\frac{\tilde{Q}^{1/2}}{2(\kappa t)^{1/2}}\left[1+\frac{a_1}{2}\epsilon+O(\epsilon^2)\right] \times\left[1-\frac{\epsilon}{8}\ln\left[\frac{\tilde{Q}\kappa t}{l^4}\right]+O(\epsilon^2)\right] \ . \tag{3.24}$$

The divergence in the limit $l\to 0$ is removed by the choice $a_1(l/\mu)=\ln(C_1\mu/l)$, where C_1 is an arbitrary constant that will not appear in the final expression for the anomalous dimension, to $O(\epsilon)$. Similar constants will appear in other a_n $(n>0)$ if the renormalization is pursued to higher order in ϵ: a proof of renormalizability or even perturbative renormalizability would show that all such constants vanish from the final result, to all orders in ϵ [6,8]. Hence,

$$h_R^{(1)}=\frac{\tilde{Q}^{1/2}}{2(\kappa t)^{1/2}}\left[1-\frac{\epsilon}{8}\ln\frac{C_1\tilde{Q}\kappa t}{\mu^4}+O(\epsilon)\right] , \tag{3.25}$$

which is independent of l and remains finite in the limit $l\to 0$.

The second term in Eq. (3.21) is independent of Q_l in the limit $t\to\infty$ (or $l\to 0$), and becomes

$$h_R^{(2)}=-\frac{r^2}{16\kappa t}+O(\epsilon) \ . \tag{3.26}$$

As anticipated in the preceding section, there is only one independent anomalous dimension.

This renormalized perturbation series may now be combined with the RG, as explained in detail in Refs. [6] and [8], to obtain the final result for the long-time behavior, to this order in ϵ:

$$h(r,t)\sim\left[\frac{A}{2(\kappa t)^{1/2+\alpha}}-\frac{r^2}{16\kappa t}\right] \times\Theta\left[\frac{A}{2(\kappa t)^{1/2+\alpha}}-\frac{r^2}{16\kappa t}\right]+O(\epsilon) \quad (t\to\infty) \ , \tag{3.27}$$

with the anomalous dimension

$$\alpha=\epsilon/8+O(\epsilon^2) \ . \tag{3.28}$$

The phenomenological parameter A is a constant of integration in the renormalization scheme [7,8] and formally has the value of $A=\lim_{l\to 0}Q_l^{(1-\epsilon/4)/2}l^{\epsilon/2}$. Note that the sequence of initial conditions obtained by taking the limit $l\to 0$ generates a generalized function, which is more singular than a Dirac δ function.

We have also extended the perturbation calculation to second order in ϵ. We find that the logarithms in the perturbation series do sum up in the way that the RG predicts, and we obtain the second order in ϵ^2 result for α:

$$\alpha=0.125\epsilon+0.096\epsilon^2+O(\epsilon^3) \ . \tag{3.29}$$

In Fig. 1, the RG calculation of α of Eq. (3.29) is compared with the value of α obtained from a numerical solution of the nonlinear eigenvalue equation [15], which we reproduced using a shooting method. The agreement is good, with a relative error of less than 3% when ϵ is as large as 0.5 We also attempted to extract the coefficient of ϵ^2 from the numerical calculation, by plotting α/ϵ as a function of ϵ, for $\epsilon<10^{-3}$: the resultant straight line had a slope of 0.07, which is somewhat smaller than the value given by the RG calculation. We were unable to determine whether or not the discrepancy is meaningful, but it should be noted that at small ϵ, the numerical calculation may be inaccurate.

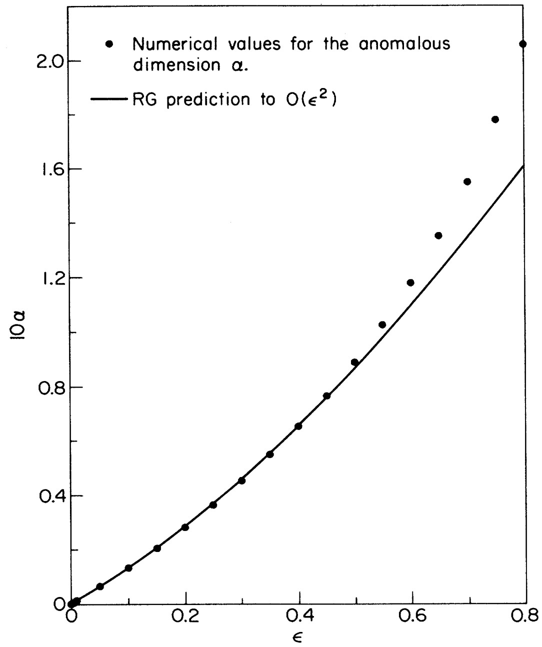

FIG. 1. The anomalous dimension α as a function of ϵ. Data points determined by numerically solving the nonlinear eigenvalue equation for α, given in Ref. [15], are denoted by ●. The continuous curve is the RG calculation of Eq. (3.29).

