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ARTICLES

Generalized coherent-state analysis of semiclassical quantum chaos for an angular momentum J in a resonant cavity

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An angular momentum J in a resonant-cavity electromagnetic field exhibits chaos in the time dependence of the expectation value of J_z. This fact is transparently demonstrated using the SU(2) generalized coherent-state picture. Using Hale's averaging theorem, the mechanism of chaos is elucidated and found to be equivalent to a periodically modulated, near-separatrix, pendulum motion. The rotating-wave approximation is also elucidated, and its nonchaos is explained. The modulations that cause chaos result from virtual transitions, and the variance of J_z initially grows exponentially.

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I. INTRODUCTION

That consensus on a definition of quantum chaos has not yet been reached is often attributed to the putative fact that the natural definition of chaos in classical systems cannot be carried over into quantum mechanics [1]. In the classical setting, one looks at phase-space trajectories and characterizes chaos in terms of the positivity of the largest Lyapunov exponent [2]. Since we are taught that quantum mechanics does not allow us to maintain the idea of a phase-space trajectory, we no longer can use the Lyapunov criterion for a definition of quantum chaos. Many researchers have concluded that the only remaining tactic is to study the quantum-mechanical properties of known, classically chaotic Hamiltonian systems; this approach has been dubbed "quantum chaology" [3].

In earlier work [4], we have argued that the Lyapunov criterion can be applied in quantum mechanics by looking at expectation value trajectories in a "quantum phase space." Heisenberg's uncertainty principle tells us that these trajectories have associated, nonvanishing variances for all of the variables. This fact does not *a priori* prevent us from using the expectation value trajectories as analogues to the classical trajectories. However, this tactic only remains sensible as long as the root-mean-square deviations from the expectations remain small compared to the expectations. It has been shown rather generally [5] that for a Hamiltonian with a semisimple Lie algebra symmetry, such that the Hamiltonian may be expressed linearly in the Lie algebra, that the required circumstance is maintained indefinitely in time. Generalized coherent states are the key.

Nevertheless, there is a significant limitation to the application of the generalized coherent-state approach that stems from the linearity of the Hamiltonian in the Lie algebra. In essence, the linearity amounts to the treatment of some part of the total system in a semiclassical manner. For example, in this paper, we will study an angular momentum J quantum object in a resonant cavity containing electromagnetic radiation that is treated semiclassically, not quantally. All of our results, which are rigorous for this treatment, become questionable if the radiation is treated quantally [4]. This is really not such a serious criticism given that the paradigm of the hydrogen atom in a microwave radiation field has become one of the central foci for quantum chaos research [1,6], and the microwave field in that case is also treated semiclassically.

Sometimes it is asserted that you cannot have chaos in quantum mechanics because the Schrödinger equation is a *linear* partial differential equation and you need *nonlinearity* for chaos. That this is wrong can be seen in two distinct ways. First, every classical Hamiltonian system can be recast by the *linear* Liouville equation for a probability distribution in phase space. This distribution can be taken initially to be a Dirac δ function (localized on the initial coordinates and momenta), and as a consequence of the first-order derivative nature of Liouville's equation, the solution will evolve as a Dirac δ function for all time. Thus, for a chaotic system, the Liouville distribution follows the chaotic trajectories precisely. Second, the semiclassical quantal problems, such as those treated here, also produce chaos in the Schrödinger description. As we will see, the *linear* Schrödinger equation implies autonomous, *nonlinear* equations for the expectation values, and the time evolution of the wave function is exactly determined by a system of coupled, *nonlinear*, ordinary differential equations.

If one does not treat part of the system semiclassically, e.g., if one quantizes the radiation field, then the symmetry group for the fully quantal problem changes. This leads to Hamiltonians not linear in the Lie algebra generators, and this in turn destroys the limitless propagation of the generalized coherent states, thereby removing the linkage with classical trajectories. Earlier, we studied the behavior of a fully quantal system, the periodically kicked pendulum [7,8]. We found that when its classical analogue was chaotic, the quantum description in terms of an expectation-value phase space became very remarkable. The variances grew exponentially fast to large size compared to the expectations, and the quantum expectation trajectories soon bore no resemblance to the classical trajectories. For example, classically, the sequence of pendulum angles from kick to kick was a chaotic sequence that jumped all over the interval $[0,2\pi]$, but quantally the expected angle quickly converged on π (the "down" position). This was a result of the variance of the angle growing so large that the quantum probability distribution for the angle became broadly spread out over all fo $[0,2\pi]$. We expect that this will generally be the case for fully quantal treatments of classically chaotic systems [9]. The semiclassical treatments are suspect for this reason.

One reason for returning to the problem of an angular momentum J in a resonant cavity is to illustrate the great simplification in the analysis achieved by using generalized coherent states. In doing so, we present a time-ordered generalization of the fundamental Lie algebra factorization identity [5] that is the key to the whole procedure. With it we are able to generalize our earlier $J=\frac{1}{2}$ results [4] to arbitrary J. Moreover, elucidation of the mechanism of chaos in this system turns out to be very natural for the generalized coherent-state representation. We again emphasize the fact that the mechanism for chaos is directly tied to virtual transitions in this quantum system [10]. Such transitions do not have a classical counterpart. Thus, we are not engaged in "quantum chaology"; rather, we demonstrate real chaos in a semiclassically treated quantum system. The correct classical analogue here is the rotating-wave approximation (RWA), and the RWA is rigorously *not chaotic.*

In Sec. II, the dynamical system is presented. The time-ordered generalization of the fundamental Lie algebra factorization identity is proved. In Sec. III the systematic procedure for treating fast processes (virtual transitions) is reviewed and used to show how a periodically modulated pendulum dynamics is embedded in this problem. This result explains the mechanism of chaos, as well as elucidating the nature of the RWA. The variance of J_z is shown to initially grow exponentially.

