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Stationary solutions for the Saffman-Taylor problem with surface tension

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We report a one-parameter family of solutions for the problem of the motion of an interface between a viscous and a nonviscous two-dimensional fluid. The solutions have the interface moving uniformly while the viscous fluid has a nontrivial potential flow. In an alternative interpretation, the existence of gravitational forces in the plane allow the interface to be at rest while the fluid is in motion. This family of solutions is a generalization of a solution reported previously [L. P. Kadanoff, Phys. Rev. Lett. **65**, 2986 (1990)]. The solutions presented here are found to be related to the traveling-wave solutions of the "Harry Dym equation," which is a completely integrable nonlinear evolution equation.

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Recently [1] one of us presented an exact solution for the Saffman-Taylor problem with surface tension. This problem [2,3] is one that involves two fluids: one with high viscosity and the other with low viscosity to be regarded as inviscid, each trapped between two glass plates separated by the small distance b. The velocity in the "two-dimensional" viscous fluid is given by Darcy's law

$$\mathbf{v}=\frac{b^2}{12\mu}\nabla p=\nabla\phi \ , \tag{1}$$

where p is the pressure, ϕ is the velocity potential, and μ is the viscosity. The pressure in the nonviscous fluid is taken to be constant and the jump in pressure across the interface is $\kappa\tau$, where τ is the surface tension and κ is the curvature of the interface. The flow in the viscous fluid is incompressible so ϕ and p obey the Laplace equation. In this paper we report a one-parameter class of exact solutions which is a generalization of the solution reported previously [1].

We first discuss some general aspects of our procedure. The dimensionality 2 of this problem makes it suitable for complex analysis techniques. Accordingly, solutions to the problem are obtained by expressing the coordinates of the (viscous) fluid, the interface, and the value of the velocity potential in terms of analytic functions of an auxiliary complex variable ω. A solution then comprises three parts: one has to specify a domain Ω, called the "physical region," in the ω plane in which the possible values of ω for this solution lie, give a function $\Phi(\omega)$ analytic in Ω to represent the complex potential, and then prescribe a conformal mapping that maps this domain onto the actual region occupied by the fluid.

Following Ref. [1] we seek solutions which have the interface moving with uniform velocity while the fluid has a nontrivial potential flow. We write the coordinates $z=x+iy$ of the fluid as the image of some physical region Ω under the map

$$z=f(\omega,t)=iV_I t+H(\omega)L \ , \tag{2}$$

where V_I is a real number, L is a length parameter which we shall determine later, and $H(\omega)$ is an analytic function of ω in Ω. The interface in the ω plane will lie on the unit circle ($|\omega|=1$) so that the coordinates $\gamma=x+iy$ lying on the actual interface will be given as

$$\gamma(s,t)=iV_I t+H(e^{is})L \ , \tag{3}$$

for some appropriate range of the parameter s. Since V_I was chosen real we have that the interface velocity is $(0,V_I)$.

Next we introduce the conjugate function $\bar{H}$ defined as $\bar{H}(\omega)=\overline{H(\bar{\omega})}$, where the overbar (on the right-hand side) stands for complex conjugation. The complex potential $\Phi=\phi+i\psi$, with ψ being the stream function, is then written via the ansatz

$$\Phi=-iV_\infty H(\omega)L-i(V_I-V_\infty)\bar{H}(\omega^{-1})L \ , \tag{4}$$

where V_∞ is a real number. The function $H(\omega)$ is picked such that $\bar{H}(\omega^{-1})$ as a function of ω is also analytic in Ω so that the velocity potential Φ itself is analytic in Ω. Hence the pressure must obey Lapace's equation in the viscous fluid. We must also satisfy the condition that on the interface the perpendicular component of the fluid velocity must be the same as the perpendicular component of the interface velocity. In the frame moving with the interface, this condition is the statement that the appropriate velocity potential in this frame, $\Phi=iV_I\gamma$, must have an imaginary part independent of s. Then the interface will be on a streamline. Equation (4) satisfies this condition by construction, since at the interface we have $\bar{H}(e^{-is})=\overline{H(e^{is})}$. Notice that Eq. (4) resembles a modified version of the circle theorem for potential flow outside an obstacle [4]. As a final step we must then check that the jump of the pressure condition at the interface as well as any additional boundary conditions are indeed satisfied. Below we follow up with this program.

