Quantum versus stochastic or hidden-variable fluctuations in two-photon interference effects

C. Su and K. Wódkiewicz*
Center for Advanced Studies and Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131
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In a series of experiments performed by Mandel and co-workers, nonclassical effects have been demonstrated in the interference of two photons generated in a process of parametric down-conversion. The nonclassical effects in the two-photon interference effects can be discussed in the framework of two different descriptions. In the first description, a stochastic theory of electromagnetic field fluctuations can be used in order to calculate the interference pattern. In the second description, a theory of hidden-variable fluctuations can be applied in order to calculate correlations of the interference pattern. A stochastic theory leads to statistical inequalities for the light intensities, while a local hidden-variable theory leads to Bell's inequalities. Using the Schwinger-boson representation of the angular momentum, we show that the two-photon interference effects can be described in terms of spin-correlated states. In particular, we show that the action of a beam splitter on the photons in a parametric down-conversion is equivalent to the production of an entangled state that is very similar to the well-known Einstein, Podolsky, and Rosen spin-singlet state. We show that the stochastic theory of two-photon fluctuations is not equivalent to a hidden-variable theory of photon correlations. We establish a range for which the stochastic theory fails but the hidden-variable theory is still possible. We compare our theoretical predictions with the experimental results and conclude that a violation of the stochastic theory has been clearly observed, while the violation of the hidden-variable theory is less pronounced.

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I. INTRODUCTION

In a series of experiments, Mandel and co-workers have studied nonclassical effects in two-photon interference [1–5]. These nonclassical effects have been demonstrated in the interference of signal and idler photons generated in a process of parametric down-conversion. In the experiment described in Ref. [3], correlation measurements of mixed idler and signal photons have been performed as a function of two linear polarizer settings, in order to observe the violation of Bell's inequality.

Nonclassical effects in these kind of experiments can be divided into two different categories. The first category of nonclassical effects follows from the assumption that in the interference effects the signal and the idler electric fields are described by classical stochastic random variables [6–8]. This stochastic description of the electromagnetic fields involved in two beam experiments lead to well-known classical inequalities for the stochastic expectation values of the fluctuating idler I_i and signal I_s intensities. Typical inequalities for this categories of effects have the following form [9]:

$$\langle I_s^2 \rangle \geq \langle I_s \rangle^2 \text{ and } \langle I_i^2 \rangle \geq \langle I_i \rangle^2 , \tag{1.1}$$

$$\langle I_s^2 \rangle + \langle I_i^2 \rangle \geq 2 \langle I_s I_i \rangle , \tag{1.2}$$

$$\langle I_s^2 \rangle \langle I_i^2 \rangle \geq \langle I_i I_s \rangle^2 . \tag{1.3}$$

An experimental observation of the violation of any of these classical inequalities is an indication of the failure of the stochastic description of electromagnetic field fluctuations in the two beam experiments. Because of this property any theory of two beams interference constrained by these inequalities will be called a stochastic theory (ST) of light interference.

The second category of nonclassical effects is much more subtle and involves tests of the locality of quantum correlations. The local versus nonlocal description of correlations is very well understood for measurements of spin polarizations for a singlet Einstein, Podolsky, and Rosen (EPR) spin state [10]. Local and objective description of the spin components based on the theory of hidden variables leads to fundamental constraints on the joint probability $p(\mathbf{a},\mathbf{b})$ involved in two spin-orientation measurements given by directions $\mathbf{a}$ and $\mathbf{b}$. These constraints take the form of inequalities, first derived and discussed by Bell. The most widely used Bell's inequality has the following form [10,11]:

$$\begin{aligned} -p(\infty,\infty) \leq p(\mathbf{a},\mathbf{b}) - p(\mathbf{a},\mathbf{b}') + p(\mathbf{a}',\mathbf{b}) + p(\mathbf{a}',\mathbf{b}') \\ - p(\mathbf{a}',\infty) - p(\infty,\mathbf{b}) \leq 0 , \end{aligned} \tag{1.4}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}'$, and $\mathbf{b}'$ are arbitrary polarizer orientations and the symbol ∞ in any of these joint probabilities indicates that the polarizer is removed.

In contrast to the EPR spin state, correlation measurements of two beams interference offer a wide variety of physical phenomena. Classical interference effects are mixed with purely quantum-mechanical effects associated with the polarization measurements or the single-photon detection. Theories of two beams interference constrained by the inequality (1.4) are called local-hidden-variable (LHV) theories. These LHV theories form the second category of nonclassical effects [8,12,13] observed or tested in two-photon parametric down-conversion.