II. AN ANGULAR MOMENTUM J IN A RESONANT CAVITY

Consider an angular momentum J in a resonant cavity such that not only does the cavity radiation affect the state of the angular momentum, but the state of the angular momentum feeds back into the state of the radiation. This may be modeled by the semiclassical Hamiltonian [4]

$$H=\Omega_0 J_z+\Gamma A(t) J_x , \tag{1}$$

in which J_x and J_z are Cartesian components of angular momentum J, Ω_0 is the size of the energy for the angular momentum, Γ is related to the electric dipole coupling strength, and $A(t)$ is the time-dependent, semiclassical cavity radiation field that we take as satisfying the Maxwell equations

$$\dot{A}=-\Omega B , \tag{2a}$$

$$\dot{B}=\Omega A+2\frac{\Gamma}{\hbar}\mathrm{Ex}(J_x) , \tag{2b}$$

in which Ω is the radiation frequency and $\mathrm{Ex}(J_x)$ denotes the expectation value of J_x. In earlier studies [4,11] of this model, the electromagnetic field A has been treated quantally, and then this model became known as the Belobrov-Zaslavskii-Tartakovskii (BZT) model. All such earlier treatments require some sort of expectation-value factorization approximation that appears most securely based [4] for the case of N noninteracting angular momenta with N sufficiently large. Here, only one is involved, but the radiation is treated semiclassically from the outset.

Introduce the raising-lowering operators $J_\pm$ by

$$J_\pm=J_x\pm iJ_y . \tag{3}$$

These satisfy the SU(2) algebra

$$[J_+,J_-]=2\hbar J_z , \quad [J_z,J_+]=\hbar J_+ , \quad [J_z,J_-]=-\hbar J_- . \tag{4}$$

The Hamiltonian becomes

$$H=\Omega_0 J_z+\tfrac{1}{2}\Gamma A(t)(J_+ +J_-) . \tag{5}$$

The corresponding Schrödinger equation is solved by the time-ordered exponential

$$\begin{aligned}\Psi(t)&=T\exp\left[-\frac{i}{\hbar}\int_0^t ds[\Omega_0 J_z+\tfrac{1}{2}\Gamma A(s)(J_+ +J_-)]\right]\Psi(0)\\ &=\exp\left[-\frac{i}{\hbar}\Omega_0 J_z t\right]T\exp\left[-\frac{i}{\hbar}\int_0^t ds\tfrac{1}{2}\Gamma A(s)\exp\left[\frac{i}{\hbar}\Omega_0 J_z s\right](J_+ +J_-)\exp\left[-\frac{i}{\hbar}\Omega_0 J_z s\right]\right]\Psi(0)\\ &=\exp\left[-\frac{i}{\hbar}\Omega_0 J_z t\right]T\exp\left[-\frac{i}{\hbar}\int_0^t ds\tfrac{1}{2}\Gamma A(s)[J_+\exp(i\Omega_0 s)+J_-\exp(-i\Omega_0 s)]\right]\Psi(0) ,\end{aligned} \tag{6}$$

in which we have used the identity

$$\exp\left[\frac{i}{\hbar}\Omega_0 J_z s\right] J_\pm \exp\left[-\frac{i}{\hbar}\Omega_0 J_z s\right] = J_\pm \exp(\pm i\Omega_0 s) \ . \quad (7)$$

The time-ordered exponential in the last equality of (6) is similar to the non-time-ordered structure that recurs in the literature on generalized coherent states [5] and that is factorized according to Helgason's identity [12]:

$$\exp(\xi J_+ - \xi^* J_-) = \exp(-\tau^* J_-)\exp\left[-\frac{1}{\hbar}\ln(1+\hbar^2\tau^*\tau)J_z\right]\exp(\tau J_+) \ , \quad (8)$$

where

$$\tau = \frac{\xi\sin(\hbar|\xi|)}{\hbar|\xi|\cos(\hbar|\xi|)} \ . \quad (9)$$

Normally, the approach to the connection [5] between quantum mechanics and classical mechanics involves using the generalized coherent states to connect the commutator algebra in the Heisenberg picture to the classical Poisson brackets. This is achieved with Helgason's identity above. Here, we will prove instead the time-ordered generalization of the factorization in (8) and use the Schrödinger picture.

Let $\xi(s)$ be defined by

$$\xi(s) = -\frac{i}{2\hbar}\Gamma\, A(s)\exp(i\Omega_0 s) \ . \quad (10)$$

The time-ordered Helgason identity is as follows:

$$T\exp\left[\int_0^t ds[\xi(s)J_+ - \xi^*(s)J_-]\right] = \exp[G(t)J_-]\exp\left[2\hbar\int_0^t ds\, G(s)\xi(s)J_z\right] \times\exp\left[\int_0^t ds\, \xi(s)\exp\left[-2\hbar^2\int_0^s ds' G(s')\xi(s')\right]J_+\right] , \quad (11)$$

where $G(t)$ solves

$$\dot G = -\hbar^2 G^2\xi - \xi^* \quad \text{with } G(0)=0 \ . \quad (12)$$

The proof is given below. Begin with

$$\begin{aligned} T\exp&\left[\int_0^t ds[\xi(s)J_+ - \xi^*(s)J_-]\right] \\ &= \exp\left[\int_0^t ds\, R(s)J_-\right] T\exp\left[\int_0^t ds\,\exp\left[-\int_0^s ds' R(s')J_-\right]\{\xi(s)J_+ - [\xi^*(s)+R(s)]J_-\}\exp\left[\int_0^s ds' R(s')J_-\right]\right] \\ &= \exp\left[\int_0^t ds\, R(s)J_-\right] T\exp\left\{\int_0^t ds\left[\xi(s)J_+ + 2\hbar\int_0^s ds' R(s')\xi(s)J_z - \hbar^2\left[\int_0^s ds' R(s')\right]^2 \xi(s)J_- - [\xi^*(s)+R(s)]J_-\right]\right\} \\ &= \exp\left[\int_0^t ds\, R(s)J_-\right] T\exp\left[\int_0^t ds\left[\xi(s)J_+ + 2\hbar\int_0^s ds' R(s')\xi(s)J_z\right]\right] , \end{aligned} \quad (13)$$

provided

$$-\hbar^2\left[\int_0^s ds' R(s')\right]^2 \xi(s) - \xi^*(s) - R(s) = 0 \ . \quad (14)$$