In order to construct our solutions we shall first give the function H

$$H(\omega)=-\int_0^\omega \frac{2\omega^2}{(\omega^4-2a\omega^2+1)^{1/2}}d\omega+iC \ , \tag{5}$$

where a is real parameter with $a>1$ and C is a constant of integration which we shall fix later. The integral in Eq. (5) is calculated along a contour in the complex ω plane connecting the origin to the point ω. It will be understood that at the lower limit, $\omega=0$, the square root in the integrand assumes the value $+1$ and that it is then varied continuously along the contour to the upper limit ω. For $a>1$ all the singularities of the integrand lie on the real axis. Denote them by $\omega=\pm\omega_1$ and $\omega=\pm\omega_2$, where $\omega_2=\omega_2^{-1}$ and $0<\omega_1<1$. For arbitrary contours the integral in Eq. (4) will be multivalued, its values depending on the position of the contour with respect to the singularities. To assure single valuedness we take for the domain D of definition of H the right half-plane $\mathrm{Re}\,\omega>0$ cut along the interval $[0,\omega_2]$ (see Fig. 1). So defined, the function $H(\omega)$ is clearly analytic in D.

We can express H in terms of standard elliptic integrals:

$$H(\omega)=\begin{cases} H_1(\omega)+iC, & \mathrm{Im}\,\omega>0 \\ H_1(\omega)+i(C-2K), & \mathrm{Im}\,\omega<0 \end{cases} \tag{6a}$$

where Im stands for the imaginary part and the function H_1 is given by

$$H_1(\omega)=2\omega_2[E(\omega/\omega_1,k)-F(\omega/\omega_1,k)] , \tag{6b}$$

where $F(\omega,k)$ and $E(\omega,k)$ are the generalized complex elliptic integrals of the first and second kind, respectively, with $k=\omega_1^2<1$ being the modulus [5]. For definetness we are taking the so-called principal value of the elliptic integrals [5], this being the one when the contour in their defining integrals is chosen to be the straight line joining the origin to the point ω. The function H_1 is double valued for ω lying on the real axis with $|\omega|>\omega_1$ but single valued otherwise. The constant K above arises from the contribution of the contour C_1 encircling the interval $[0,\omega_2]$ (see Fig. 1). Performing this integration one finds

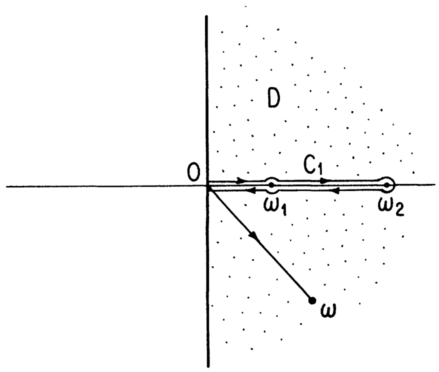

FIG. 1. The domain D of definition of the function $H(\omega)$ [see Eq. (5)]. Also shown is a contour of integration for the case when $\mathrm{Im}\,\omega<0$.

$$i2K=\int_{C_1}\frac{2\omega^2}{(\omega^4-2a\omega^2+1)^{1/2}}d\omega$$
$$=4i\int_{\omega_1}^{\omega_2}\frac{\lambda^2}{[(\lambda^2-\omega_1^2)(\omega_2^2-\lambda^2)]^{1/2}}d\lambda . \tag{7}$$

The last integral can be written in terms of the complete elliptic integrals [6]. We find then that

$$K=2\omega_2E(\pi/2,k') . \tag{8}$$

We now specify the physical region Ω. A solution is obtained by choosing the first quadrant in D, with the condition $|\omega|>1$, as the domain Ω (see Fig. 2). The coordinates $\gamma=x+iy$, which lie on the interface, are given as the image of the first quadrant of the unit circle under the map (3). Performing the corresponding integrations one can write the interface function $\gamma(s,t)$ explicitly. One finds