At this point it is fair to raise the issue of complementarity between these two categories of theories. Are the violations of inequalities (1.1)–(1.3) a consequence of the violations of Bell's inequality (1.4)? Is the violation of the Bell's inequality (1.4) inevitably followed by a violation of the stochastic inequalities? It is possible to have a LHV description and violate the stochastic picture? It is possible to violate Bell's inequality and still have an adequate stochastic description? Which aspects of the ST or the LHV description of two-photon correlations have been tested experimentally and which have not?

It is the purpose of this paper to study these problems and to provide answers to these questions. This paper is organized in the following way. In Sec. II we give a theoretical description of the interference of two photons using the Schwinger representation for angular momentum. We show that in this representation the interference of the signal and idler photons can be described as correlations of spin components. Because of this description we can establish a connection between the two-photon interference effects and the EPR spin correlations. We offer in this section a ST of the interference and a LHV theory of photon correlations based on local realism. We compare the ST and the LHV theory predictions with quantum mechanics. The nonlocal and the jumplike character of two-photon correlations is established.

In Sec. III we study the interference if the signal and the idler photons are mixed by a beam splitter. Using the angular momentum description of these photons we show that the action of the beam splitter is equivalent to a production of the EPR entangled state from two uncorrelated spin states. In this framework we provide a nonlocal description of photon correlations with a beam splitter. The ST and the LHV theory of such correlations are compared with quantum-mechanical predictions and the experimental results. In Sec. IV some final conclusions and remarks are presented.

II. INTERFERENCE OF TWO PHOTONS

A. Angular momentum description of the interference

It is the purpose of this section to give a complete discussion of the interference of two correlated photons from the point of view of a LHV description and from the point of view of a ST of two beams fluctuations. We start our discussion by reformulating the two-photon interference in terms of spin-variable correlations.

We consider a degenerate parametric down-conversion process in which a signal and idler photons are produced. Measuring the joint probability for the detection of these two photons at two positions of the photodetectors, an interference pattern has been observed. Following the theoretical description of this effect, presented in Ref. [8], we assume that the down-conversion process is generating a two-photon state $|1_A,1_B\rangle$, where A and B correspond to single-mode signal and idler photons described by boson $\hat{a},\hat{a}^\dagger$ and $\hat{b},\hat{b}^\dagger$ annihilation and creation operators, respectively.

At the detector the positive-frequency part of the electric field can be expressed by the following formula:

$$\hat{E}^{(+)}(\phi)=\frac{1}{\sqrt{2}}(\hat{a}+\hat{b}e^{-i\phi}) , \tag{2.1}$$

where the phase ϕ corresponds to the geometrical spacing of the detector and the numerical prefactor has been selected for later convenience. The field intensity operator at the screen is given by the following formula:

$$\hat{I}(\phi)=\hat{E}^{(-)}(\phi)\hat{E}^{(+)}(\phi)$$
$$=\tfrac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}+\hat{b}^\dagger\hat{a}e^{i\phi}+\hat{a}^\dagger\hat{b}e^{-i\phi}). \tag{2.2}$$

The two independent boson excitations of the signal and idler field can be used as a Schwinger's representation [14] of a fictitious angular momentum $\hat{\mathbf{J}}$. In this representation we have the following formulas for the angular momentum operators:

$$\hat{J}_+=\hat{a}^\dagger\hat{b},\quad \hat{J}_-=\hat{b}^\dagger\hat{a} , \tag{2.3a}$$

$$\hat{J}_z=\tfrac{1}{2}(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}),\quad \hat{N}=\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b} , \tag{2.3b}$$

$$\mathbf{J}^2=\frac{N}{2}\left[\frac{N}{2}+1\right]=j(j+1) , \tag{2.3c}$$

where $j=N/2$ is the total angular momentum (spin) of the system.

Using the relations (2.3), we can rewrite Eq. (2.2) in the following form:

$$\hat{I}=-\tfrac{1}{2}(\hat{N}+\hat{J}_-e^{i\phi}+\hat{J}_+e^{-i\phi}) . \tag{2.4}$$

It is easily shown that for the initial number state $|1_A,1_B\rangle$, the only possible states generated by $\hat{I}$ are $|1_A,1_B\rangle$, $|2_A,0_B\rangle$, and $|0_A,2_B\rangle$.