Introduce $G(t)$ by

$$G(t) = \int_0^t ds\, R(s) \ , \quad (15)$$

so that (14) becomes

$$\dot G = -\hbar^2 G^2\xi - \xi^* \quad \text{with } G(0)=0 \ . \quad (16)$$

Equation (13) may be written as

$$\begin{aligned} T\exp\left[\int_0^t ds[\xi(s)J_+ - \xi^*(s)J_-]\right] &= \exp[G(t)J_-]T\exp\left[\int_0^t ds[\xi(s)J_+ + 2\hbar G(s)\xi(s)J_z]\right] \\ &= \exp[G(t)J_-]\exp\left[2\hbar\int_0^t ds\, G(s)\xi(s)J_z\right] \\ &\quad\times\exp\left[\int_0^t ds\, \xi(s)\exp\left[-2\hbar^2\int_0^s ds' G(s')\xi(s')\right]J_+\right] , \end{aligned} \quad (17)$$

which concludes the proof.

This proof in no way depends upon the specific form of (10), which was introduced to make the connection with (6). We may easily check that for the special case where $\xi(s)$ is a constant, (11) reduces to (8) and (9) when $t=1$. In this case, (16) is readily integrated, yielding

$$G(t)=\frac{i\xi^*[\exp(i2\hbar|\xi|t)-1]}{\hbar|\xi|[\exp(i2\hbar|\xi|t)+1]}\ , \tag{18}$$

which for $t=1$ gives

$$G(1)=-\frac{\xi^*\tan(\hbar|\xi|)}{\hbar|\xi|}=-\tau^*\ . \tag{19}$$

This implies that

$$\begin{aligned}\int_0^1 ds\, G(s)\xi(s)&=\frac{i|\xi|}{\hbar}\int_0^1 ds\frac{[\exp(i2\hbar|\xi|s)-1]}{[\exp(i2\hbar|\xi|s)+1]}\\&=-\frac{|\xi|}{\hbar}\int_0^1 ds\tan(\hbar|\xi|s)\\&=\frac{1}{\hbar^2}\ln\cos(\hbar|\xi|)\\&=-\frac{1}{2\hbar^2}\ln(1+\hbar^2\tau^*\tau)\ .\end{aligned} \tag{20}$$

Finally, we also get

$$\begin{aligned}\int_0^1 ds\,\xi(s)\exp\left[-2\hbar^2\int_0^s ds'\,G(s')\xi(s')\right]&=\int_0^1 ds\,\xi\exp[-2\ln\cos(\hbar|\xi|s)]\\&=\int_0^1 ds\,\xi\cos^{-2}(\hbar|\xi|s)\\&=\frac{\xi\tan(\hbar|\xi|)}{\hbar|\xi|}=\tau\ .\end{aligned} \tag{21}$$

These results reduce (11) to (8) as claimed.

We have expressed the time evolution of the system by the unitary operator on the right-hand side of (6). Together with (11) we obtain a fully factorized time evolution given by

$$\Psi(t)=\exp\left[-\frac{i}{\hbar}\Omega_0J_zt\right]\exp[G(t)J_-]\exp\left[2\hbar\int_0^t ds\,G(s)\xi(s)J_z\right]\exp\left[\int_0^t ds\,\xi(s)\exp\left[-2\hbar^2\int_0^s ds'\,G(s')\xi(s')\right]J_+\right]\Psi(0)\ , \tag{22}$$

in which

$$\xi(s)=-\frac{i\Gamma}{2\hbar}A(s)\exp(i\Omega_0 s)\ , \tag{23a}$$

$$\dot G=-\hbar^2G^2\xi-\xi^* \quad\text{with } G(0)=0\ , \tag{23b}$$

$$\dot A=-\Omega B\ , \tag{23c}$$

$$\dot B=\Omega A+\frac{\Gamma}{\hbar}\text{Ex}(J_++J_-)\ . \tag{23d}$$

Since $\text{Ex}(J_\pm)$ can be computed from $\Psi(t)$, the dynamics has been reduced to a closed system of coupled, ordinary differential equations [in five real variables: A, B, ξ, (which is complex, but its real and imaginary parts are not independent), and the real and imaginary parts of G]. While an arbitrary initial state $\Psi(0)$ may be used in (22), a special choice [5] leads to generalized coherent states in this case, and it is precisely this choice that yields chaos in this example.

Let the eigenstates for angular momentum J be denoted by $|j,m\rangle$ such that

$$J_z|j,m\rangle=m\hbar|j,m\rangle\ , \tag{24a}$$

$$J^2|j,m\rangle=\hbar^2j(j+1)|j,m\rangle\ . \tag{24b}$$

The state with "highest weight" is denoted [5] by $|j,j\rangle$, i.e., $m=j$. This state has the property

$$J_+|j,j\rangle=0\ , \tag{25}$$

in addition to (24a). We choose for $\Psi(0)$ this state $|j,j\rangle$ and obtain the generalized coherent state

$$|\Psi(t)\rangle=\exp\left[-\frac{i}{\hbar}\Omega_0J_zt\right]\exp[G(t)J_-]|j,j\rangle\times\exp\left[2\hbar^2j\int_0^t ds\,G(s)\xi(s)\right]\ . \tag{26}$$