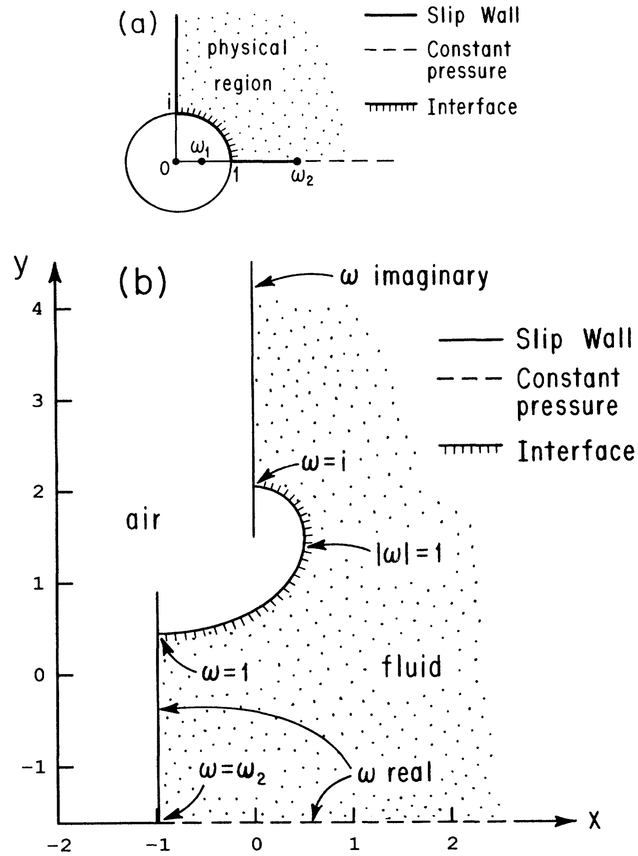

FIG. 2. The geometry of the first solution. Panel (a) shows the physical region in the ω plane and (b) the plane $z=x+iy$, where x and y are coordinates of points in the fluid; (a) is time independent and in (b) the slip walls, the constant-pressure inlet and the interface are all moving towards the top of the sheet with velocity V_I.

$$L^{-1}[\gamma(s,t)-iV_I t]=i\sqrt{2(a-\cos 2s)}+\sqrt{2}\int_{\pi/2}^{s}\frac{\cos 2s}{\sqrt{a-\cos 2s}}ds\ , \tag{9}$$

for $0\le s\le \pi/2$. Here the constant C in Eq. (5) was conveniently chosen to be

$$C=\sqrt{2(1+a)}-\int_0^1\frac{2\lambda^2}{(\lambda^4+2a\lambda^2+1)^{1/2}}d\lambda\ . \tag{10a}$$

In the case $a>1$ this reduces to

$$C=\omega_2[2E(\beta,k')-k'^2]=\tfrac{1}{2}K\ , \tag{10b}$$

where $\beta=\arctan(1/\sqrt{k}\,)$, and $E(\beta,k')$ is the standard real elliptic integral of the second kind [6]. The last equality is obtained after some manipulation with the elliptic integrals. The integral in Eq. (9) can be further expressed in terms of real elliptic integrals [6], but we will not pursue this detail here. Using Eq. (9) we can easily compute the curvature $\kappa(s)$ of the interface. After a brief calculation we find

$$\kappa(s)=|\partial_s\gamma|^{-1}\mathrm{Im}[\partial_s\ln(\partial_s\gamma)]=\frac{1}{L}\sqrt{2(a-\cos 2s)}=\frac{1}{L^2}\mathrm{Im}(\gamma-iV_i t)\ . \tag{11}$$

It is easy to check from Eq. (6b) that $\overline{H}_1(\omega)=H_1(\omega)$. Using this fact and Eqs. (6a) and (10b), we obtain that

$$\overline{H}(\omega)=H(\omega)+iK\ . \tag{12}$$

Hence $\overline{H}(\omega^{-1})$ as a function of ω is indeed analytic in Ω as advertised. [Note, however, that $\overline{H}(\omega^{-1})$ is not analytic in D.] With the explicit form of $\overline{H}(\omega)$, we can now compute the velocity field of the fluid in terms of the auxiliary variable ω. Using Eqs. (2), (4), and (12), we find, for the gradient of Φ,

$$v_x-iv_y=\frac{\partial_\omega\Phi}{\partial_\omega H}=-iV_\infty-i(V_I-V_\infty)\frac{1}{\omega^4}\ . \tag{13}$$

According to Fig. 2, the region of large $|w|$ is the region far from the interface. Thus V_∞ has the interpretation that the far-field velocity is $(0,V_\infty)$.