Using the definitions (2.3) it is easy to check that these three states correspond to an angular momentum (spin) equal to 1 ($N=2$) and three different spin projections $m=0,1,-1$. From the point of view of the spin variables, the two-photon interference is equivalent to a sin-1 system. The three-photon states quoted above shall be described using the angular notation $|0\rangle$, $|+\rangle$, and $|-\rangle$, with $m=0,\pm1$ and where we have dropped the index $j=1$.

In spin variables the photon intensity (2.2) has the following form:

$$\hat{I}(\phi)=1+\hat{J}(\phi) , \tag{2.5a}$$

where

$$\hat{J}(\phi)=\hat{\mathbf{J}}\cdot\mathbf{n} , \tag{2.5b}$$

with

$$\mathbf{n}=(\cos\phi,\sin\phi,0) . \tag{2.6}$$

From the definition of $\hat{E}(\phi)$ and $\hat{J}(\phi)$, we obtain the following two commutation relations:

$$[\hat{E}^{(+)}(\phi_1),\hat{E}^{(-)}(\phi_2)]=\tfrac{1}{2}(1+e^{i(\phi_1-\phi_2)}) , \tag{2.7}$$

$$[\hat{J}(\phi_1),\hat{J}(\phi_2)]=i\hat{J}_z\sin(\phi_1-\phi_2) . \tag{2.8}$$

The joint probability of photodetection at the ϕ_1 and ϕ_2 locations of the interference experiment is given by

$$p_{12}(\phi_1,\phi_2)=\langle \hat{E}^{(-)}(\phi_1)\hat{E}^{(-)}(\phi_2)\hat{E}^{(+)}(\phi_2)\hat{E}^{(+)}(\phi_1)\rangle$$
$$=\langle :\hat{I}(\phi_1)\hat{I}(\phi_2):\rangle \;, \quad (2.9a)$$

where :: denotes the normal order of the electric-field operators, and the average is over state $|0\rangle$. We shall assume in the following perfect detectors so all the quantum efficiency factors can be set to unity. It is readily verified that the normally ordered intensities are

$$\langle :\hat{I}(\phi_1)\hat{I}(\phi_2):\rangle$$
$$=\langle \hat{I}(\phi_1)\hat{I}(\phi_2)\rangle-\langle \hat{E}^{(-)}(\phi_1)\hat{E}^{(+)}(\phi_2)\rangle \times[\hat{E}^{(+)}(\phi_1),\hat{E}^{(-)}(\phi_2)] \;. \quad (2.9b)$$

The first term of this expression can be easily calculated. We note first that due to the commutation relation (2.8) the angular momentum $\hat{J}(\phi)$ commutes for different values of ϕ if applied to the state $|0\rangle$. As a result of this we obtain the following expression:

$$\langle \hat{I}(\phi_1)\hat{I}(\phi_2)\rangle=1+\cos(\phi_1-\phi_2) \;. \quad (2.10)$$

The second contribution to the expression (2.9) involves a commutator of the electric-field operators. For the two-photon state of the idler and signal we obtain that

$$\langle \hat{E}^{(-)}(\phi_1)\hat{E}^{(+)}(\phi_2)\rangle=\tfrac{1}{2}(1+e^{i(\phi_1-\phi_2)}) \;, \quad (2.11a)$$

$$\langle \hat{E}^{(-)}(\phi_1)\hat{E}^{(+)}(\phi_2)\rangle=[\hat{E}^{(+)}(\phi_1),\hat{E}^{(-)}(\phi_2)]^* \;, \quad (2.11b)$$

and

$$\langle \hat{I}(\phi_1)\hat{I}(\phi_2)\rangle=2|[\hat{E}^{(+)}(\phi_1),\hat{E}^{(-)}(\phi_2)]|^2 \;. \quad (2.11c)$$

Because of these relations we obtain a remarkable simplification of the formula (2.9):

$$\langle :\hat{I}(\phi_1)\hat{I}(\phi_2):\rangle=\tfrac{1}{2}\langle \hat{I}(\phi_1)\hat{I}(\phi_2)\rangle$$
$$=\tfrac{1}{2}[1+\cos(\phi_1-\phi_2)] \;. \quad (2.12)$$

From this formula it follows that for this particular state of the field the intensity operators $\hat{I}(\phi_1)$ and $\hat{I}(\phi_2)$ commute and that the normal ordering of these operators reduces to a trivial factor of $\frac{1}{2}$.