Now, define $z(t)$ and $S_\pm(t)$ by

$$z(t)=\langle\Psi(t)|J_z|\Psi(t)\rangle\ , \tag{27a}$$

$$S_\pm(t)=\langle\Psi(t)|J_\pm|\Psi(t)\rangle\ . \tag{27b}$$

Each of these expectations contains the factor

$$\exp\left[2\hbar^2j\int_0^t ds[G(s)\xi(s)+G^*(s)\xi^*(s)]\right]=[1+\hbar^2|G(s)|^2]^{-2j}\ , \tag{28}$$

which follows from (12) and its complex conjugate, which yield

$$G(s)\xi(s)+G^*(s)\xi^*(s)=-(1+\hbar^2|G|^2)^{-1}\frac{d}{ds}|G(s)|^2 = -\frac{1}{\hbar^2}\frac{d}{ds}\ln[1+\hbar^2|G(s)|^2] . \quad (29)$$

Using the commutation rules (4), we obtain

$$z(t)=[1+\hbar^2|G(t)|^2]^{-2j}\langle j,j|\exp[G^*(t)J_+] \times J_z\exp[G(t)J_-]|j,j\rangle , \quad (30a)$$

$$S_\pm(t)=[1+\hbar^2|G(t)|^2]^{-2j}\langle j,j|\exp[G(t)^*J_+] \times J_\pm\exp[G(t)J_-]|j,j\rangle\exp(\pm i\Omega_0 t) . \quad (30b)$$

In particular, $\mathrm{Ex}(J_x)$, needed in (23d), is just $\frac{1}{2}[S_+(t)+S_-(t)]$.

Explicit reduction of these expressions, eliminating all remaining operators, requires the identity

$$J_-^k|j,j\rangle=\hbar^k\left[\frac{(2j)!k!}{(2j-k)!}\right]^{1/2}|j,j-k\rangle \quad \text{for } k=0,1,2,\ldots,2j \quad (31)$$

and its adjoint. Expanding the exponentials in (30a) and (30b) yields

$$z(t)=[1+\hbar^2|G(t)|^2]^{-2j} \times\sum_{k=0}^{2j}\hbar(j-k)\frac{(2j)!}{(2j-k)!k!}\hbar^{2k}|G(t)|^{2k} , \quad (32a)$$

$$S_+(t)=[1+\hbar^2|G(t)|^2]^{-2j} \times\sum_{k=0}^{2j-1}\hbar(2j-k)\frac{(2j)!}{(2j-k)!k!}\hbar^{2k}|G(t)|^{2k} \times\hbar G(t)\exp(i\Omega_0 t) , \quad (32b)$$

$$S_-(t)=[1+\hbar^2|G(t)|^2]^{-2j} \times\sum_{k=0}^{2j-1}\hbar(2j-k)\frac{(2j)!}{(2j-k)!k!}\hbar^{2k}|G(t)|^{2k} \times\hbar G^*(t)\exp(-i\Omega_0 t) . \quad (32c)$$

Using the identities

$$\sum_{k=0}^{2j-1}(2j-k)\frac{(2j)!}{(2j-k)!k!}x^{2k}=2j(1+x^2)^{2j-1} , \quad (33a)$$

$$\sum_{k=0}^{2j}(j-k)\frac{(2j)!}{(2j-k)!k!}x^{2k}=j(1-x^2)(1+x^2)^{2j-1} \quad (33b)$$

gives

$$z(t)=j\hbar\frac{1-\hbar^2|G(t)|^2}{1+\hbar^2|G(t)|^2} , \quad (34a)$$

$$S_+(t)=2j\hbar^2\frac{G(t)\exp(i\Omega_0 t)}{1+\hbar^2|G(t)|^2} , \quad (34b)$$

$$S_-(t)=2j\hbar^2\frac{G^*(t)\exp(-i\Omega_0 t)}{1+\hbar^2|G(t)|^2} . \quad (34c)$$

It is immediately clear that the conservation law

$$z^2+S_+S_-=\hbar^2j^2 \quad (35)$$

is satisfied. This should not be confused with the identity

$$\mathrm{Ex}(J^2)=\mathrm{Ex}(J_z^2)+\tfrac{1}{2}\mathrm{Ex}(J_+J_-)+\tfrac{1}{2}\mathrm{Ex}(J_-J_+) =\hbar^2 j(j+1) , \quad (36)$$

which holds for expectations with respect to any state whatsoever, whereas (35) refers to the particular initial state, $|\Psi(0)\rangle=|j,j\rangle$, and is quadratic in the expectations.

In order to justify the claim that this system exhibits chaos, it is useful to determine the rate equations for z and $S_\pm$. Repeated application of (16) and its complex conjugate, definition (10), and Eqs. (34a)–(34c) yield

$$\dot z=\frac{i\Gamma}{2}A(S_- - S_+) , \quad (37a)$$

$$\dot S_\pm=\pm i\Omega_0 S_+ \mp iGAz . \quad (37b)$$

It is readily verified that these are consistent with (35). They are solved together with

$$\dot A=-\Omega B , \quad (38a)$$

$$\dot B=\Omega A+\frac{\Gamma}{\hbar}(S_+ + S_-) . \quad (38b)$$

In addition to the conservation law (35), we also find the second conservation law:

$$\Omega_0 z+\tfrac{1}{4}\hbar\Omega(A^2+B^2)+\tfrac{1}{2}\Gamma A(S_+ + S_-)=\text{const} . \quad (39)$$

Thus, we have coupled, ordinary differential equations in five real variables (z, A, B, and the real and imaginary parts of $S_\pm$) and two conservation laws. This leaves three independent variables, the minimum required for chaos (cf. the Poincaré-Bendixson theorem [2]).