At the interface we must satisfy a discontinuity condition on the pressure. This is the statement that the real part of Φ must be proportional to the curvature, specifically

$$\mathrm{Re}\ \Phi=\frac{b^2}{12\mu}\kappa\tau\ , \tag{14}$$

where the pressure in the nonviscous fluid has been set equal to zero by the choice of C in Eqs. (10). A substitution of (4) and (11) into (14) gives the length L:

$$L=\left[\frac{b^2}{12\mu}\frac{\tau}{2V_\infty-V_I}\right]^{1/2}. \tag{15}$$

For the solution to make sense, L must be real. Within this constraint, V_∞ and V_I may be independently varied. It is worth pointing out that in the frame where the interface is at rest the flow given by Eq. (13) will generally have stagnation points corresponding to the values $\omega=\pm 1$ and $\omega=\pm i$, whenever any of these points lies in the physical region. However, there is no stagnation point in the original frame since the zeros of Eq. (13) all lie witihin the unphysical region $|\omega|<1$, as one can easily check using the constraint $2V_\infty-V_I>0$ from Eq. (15).

The last step of the calculation is simply checking the remaining boundary conditions at the edges of the physical region. On these surfaces ω is either real or pure imaginary. From Eq. (13), in both cases, the velocity points in the y direction. The interpretation of these surfaces are as follows.

(a) Slip walls. These are the surfaces $\omega=i\lambda$, $\lambda>1$; and $\omega=\lambda$, $1<\lambda<\omega_2$ (see Fig. 2). In both cases the flow is tangential to the walls, so we simply apply a slip boundary condition on these walls.

(b) Constant-pressure inlet. Here ω is real and bigger than ω_2. The flow being perpendicular to the wall means that the pressure is constant along this wall. Hence we interpret it as an inlet for fluid held at a fixed pressure.

In our solution, not only does the interface move with velocity $(0,V_I)$, but the walls do also. From an experimentalist's point of view, this inconvenience can be "fixed" by tilting a setup like the one shown in Fig. 2, so that the top of the picture is higher than the bottom, by an angle θ given by [1]

$$\sin\theta=\frac{12\mu V_I}{gb^2\Delta\rho}\ . \tag{16}$$

Here $\Delta\rho$ is the density difference between the two fluids and g is the gravitational acceleration. Then the solution will be one in which the walls and interface are at rest. Figure 3 shows some streamlines in this frame for the case $a=1.1$. This solution approaches the one given in

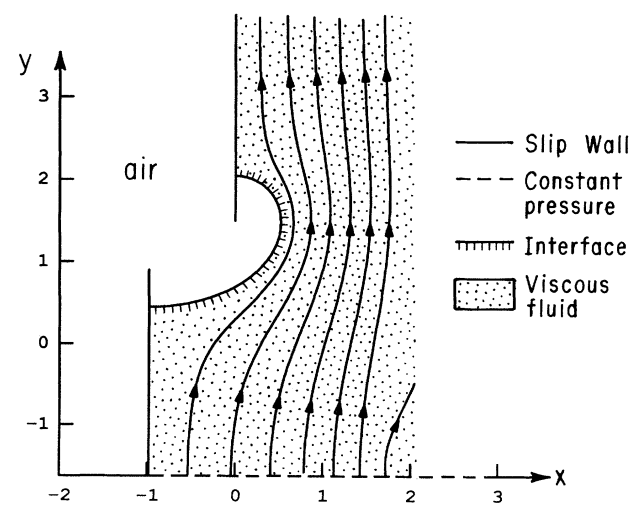

FIG. 3. Some streamlines for the first class of solutions in the "tilted" frame (see text) where the interface and walls are at rest, and the viscous fluid has a nontrivial flow. Shown here is the case $a=1.1$.

Ref. [1] when we take the limit $a \to 1$.