The angular momentum average with a state of given m is of course equal to zero. As a result of this we obtain that the individual intensity averages are

$$\langle \hat{I}(\phi_1)\rangle=\langle \hat{I}(\phi_2)\rangle=1. \quad (2.13)$$

From Eqs. (2.12) and (2.13) we can calculate the normalized second-order coherence function:

$$g^{(2)}(\phi_1,\phi_2)=\frac{\langle :\hat{I}(\phi_1)\hat{I}(\phi_2):\rangle}{\langle \hat{I}(\phi_1)\rangle\langle \hat{I}(\phi_2)\rangle}=\tfrac{1}{2}[1+\cos(\phi_1-\phi_2)] \;, \quad (2.14)$$

which due to our normalization is identical to the joint probability given by Eq. (2.14).

We conclude this part noting that the use of Schwinger's angular momentum representation has helped us to rewrite and to reinterpret the quantum-mechanical intensity correlations in terms of spin-1 angular momentum variables. We shall dwell on this description in the following part in which the nonlocal character of quantum correlations will be discussed.

B. Nonlocal description of the interference

In Sec. II A we have derived the normally ordered correlation functions of the electric-field operators using Schwinger's angular momentum representation. In this part we shall explore the spin-1 picture of two-photon correlations in order to formulate and to discuss the nonlocal character of these correlations. From Eqs. (2.9) and (2.14) we concluded that

$$g^{(2)}(\phi_1,\phi_2)=p_{12}(\phi_1,\phi_2)=\tfrac{1}{2}[1+\cos(\phi_1-\phi_2)] \;. \quad (2.15)$$

In the following we shall give a spin-correlation interpretation of this formula.

Let us consider the operator $\hat{P}(\phi)$ which is defined as

$$\hat{P}(\phi)=\tfrac{1}{2}[1+\hat{J}(\phi)]=\frac{\hat{I}(\phi)}{2}, \quad (2.16a)$$

where $\hat{I}(\phi)$ is the intensity operator (2.2). If $\hat{P}(\phi)$ operates on the state $|0\rangle$ it is easy to show that

$$\langle \hat{P}(\phi)\rangle=\langle \hat{P}^2(\phi)\rangle \;, \quad (2.16b)$$

i.e., $\hat{P}(\phi)$ acts like a projection operator on $|0\rangle$. In addition, we obtain from the definition (2.16a) the following decomposition of unity:

$$\int_0^{2\pi}\frac{d\phi}{\pi}\hat{P}(\phi)=1 \;. \quad (2.17)$$

According to these formulas the operator $\hat{P}(\phi)$ can be interpreted as a projection of the spin on a direction given by the unit vector (2.6). A joint measurement involving two different orientations ϕ_1 and ϕ_2 is described in this case by the following probability:

$$p(\phi_1,\phi_2)=\langle \hat{P}(\phi_1)\hat{P}(\phi_2)\rangle=\tfrac{1}{4}\langle \hat{I}(\phi_1)\hat{I}(\phi_2)\rangle \;. \quad (2.18)$$

Using the result (2.10), Eq. (2.18) becomes

$$p(\phi_1,\phi_2)=\tfrac{1}{4}[1+\cos(\phi_1-\phi_2)] \;, \quad (2.19)$$

i.e., an expression identical to a joint measurement of spin orientations in EPR correlation. Because of the relation (2.12) we obtain that a spin joint probability $p(\phi_1,\phi_1)$ is just $\frac{1}{2}$ of the normally ordered electric fields joint probability $p_{12}(\phi_1,\phi_2)$. Performing the marginal average of this distribution, with the help of Eq. (2.17) we obtain the onefold distribution function which in this case is equal to the probability with one of the polarizers removed:

$$p(\phi_1,\infty)=p(\phi_1)=\int_0^{2\pi}\frac{d\phi_2}{\pi}p(\phi_1,\phi_2)=\tfrac{1}{2} \;. \quad (2.20a)$$

Similarly

$$p(\infty,\phi_2)=p(\phi_2)=\tfrac{1}{2} \;. \quad (2.20b)$$

Because of the factor $\frac{1}{2}$ involved in the expressions (2.15) and (2.19), it is important to note that the marginals of $p_{12}(\phi_1,\phi_2)$ do not lead to marginal distributions (2.20). This is because the joint distribution p_{12} involves normally ordered operators which are not projection operators. Only averages of projection operators have the right marginal distributions. In the present physical situation the difference is in a trivial factor of $\frac{1}{2}$, but it reflects again the fact that the definitions (2.9) do not involve projection operators.