For $J=\frac{1}{2}$, the preceding description reduces to the case previously treated as the BZT model [4]. Thus, we already know that Eqs. (37a), (37b), (38a), and (38b) produce chaos. However, the use of the generalized coherent states has also produced the explicit expressions in (34a)–(34c) in terms of $G(t)$. Moreover, the form of $S_\pm$ in (34b) and (34c) is automatically the so-called rotating-frame representation, on account of the explicit $\exp(\pm i\Omega_0 t)$ dependence. The rotating-frame transformation may also be used for the field variables A and B as follows:

$$E_\pm=A\pm iB , \quad (40a)$$

$$F_\pm=E_\pm\exp(\pm i\Omega_0 t) . \quad (40b)$$

If we simply set $F=F_+$, then $F_-=F^*$, and (34b) and (34c) combined with (38a) and (38b) yields

$$\dot F=i2j\hbar\Gamma\exp(-i\Omega t)\frac{G\exp(i\Omega_0 t)+G^*\exp(-i\Omega_0 t)}{1+\hbar^2|G|^2} . \quad (41)$$

This must be solved together with (10) and (12) which may be rendered

$$\xi=-\frac{i\Gamma}{4\hbar}[F\exp(i\Omega t)+F^*\exp(-i\Omega t)]\exp(i\Omega_0 t)\ , \quad (42a)$$

$$\dot{G}=-\hbar^2G^2\xi-\xi^* \text{ with } G(0)=0\ . \quad (42b)$$

Equations (41) and (42) provide a closed description in terms of four real quantities (real and imaginary parts of F and G) with one conservation law, Eq. (39), rendered as

$$j\hbar\Omega_0\frac{1-\hbar^2|G(t)|^2}{1+\hbar^2|G(t)|^2}+\tfrac{1}{4}\hbar\Omega|F|^2+\tfrac{1}{4}2j\hbar^2\Gamma\frac{[F\exp(i\Omega t)+F^*\exp(-i\Omega t)][G\exp(i\Omega_0 t)+G^*\exp(-i\Omega_0 t)]}{1+\hbar^2|G|^2}=\text{const}\ . \quad (43)$$

III. THE CHAOS MECHANISM

Equations (41)–(43) are an exact consequence of the generalized coherent-state treatment of the semiclassical quantum problem posed by Hamiltonian (1) and Maxwell equations (2a) and (2b). All properties of the state of the angular momentum J are determined by the function G, which is automatically in the Ω_0–rotating-frame representation. All properties of the field are determined by F, which is in the Ω–rotating-frame representation by construction. These rotating-frame representations greatly simplify the elucidation of dynamical chaos in this system. Equations (41) and (42a) make it manifest that there are two widely separated time scales in this problem whenever Ω_0 and Ω are near or in resonance, i.e., a long-time scale $|\Omega-\Omega_0|^{-1}$ and a short-time scale $(\Omega+\Omega_0)^{-1}$.

The rotating-wave approximation (RWA) is justified by arguing that if we average over a time that is long compared to $(\Omega+\Omega_0)^{-1}$ but still short compared to $|\Omega-\Omega_0|^{-1}$, then the fast oscillations in the factors $\exp[\pm i(\Omega_0+\Omega)t]$ will average to zero. This is a heuristic argument, but it can be made rigorous using Hale's averaging theorem [13,14]. In any case, it amounts to replacing Eqs. (41), (42a), and (42b) by

$$\xi=-\frac{i\Gamma}{4\hbar}F^*\ , \quad (44a)$$

$$\dot{G}=-\hbar^2G^2\xi-\xi^* \text{ with } G(0)=0\ , \quad (44b)$$

$$\dot{F}=i2j\hbar\Gamma\frac{G}{1+\hbar^2|G|^2}\ . \quad (44c)$$

Furthermore, we will assume precise resonance ($\Omega_0=\Omega$) for simplicity. In this approximation (RWA), the conservation law (43) becomes two conservation laws:

$$j\hbar\Omega\frac{1-\hbar^2|G(t)|^2}{1+\hbar^2|G(t)|^2}+\tfrac{1}{4}\hbar\Omega|F|^2=C_1\ , \quad (45a)$$

$$\frac{FG^*+F^*G}{1+\hbar^2|G|^2}=C_2\ , \quad (45b)$$

which are readily verified using (44a)–(44c). This, of course, means that the problem has been reduced to two independent real variables and the chaos is no longer a possibility [2] in the RWA. The rapid oscillations we have ignored correspond to virtual transitions [4,10] in the fully quantal treatment wherein the field is also quantized. By using Hale's averaging theorem, we not only rigorously justify the RWA as the lowest order of averaging, but also rigorously obtain the next-order averaging corrections caused by the fast oscillations. These corrections will break conservation law (45b), and once again allow the possibility [2] of chaos. Below, we show that the RWA is equivalent to a near-separatrix pendulum, and that the first-order corrections amount to a periodically modulated, near-separatrix pendulum, known to be a paradigm generic for chaos. The periodic modulations are caused by the virtual transitions.

The key to seeing the preceding connections is the transformation [15]

$$z=j\hbar\cos\phi\ . \quad (46)$$

Now,

$$\ddot{z}=-j\hbar(\ddot{\phi}\sin\phi+\dot{\phi}^2\cos\phi)\ , \quad (47)$$

and from (44a)–(44c)

$$\dot{z}=-\tfrac{1}{2}j\hbar\hbar\Gamma\frac{GF^*-G^*F}{1+\hbar^2|G|^2}=-j\hbar\dot{\phi}\sin\phi\ . \quad (48)$$

Equations (44a)–(44c) and (48) also give

$$\begin{aligned}\ddot{z}&=-\frac{\Gamma^2}{4}j\hbar|F|^2\frac{1-\hbar^2|G|^2}{1+\hbar^2|G|^2}-2j^2\hbar^2\hbar\Gamma^2\frac{|G|^2}{(1+\hbar^2|G|^2)^2}\\&=-\frac{\Gamma^2}{\hbar}z\left[\frac{C_1}{\Omega}-z\right]-\frac{1}{2}\frac{\Gamma^2}{\hbar}(j^2\hbar^2-z^2)\\&=-\frac{\Gamma^2}{2\hbar}\left[j^2\hbar^2+\frac{2C_1}{\Omega}z-3z^2\right]\ . \end{aligned}\quad (49)$$