We can obtain a second solution for this problem by choosing a different physical region, as follows: we take Ω to be the full subregion of D which lies outside the unit circle ($|\omega|>1$) (see Fig. 4). There is now an additional interface corresponding to the fourth quadrant of the unit circle. According to Eqs. (3), (10b), and (12) the interface function $\gamma'(s,t)=iV_I t+H(e^{-is})L$ for this second interface will be given by

$$\gamma'(s,t)=\overline{\gamma(s,t)}+i2(V_I t-CL), \quad 0\le s\le \pi/2 . \tag{17}$$

Thus γ' is just the mirror reflection of γ with respect to the axis $y=V_I t-CL$. The curvature κ' of this interface is obviously $\kappa'(s)=-\kappa(s)$. The complex potential for this solution is written as

$$\Phi=-iV_\infty H(\omega)L-i(V_I-V_\infty)[H_1(\omega^{-1})-iC]L , \tag{4'}$$

which is simply the analytic continuation of the previous one to the new physical region. From the jump of pressure condition we find that the nonviscous fluid at the second interface must be kept at a higher pressure p_2 given by

$$p_2=\frac{12\mu}{b^2}LV_I K . \tag{18}$$

(Recall that the pressure p_1 of the nonviscous fluid at the first interface was set equal to zero.) The remaining boundary conditions are clearly satisfied since we only have slip walls at $\omega=\pm i\lambda$, $\lambda>1$, and at $\omega=\lambda$, $1<\lambda<\omega_2$ (see Fig. 4). Figure 5 shows some streamlines in the frame where the interfaces and walls are at rest. This second solution has the advantage that the end boundaries of the Hele-Shaw cell can be placed far away from the interface, making this setup a better candidate for an experimental verification.

We now examine a third family of solutions for which the parameter a falls within the range $|a|<1$. As before, we define the function H via the contour integral

$$H(\omega)=\int_0^\omega \frac{2\omega^2}{(\omega^4-2a\omega^2+1)^{1/2}}d\omega-iC , \tag{19}$$

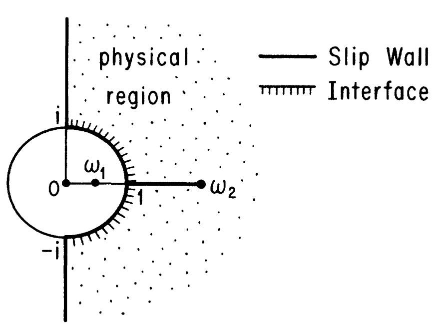

FIG. 4. The geometry in the ω plane of the second class of solutions.

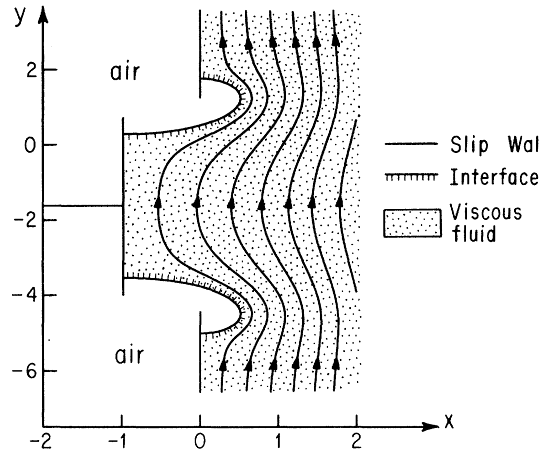

FIG. 5. Some streamlines for the second class of solutions, for the case $a=1.1$, shown in the frame where the interfaces and walls are at rest. The nonviscous fluid, "air," in the bottom region is kept at a pressure higher than the pressure of the top region by an amount p_2 which is given in Eq. (18).

where C is as given in Eq. (10a) with $|a|<1$. In this case the singularities of the integrand all lie on the unit circle; that is, $\omega_1=e^{i\alpha}$ and $\omega_2=e^{-i\alpha}$, where $\cos 2\alpha=a$. The domain D of the definition of H is chosen to be the right half-plane cut along the two segments of the unit circle given by $\omega=e^{\pm is}$, $\alpha\le s\le \pi/2$ (see Fig. 6). Since D is simply connected and does not encircle any singularity it follows that $H(\omega)$ is analytic in D.