IV. THE MODIFIED POROUS-MEDIUM EQUATION: ARBITRARY n AND d

In this section, we report our results for the modified porous-medium equation (1.3) with arbitrary n and d:

$$\frac{\partial u}{\partial t}=\frac{D}{r^{d-1}}\frac{\partial}{\partial r}\left[r^{d-1}\frac{\partial}{\partial r}u^{n+1}\right] \tag{4.1}$$

with $D=1$ for $\partial_t u \le 0$, and $D=1+\epsilon$ for $\partial_t u \ge 0$. We construct the initial condition as in the preceding section: this gives

$$u_0(r,0)=\frac{Q_l A}{l^d}\left[\xi_0^2-\frac{r^2}{l^2}\right]^{1/n}\Theta(\xi_0 l-r) , \tag{4.2}$$

where ξ_0 is given by

$$\xi_0=\left[S_d\left[\frac{n\theta}{2(n+1)}\right]^{1/n}\times\int_0^1 t^{d-1}(1-t^2)^{1/n}dt\right]^{-n\theta} . \tag{4.3}$$

In order to perform the perturbation theory, it is convenient to make the transformation to the "Hamiltonian-Jacobi" form of Eq. (1.3), using the change of variables [22]

$$v\equiv\frac{n+1}{n}u^n . \tag{4.4}$$

The resultant equation for v is

$$\partial_t v=D[(\nabla v)^2+nv\Delta v] . \tag{4.5}$$

Assuming a perturbation theory of the form $v=v_0+\epsilon v_1+\cdots$, we obtain

$$v_0(r,t)=\frac{\theta}{2}\frac{Q^n\xi_0^{1/\theta}}{s^{nd/2}}\left[1-\frac{r^2}{s}\right]\Theta(s-r^2) , \tag{4.6}$$

where

$$s\equiv(Q^n t\xi_0^{1/\theta}+l^{1/\theta})^{2\theta} . \tag{4.7}$$

The governing equation for v_1 is rather complicated, but may be simplified through use of the variable $y\equiv r^2/s$. The resulting equation is then

$$(s\partial_s-nL)v_1=\frac{Q^n\xi_0^{1/\theta}\theta}{2s^{nd/2}}\left[y\left[\frac{nd}{2}\right]-\frac{nd}{2}\right]\Theta(nd\theta-y) \tag{4.8}$$

with the operator L given by

$$L\equiv y(1-y)\partial_y^2+\left[\frac{d}{2}-\left[\frac{d}{2}+\frac{1}{n}\right]y\right]\partial_y-\frac{d}{2} . \tag{4.9}$$

The Green function for this operator may be expressed as a sum over Jacobi polynomials, and the singular part of v_1 extracted from the lowest-order term, when integrated with the right-hand side of Eq.(4.8).

After renormalization, we finally obtain the result

$$u(r,t)\sim\left[\frac{A}{(Q^n t)^{dn\theta+\alpha}}-\frac{n\theta r^2}{2(n+1)t}\right]^{1/n}+O(\epsilon) , \tag{4.10}$$

where A is a constant. The anomalous dimension $\alpha=\lambda\epsilon+O(\epsilon^2)$, where

$$\lambda=\frac{4}{d(d+2)}\frac{\Gamma(d/2+1/n)}{\Gamma(1/n)\Gamma(d/2)}(nd\theta)^{d/2+1}\times F(1-1/n,d/2;d/2+2;dn/dn+2) \tag{4.11}$$

and F and Γ are the hypergeometric function and gamma functions, respectively.

As a check on our results, we can examine special limits of Eq. (4.11). For the case $d=2$ and $n=1$, we recovered the correct result $\alpha=\epsilon/8+O(\epsilon^2)$ for groundwater spreading, and for the case $d=1$ and $n\to 0$, we obtain $\alpha=\epsilon/\sqrt{2\pi e}+O(\epsilon^2)$ and the long-time asymptotic form

$$u(x,t)\sim\frac{A}{(\kappa t)^{1/2+\alpha}}e^{-x^2/4\kappa t}+O(\epsilon) \tag{4.12}$$

as found earlier [6,7].

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