It is well known in the context of a LHV theory of spin correlations, that the joint probability (2.19) will violate Bell's inequality (1.4) for some values of ϕ_1 and ϕ_2. It means that an interference measurement involving a two-photon state can violate Bell's inequality if different positions of the interference pattern are investigated. This violation follows of course from the wrong assumption that spin orientations are objective local realities subjected only to local hidden parameters denoted here by λ_1 and λ_2.

In the following we shall present arguments showing that quantum-mechanical correlations violate this locality assumption [15]. In fact, we shall derive an explicit formula underlying the nonlocal character in two beams interference.

Using a trivial identity

$$\hat{I}(\phi)=\int d\lambda\,\lambda\delta(\lambda-\hat{P}(\phi)) \tag{2.21}$$

we can rewrite the formula (2.18) in the following form:

$$p(\phi_1,\phi_2)=\int d\lambda_1\int d\lambda_2\lambda_1\lambda_2 p(\phi_1,\lambda_1;\phi_2,\lambda_2)\ , \tag{2.22}$$

where the distribution function $p(\phi_1,\lambda_1;\phi_2,\lambda_2)$ is defined as

$$p(\phi_1,\lambda_1;\phi_2,\lambda_2)=\langle\delta(\lambda_1-\hat{P}(\phi_1))\delta(\lambda_2-\hat{P}(\phi_2))\rangle\ . \tag{2.23}$$

After some simple algebra, Eq. (2.23) yields

$$p(\phi_1,\lambda_1;\phi_2,\lambda_2)=\tfrac{1}{4}[1+\cos(\phi_1-\phi_2)][\delta(\lambda_1)\delta(\lambda_2)+\delta(\lambda_1-1)\delta(\lambda_2-1)] + \tfrac{1}{4}[1-\cos(\phi_1-\phi_2)][\delta(\lambda_1-1)\delta(\lambda_2)+\delta(\lambda_1)\delta(\lambda_2-1)]\ . \tag{2.24}$$

The distribution function is bivalued, with the parameters λ_1 and λ_2 being random numbers that can be 0 or 1. If the random number is 0 no photon is detected. If the value of this number is 1 a photon has been detected. The expression (2.22) clearly has the form of a hidden-variable theory. But there is a fundamental difference, because the distribution (2.24) depends in addition on the geometrical location of the detectors given by the angles ϕ_1 and ϕ_2. In this sense this distribution is nonlocal because it depends on arbitrary and even remote positions of the photodetectors [16]. This nonlocal character of the distribution follows from quantum mechanics and results in the violation of Bell's inequality (1.4). The derivation of this inequality is based on the locality assumption which requires that the distribution of hidden variables is independent from the orientation (position) of the detectors. From the nonlocal quantum-mechanical distribution (2.24) one can easily establish joint distributions involving, for example, a photon detection at ϕ_1 and no photon detection at ϕ_2, or a no photon detection at ϕ_1 and no photon detection at ϕ_2. We shall denote these joint distributions by $p(i;j)$ with $i,j=0,1$, where the first digit corresponds to yes or no detection at the position ϕ_2 and the second represents a yes or no detection at the location ϕ_1. From Eq. (2.24), it is easily verified that

$$p(0;0)=p(1;1)=\tfrac{1}{4}[1+\cos(\phi_1-\phi_2)]\ , \tag{2.25a}$$

$$p(0;1)=p(1;0)=\tfrac{1}{4}[1-\cos(\phi_1-\phi_2)]\ . \tag{2.25b}$$

We can rewrite the joint-probability distribution (2.24) in terms of a conditional probability and a onefold distribution according to the Bayes formula

$$p(\phi_1,\lambda_1;\phi_2,\lambda_2)=p(\phi_2,\lambda_2|\phi_1,\lambda_1)p(\phi_1,\lambda_1), \tag{2.26}$$

and where the marginal distribution is

$$p(\phi_1,\lambda_1)=\int d\lambda_2 p(\phi_1,\lambda_1;\phi_2,\lambda_2) =\tfrac{1}{2}\delta(\lambda_1-1)+\tfrac{1}{2}\delta(\lambda_1)\ . \tag{2.27}$$