The physical region Ω is taken to be the first quadrant within D (see Fig. 7). The interface is the image under Eq. (3) of the arc segment $\omega=e^{is}$, $\alpha\le s\le\pi/2$. This arc segment has two "sides" depending on whether we ap-

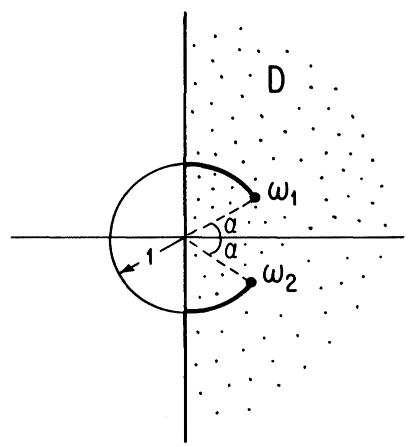

FIG. 6. The domain D of definition of the function $H(\omega)$ for the third class of solutions, showing the two cuts along the unit circle [see Eq. (19)].

proach it from above or from below. Each of these sides gets mapped onto an interface which we denote by γ_U and γ_L, respectively. More precisely, we write

$$\gamma_U(s,t)=iV_I t+H(\omega)L, \quad \omega\searrow e^{is}$$
$$\gamma_L(s,t)=iV_I t+H(\omega)L, \quad \omega\nearrow e^{is} \tag{20}$$

where the downward (upward) arrow means that ω approaches e^{is} from above (below). Repeating the previous analysis we find, for the lower interface,

$$\gamma_L(s,t)=-\gamma(s,t), \quad \alpha\le s\le\pi/2 \tag{21}$$

where $\gamma(s,t)$ has the same formal expression as in Eq. (9) but with $|a|<1$. Similarly, for the upper interface we find

$$\gamma_U(s,t)=2\gamma(\alpha,t)-\gamma_L(s,t)\ , \tag{22}$$

where $\gamma(\alpha,t)\equiv\gamma_L(\alpha,t)=\gamma_U(\alpha,t)$ is a real number. Hence these interfaces are the reflections of one another with respect to the point $\gamma(\alpha,t)$; see Fig. 7(b). The curvatures of the interfaces are clearly $\kappa_U(s)=-\kappa_L(s)=\kappa(s)$, for $\alpha\le s\le\pi/2$, where $\kappa(s)$ is as given in Eq. (11) with $|a|<1$.

The complex potential Φ is written in a slightly modified version of Eq. (4):

$$\Phi=-iV_\infty H(\omega)L+i(V_I-V_\infty)\bar{H}(\omega^{-1})L\ , \tag{4''}$$

where $\bar{H}(\omega)=H(\omega)+i2C$, as before. The choice of a plus sign in the second term of Eq. (4'') is to compensate for the fact that at the interfaces we now have $\bar{H}(1/e^{is})=-H(e^{is})$. This velocity potential is clearly analytic in Ω. It is easy to check that the boundary conditions at both upper and lower interfaces are indeed satisfied with the parameter L being the same as in Eq. (15). We have three other boundary conditions at the edges of the physical region: (a) two slip walls, one at $\omega=i\lambda$, $\lambda>1$, and another at $\omega=i\lambda$, $0<\lambda<1$, and (b) constant pressure inlet or outlet at $\omega=\lambda$, $\lambda>0$ [see Fig. 7(b)].

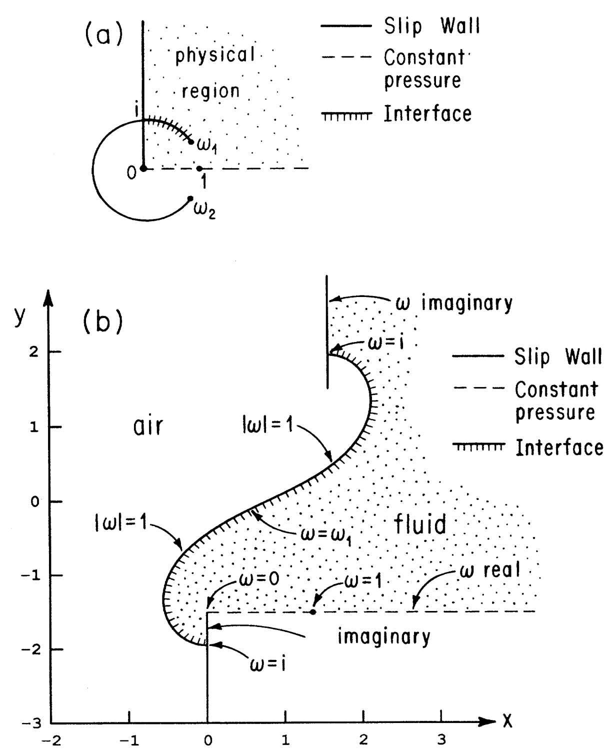

FIG. 7. The geometry of the third solution. Panel (a) shows the ω plane and (b) the z plane. Note the two-"sided" interface in (a) and the corresponding upper and lower interfaces in (b).