Solving (48) for $\dot{\phi}$ gives

$$\begin{aligned}\dot{\phi}^2\sin^2\phi&=-\frac{\hbar^2}{4}\Gamma^2\frac{G^2F^{*2}+G^{*2}F^2-2|G|^2|F|^2}{(1+\hbar^2|G|^2)^2}\\&=-\frac{\hbar^2}{4}\Gamma^2\left[C_2^2-\frac{4|G|^2|F|^2}{(1+\hbar^2|G|^2)^2}\right]\\&=-\frac{\hbar^2}{4}\Gamma^2\left[C_2^2-\frac{1}{\hbar^4j^2}(j^2\hbar^2-z^2)\frac{4}{\hbar}\left[\frac{C_1}{\Omega}-z\right]\right]\ .\end{aligned}\quad (50)$$

Putting (50) into (47) and using (49) yields

$$\ddot{\phi}=\Gamma^2\sin\phi+\Gamma^2\frac{\hbar^2C_2^2}{4}\frac{\cos\phi}{\sin^3\phi}\ . \quad (51)$$

This is the equation for a spherical pendulum with the upward vertical position corresponding to $\phi=0$. C_2 measures the azimuthal angular momentum. Our choice of initial conditions for the generalized coherent state required that $G(0)=0$, so that $C_2=0$. Thus, we have a *planar* pendulum in the RWA. Moreover, $G(0)=0$ implies $\phi(0)=0$ as well. However, $\dot{\phi}(0)\neq 0$, as follows from (50), which implies $\dot{\phi}(0)=\pm\Gamma[(C_1/\hbar\Omega)-j]^{1/2}$, which is a measure of the initial energy in the field. These particulars correspond to a near-separatrix motion of a planar pendulum. Equation (49), with initial conditions $z(0)=jh$ and $\dot{z}(0)=0$, describes the very same thing (in this form the solutions are given naturally in terms of Jacobian elliptic functions).

Hale's averaging theorem may be applied to (41), (42a), and (42b) directly. Since this procedure was implemented in an earlier paper [4] on the BZT model, and since the equations there are closely related to those here, we will relegate the details to the Appendix and only quote the outcome of the lengthy computations in the text. The restriction to the resonant case is maintained.

Introduce the dimensionless time $t'=\Omega t$, and the parameter $\epsilon=\Gamma/\Omega$, which is also dimensionless. From now on, all time derivatives refer to t' and we immediately drop explicit use of the prime in t'. Equations (41), (42a), and (42b) become

$$\dot{G}=i\frac{\hbar\epsilon}{4}G^2[F\exp(2it)+F^*]-\frac{i\epsilon}{4\hbar}[F^*\exp(-2it)+F]\ , \tag{52a}$$

$$\dot{F}=i2j\hbar\epsilon[G+G^*\exp(-2it)](1+\hbar^2|G|^2)^{-1}\ . \tag{52b}$$

Every term on the right-hand sides is of order ϵ, but some terms are rapidly oscillating with the frequency 2 (2Ω scaled by Ω^{-1}). Hale's theorem produces equations that incorporate the rapid oscillations systematically to various orders in ϵ.

Denote by $\bar{G}$ and $\bar{F}$ the time averages of G and F over one period of the rapid oscillations. After a lengthy calculation [see (A7) of the Appendix], Hale's averaging theorem yields

$$\begin{aligned}\dot{\bar{G}}=\epsilon&\left[i\frac{\hbar}{4}\bar{G}^2\bar{F}^*-\frac{i}{4\hbar}\bar{F}\right]\\
&+\epsilon^2\Bigg[\left[i\frac{\hbar}{2}\bar{G}[\bar{F}\exp(2it)+\bar{F}^*]\right]\left[\frac{\hbar}{8}\bar{G}^2\bar{F}[\exp(2it)-1]+\frac{1}{8\hbar}\bar{F}^*[\exp(-2it)-1]\right]\\
&-\left[\frac{i\hbar}{4}\bar{G}^2[\exp(2it)+1]-\frac{i}{4\hbar}[\exp(-2it)+1]\right]\{j\hbar\bar{G}^*[\exp(-2it)-1](1+\hbar^2|\bar{G}|^2)^{-1}\}\\
&-\left[\frac{\hbar}{4}\bar{G}\,\bar{F}[\exp(2it)-1]\right]\left[i\frac{\hbar}{4}\bar{G}^2\bar{F}^*-\frac{i}{4\hbar}\bar{F}\right]-\left[\frac{\hbar}{8}\bar{G}^2[\exp(2it)-1]\right][i2j\hbar\bar{G}(1+\hbar^2|\bar{G}|^2)^{-1}]\\
&+\left[\frac{1}{8\hbar}[\exp(-2it)-1]\right][i2j\hbar\bar{G}^*(1+\hbar^2|\bar{G}|^2)^{-1}]\Bigg]\ ,\end{aligned} \tag{53a}$$