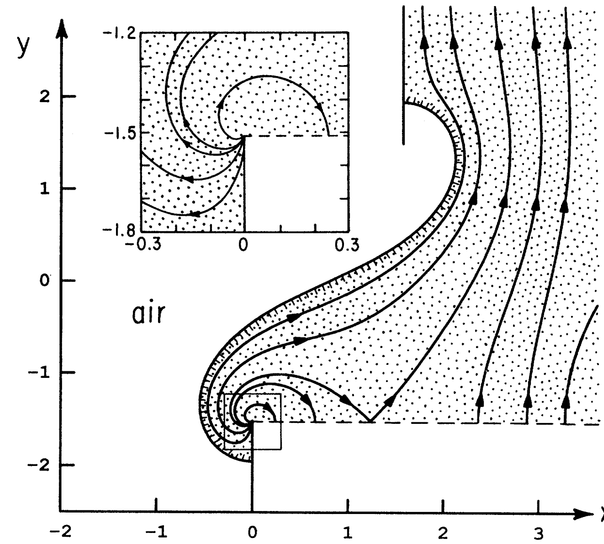

FIG. 8. Some streamlines (in the "rest" frame) for the third family of solutions; shown here is the case $a=0.9$. The inset shows closer details of the behavior near the $r^{-4/3}$ point source (see text).

This last solution has the interesting feature that the origin $\omega=0$ no longer lies in the unphysical region but rather at the edge of the physical region. According to Eq. (13) this means that the velocity field has a singularity at the point $z_c=H(0)L=-iCL$, that is, the velocity becomes infinitely large as z approaches z_c. One can determine the nature of this singularity by looking at the behavior of the solution in a small neighborhood of z_c. One then finds that the velocity field diverges at $z=z_c$ as a 4/3 power pole. We interpret this by saying that for this solution we have an $r^{-4/3}$ point source sitting at $z=z_c$. Figure 8 shows some streamlines for the case $a=0.9$.

As mentioned previously [1], there seems to be a close connection between the Saffman-Taylor problem and the "Harry Dym equation" (HDE) [7,8]

$$\frac{\partial}{\partial t}F^{-2}=c\left[\frac{\partial}{\partial x}\right]^3 F\ , \tag{23}$$

which is a nonlinear evolution equation related to the classical string problem and known to be completely integrable [8]. The results given here are related to the traveling-wave-type solutions of the HDE. If we write

$F(x,t)=u(x-vt)$, then u obeys the following nonlinear ordinary differential equation (ODE):

$$u'=\frac{c}{2v}u^3u''' , \tag{24}$$

where primes indicate differentiation with respect to the function argument. Equation (24) can be integrated and gives implicit solutions for u in the form

$$x=vt+\frac{2}{c}\int\left[\frac{u}{Au^2+Bu+v}\right]^{1/2}du , \tag{25}$$

where B and A are constants to be determined from the behavior of the solutions at infinity. The similarities between (25) and the solutions reported here, as given in Eqs. (5) and (19), are evident. For instance, we interpret the $\frac{4}{3}$-power pole present in one of our solutions as a physical realization of the $\frac{2}{3}$-power branch point characteristic singularity of the HDE [9,10]. A detailed analysis of the HDE and its connections to the Saffman-Taylor problem is beyond the scope of this paper. Here we will just mention, as a final comment, that if one treats the kind of steady flows considered in this paper, i.e., Hele-Shaw flows for which $\Phi(z)\sim z$ as $z\rightarrow\infty$, using the formalism of the Schwarz function [11], then one obtains that the derivative of the Schwarz function obeys an ODE similar to Eq. (24) for the traveling-wave solutions of the HDE. We are currently investigating whether other solutions [10] of the Harry Dym equation would also correspond to solutions of the interface problem. While this paper was being reviewed, we became aware of Ref. [12], where a relation between Hele-Shaw flows and a forced Harry Dym equation is also discussed within the formalism of the Schwarz function.

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