$$\begin{aligned}\dot{\bar{F}}=\epsilon&[i2j\hbar\bar{G}(1+\hbar^2|\bar{G}|^2)^{-1}]\\
&+\epsilon^2\Bigg[\left[i2j\hbar[1+\exp(-2it)](1+\hbar^2|\bar{G}|^2)^{-1}-i2j\hbar^3[\bar{G}+\bar{G}^*\exp(-2it)]\frac{\bar{G}+\bar{G}^*}{(1+\hbar^2|\bar{G}|^2)^2}\right]\\
&\times\frac{1}{2}\left[\frac{\hbar}{8}\bar{G}^2\bar{F}[\exp(2it)-1]+\frac{1}{8\hbar}\bar{F}^*[\exp(-2it)-1]+\text{c.c.}\right]\\
&-\left[2j\hbar[1-\exp(-2it)](1+\hbar^2|\bar{G}|^2)^{-1}+2j\hbar^3[\bar{G}+\bar{G}^*\exp(-2it)]\frac{\bar{G}-\bar{G}^*}{(1+\hbar^2|\bar{G}|^2)^2}\right]\\
&\times\frac{1}{2i}\left[\frac{\hbar}{8}\bar{G}^2\bar{F}[\exp(2it)-1]+\frac{1}{8\hbar}\bar{F}^*[\exp(-2it)-1]-\text{c.c.}\right]\\
&-\left[-j\hbar[\exp(-2it)-1](1+\hbar^2|\bar{G}|^2)^{-1}+j\hbar^3\bar{G}^*[\exp(-2it)-1]\frac{\bar{G}+\bar{G}^*}{(1+\hbar^2|\bar{G}|^2)^2}\right]\frac{1}{2}\left[i\frac{\hbar}{4}\bar{G}^2\bar{F}^*-\frac{i}{4\hbar}\bar{F}+\text{c.c.}\right]\\
&-\left[ij\hbar[\exp(-2it)-1](1+\hbar^2|\bar{G}|^2)^{-1}-ij\hbar^3\bar{G}^*[\exp(-2it)-1]\frac{\bar{G}-\bar{G}^*}{(1+\hbar^2|\bar{G}|^2)^2}\right]\frac{1}{2i}\left[i\frac{\hbar}{4}\bar{G}^2\bar{F}^*-\frac{i}{4\hbar}\bar{F}-\text{c.c.}\right]\Bigg]\ .\end{aligned} \tag{53b}$$

The order ϵ terms correspond precisely to the RWA given by (44a)–(44c). The order ϵ^2 terms contain $\bar{G}$, $\bar{F}$, and exp($\pm$2it) rapid oscillations. The averaging theorem implies [see (A9) of the Appendix]

$$G=\bar{G}+\epsilon\left[\frac{\hbar}{8}\bar{G}^2\bar{F}[\exp(2it)-1]+\frac{1}{8\hbar}\bar{F}^*[\exp(-2it)-1]\right]+O(\epsilon^2)\ , \tag{54a}$$

$$F=\bar{F}+\epsilon\{-j\hbar\bar{G}^*[\exp(-2it)-1]\times(1+\hbar^2|\bar{G}|^2)^{-1}\}+O(\epsilon^2)\ . \tag{54b}$$

Thus, we may write

$$z=\bar{z}+\epsilon\frac{\hbar^2}{4}\{\bar{F}\,\bar{G}[1-\exp(2it)]+\text{c.c.}\}(1+\hbar^2|\bar{G}|^2)^{-1}+O(\epsilon^2)\ . \tag{55}$$

Taking two time derivatives and keeping only the lowest-order terms for the z part $[O(\epsilon^2)]$ and for the rapid oscillations $[O(\epsilon)]$ yields

$$\ddot{z}=-\frac{\epsilon^2}{2\hbar}\left[j^2\hbar^2+\frac{2C_1}{\Omega}z-3z^2\right]+\epsilon\hbar^2(1+\hbar^2|\bar{G}|^2)^{-1}[\bar{F}\,\bar{G}\exp(2it)+\text{c.c.}]\ , \tag{56}$$

in which $\bar{F}$ and $\bar{G}$ satisfy the order ϵ equation (RWA) implicit in (53a) and (53b). This is the modulated, near-separatrix, planar pendulum that is generically chaotic. Our earlier work [4] showed that numerical integrations of (56) for $\epsilon=0.05$ are in very good quantitative agreement with numerical integrations of the complete set of quantum equations (41), (42a), and (42b).

It is straightforward to obtain the variance of J_z in closed form:

$$\text{Ex}([J_z-\text{Ex}(J_z)]^2)=2j\hbar^2\frac{\hbar^2|G|^2}{(1+\hbar^2|G|^2)^2}=\frac{1}{2j}[(j\hbar)^2-z^2]=\tfrac{1}{2}\hbar^2 j\sin^2\phi\ , \tag{57}$$

where the second equality follows from (34a) and the third from (46). Initially, $\phi=0$, and for $C_2=0$, Eq. (51) implies an initially exponential growth of ϕ $[\dot{\phi}(0)\neq 0]$. Thus, as long as ϕ is in the regime where $\sin\phi\sim\phi$ to at least 1%, say, i.e., $\phi<0.3$ (in radians), then the variance of J_z grows exponentially. This can be seen explicitly for the parameter values used in our earlier work [4,10]. During the initial stages, the solution for $\phi(t)$ is

$$\phi(t)=\frac{\dot{\phi}(0)}{2\epsilon}[\exp(\epsilon t)-\exp(-\epsilon t)]\ . \tag{58}$$

The parameter values are $\epsilon=0.05$ and $\dot{\phi}(0)=-10^{-5}$. There is an induction stage lasting until $t=46$ during which ϕ grows from zero to -9×10^{-5} and at the end of which the exponential $\exp(-\epsilon t)$ is only 1% of the size of the exponential $\exp(\epsilon t)$. For the next 162 units of time, ϕ grows as an almost pure exponential over more than three orders of magnitude (the variance grows by *seven* orders of magnitude) until it reaches 0.3 rad. After this, of course, the nonlinear terms in $\sin\phi$ become important and the growth saturates. This is in keeping with our earlier observations [7,8] for the chaotic, periodically kicked pendulum treated quantum mechanically.

In summary, we have shown the following.

(1) Equations (37a) and (37b) and (38a) and (38b) provide an exact description of this semiclassical quantum problem and produce chaotic time dependence with a positive Liapunov exponent. An equivalent set of equations is given by Eqs. (41), (42a), (42b), and (43) in terms of the natural, rotating-frame quantities F and G.

(2) The chaotic physical quantities $z(t)$, $S_+(t)$, and $S_-(t)$ may be expressed in closed form in terms of the function $G(t)$ as a result of the time-ordered generalization of Helgason's factorization identity.

(3) The mechanism of the chaos is that of a periodically perturbed, near-separatrix planar pendulum as is exhibited by Eq. (56). The periodic perturbations have their origin in virtual quantum transitions that have no classical analogue.

(4) The variance of J_z shows an initial exponential growth over seven orders of magnitude. Such exponential growth of the variance appears to be characteristic of chaos in quantum systems and is very different from the usual quadratic-in-time growth of the variance of a quantum operator during nonchaotic dynamics.

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APPENDIX

Application of Hale's theorem [4,13,14] requires putting Eqs. (41), (42a), and (42b) into real form. We do this by taking the real and imaginary parts of (52a) and (52b):

$$\dot{G}_r=\tfrac{1}{2}\left[i\frac{\hbar\epsilon}{4}G^2[F\exp(2it)+F^*]-\frac{i\epsilon}{4\hbar}[F^*\exp(-2it)+F]+\text{c.c.}\right]\ , \tag{A1a}$$

$$\dot{G}_i=\frac{1}{2i}\left[i\frac{\hbar\epsilon}{4}G^2[F\exp(2it)+F^*]-\frac{i\epsilon}{4\hbar}[F^*\exp(-2it)+F]-\text{c.c.}\right]\ , \tag{A1b}$$

$$\dot{F}_r=\tfrac{1}{2}\{i2j\hbar\epsilon[G+G^*\exp(-2it)](1+\hbar^2|G|^2)^{-1}+\text{c.c.}\}\ , \tag{A1c}$$

$$\dot{F}=\frac{1}{2i}\{i2j\hbar\epsilon[G+G^*\exp(-2it)](1+\hbar^2|G|^2)^{-1}-\text{c.c.}\}\ . \tag{A1d}$$

Following Hale's procedure and using the notation from our 1987 paper [4], we obtain

$$\bar{f}_1=\frac{1}{2}\left[i\frac{\hbar}{4}\bar{G}^2\bar{F}^*-\frac{i}{4\hbar}\bar{F}+\text{c.c.}\right], \tag{A2a}$$

$$\bar{f}_2=\frac{1}{2i}\left[i\frac{\hbar}{4}\bar{G}^2\bar{F}^*-\frac{i}{4\hbar}\bar{F}-\text{c.c.}\right], \tag{A2b}$$

$$\bar{f}_3-\tfrac{1}{2}[i2j\hbar\bar{G}(1+\hbar^2|\bar{G}|^2)^{-1}+\text{c.c.}], \tag{A2c}$$

$$\bar{f}_4=\frac{1}{2i}[i2j\hbar\bar{G}(1+\hbar^2|\bar{G}|^2)^{-1}-\text{c.c.}], \tag{A2d}$$

$$\tilde{f}_1=\frac{1}{2}\left[i\frac{\hbar}{4}\bar{G}^2\bar{F}\exp(2it)-\frac{i}{4\hbar}\bar{F}^*\exp(-2it)+\text{c.c.}\right], \tag{A3a}$$

$$\tilde{f}_2=\frac{1}{2i}\left[i\frac{\hbar}{4}\bar{G}^2\bar{F}\exp(2it)-\frac{i}{4\hbar}\bar{F}^*\exp(-2it)-\text{c.c.}\right], \tag{A3b}$$

$$\tilde{f}_3=\tfrac{1}{2}[i2j\hbar\bar{G}^*\exp(-2it)(1+\hbar^2|\bar{G}|^2)^{-1}+\text{c.c.}], \tag{A3c}$$

$$\tilde{f}_4=\frac{1}{2i}[i2j\hbar\bar{G}^*\exp(-2it)(1+\hbar^2|\bar{G}|^2)^{-1}-\text{c.c.}], \tag{A3d}$$

and

$$\omega_1=\frac{1}{2}\left[\frac{\hbar}{8}\bar{G}^2\bar{F}[\exp(2it)-1]+\frac{1}{8\hbar}\bar{F}^*[\exp(-2it)-1]+\text{c.c.}\right], \tag{A4a}$$

$$\omega_2=\frac{1}{2i}\left[\frac{\hbar}{8}\bar{G}^2\bar{F}[\exp(2it)-1]+\frac{1}{8\hbar}\bar{F}^*[\exp(-2it)-1]-\text{c.c.}\right], \tag{A4b}$$

$$\omega_3=\tfrac{1}{2}\{-j\hbar\bar{G}^*[\exp(-2it)-1](1+\hbar^2|\bar{G}|^2)^{-1}+\text{c.c.}\}, \tag{A4c}$$

$$\omega_4=\frac{1}{2i}\{-j\hbar\bar{G}^*[\exp(-2it)-1](1+\hbar^2|\bar{G}|^2)^{-1}-\text{c.c.}\}. \tag{A4d}$$

Now we let y_i be defined by

$$y_1=\bar{G}_r, \tag{A5a}$$

$$y_2=\bar{G}_i, \tag{A5b}$$

$$y_3=\bar{F}_r, \tag{A5c}$$

$$y_4=\bar{F}_i. \tag{A5d}$$

We also write

$$f_i=\bar{f}_i+\tilde{f}_i. \tag{A6}$$

Hale's theorem implies

$$y_i=\epsilon\bar{f}_i+\epsilon^2\left[\frac{\partial f_i}{\partial y_j}\omega_j-\frac{\partial\omega_i}{\partial y_j}\bar{f}_j\right]. \tag{A7}$$

The original variables that appear in (A1a)–(A1d) may be denoted by x_i and are given by

$$x_1=G_r, \tag{A8a}$$

$$x_2=G_i, \tag{A8b}$$

$$x_3=F_r, \tag{A8c}$$

$$x_4=F_i. \tag{A8d}$$

After computing all of the derivatives in (A7) we finally get Eqs. (53a) and (53b) of the text. Hale's theorem also implies that

$$x_i=y_i+\epsilon\omega_i+O(\epsilon^2), \tag{A9}$$

which is used to write Eqs. (54a) and (54b) of the text.

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