From these relations we derive the following conditional probabilities involving combinations of yes and no detections:

$$p(0|0)=p(1|1)=\tfrac{1}{2}[1+\cos(\phi_1-\phi_2)]\ , \tag{2.28a}$$

$$p(0|1)=p(1|0)=\tfrac{1}{2}[1-\cos(\phi_1-\phi_2)]\ . \tag{2.28b}$$

Comparing Eqs. (2.28a) and (2.21) we can conclude that the second-order coherence function can be identified with a conditional probability of yes-yes and no-no detection. It means that the probability of photon detection at ϕ_2 under the condition of zero photon detection at ϕ_1 is given by the formula (2.28a). The same relation holds for no photon detection at ϕ_2 under the condition of no photon detection at ϕ_1. If the photodetectors are at the same position, i.e., $\phi_1=\phi_2$ once a photon has been detected the detection of the second one is a certainty [$p(1,1)=1$ for $\phi_2=\phi_2$]. If $\phi_1-\phi_2=\pi,3\pi,\ldots$ once a photon has been detected at ϕ_1 there is a certainty that the next photon will not be detected [$p(1,1)=0$ for $\phi_1-\phi_2=\pi$]. The outcome of each detector is represented by a perfect random series of 0 and 1 equally distributed. The nonlocal character of these two random sequences is reflected in their correlations. For example, if the outcome of the photodetection at ϕ_1 is given by the following random sequence: $\lambda_1=(0,1,1,0,0,\ldots)$ the outcome at

$\phi_2=\phi_1+\pi$ can be predicted with certainty and is equal to $\lambda_2=(1,0,0,1,1,\ldots)$, i.e., no correlations of digits $1\rightarrow1$ or $0\rightarrow0$ can be observed. Changing the value of ϕ_2 we will change the outcome of the experiment according to the formula (2.28a). This example illustrates the nonlocal character of quantum correlations in which two perfectly random sequences λ_1 and λ_2 are correlated with changing position of photodetectors. This is how quantum mechanics works.

In conclusion we note that the nonlocal character of quantum fluctuations leading to the violation of Bell's inequality cannot be tested in a single experiment. A demolition of local realism always requires a series of experiments involving different locations of ϕ_1 and ϕ_2 according to the inequality (1.4). This situation is quite different from the stochastic description involving classical inequalities (1.1)–(1.3) which are completely independent from the location of the photodetectors.

In the following we shall investigate the stochastic picture of the interference of two photons.

C. Stochastic description of the interference

In classical wave optics, as we know, the electrical field is just a c number instead of an operator. In the two-photon interference case, we can express the field at the detector as a linear superposition of the signal ($\mathcal{E}_s$) and idler ($\mathcal{E}_i$) fields. For the positive-frequency part of this field we have

$$\mathcal{E}^{(+)}(\phi)=\mathcal{E}_s+\mathcal{E}_i e^{-i\phi} . \tag{2.29}$$

The instantaneous intensity of this field is

$$I(\phi)=I_s+I_i+\mathcal{E}_s\mathcal{E}_i^* e^{i\phi}+\mathcal{E}_s^*\mathcal{E}_i e^{-i\phi} , \tag{2.30}$$

where $I_s=|\mathcal{E}_s|^2$, $I_i=|\mathcal{E}_i|^2$.

In a stochastic description, the electric fields $\mathcal{E}_s$ and $\mathcal{E}_i$ become random variables described by a classical distribution function. The outcome of the measurement at the detector is given as an ensemble average over different statistical realizations of these fields. In addition, we shall disregard all terms nonstationary in the phase difference ϕ_1 and ϕ_2. As a result of these assumptions we obtain that the mean intensity at the detector is

$$\langle I(\phi)\rangle=\langle I_s\rangle+\langle I_i\rangle . \tag{2.31}$$

Following the same procedure we obtain the following intensity correlation function at two different locations ϕ_1 and ϕ_2:

$$\langle I(\phi_1)I(\phi_2)\rangle=\langle I_s^2\rangle+\langle I_i^2\rangle+2\langle I_sI_i\rangle[1+\cos(\phi_1-\phi_2)] . \tag{2.32}$$

The degree of second-order coherence becomes

$$g^{(2)}(\phi_2,\phi_1)=\frac{\langle I(\phi_1)I(\phi_2)\rangle}{\langle I(\phi_1)\rangle\langle I(\phi_2)\rangle} = A_{\mathrm{ST}}[1+\eta_{\mathrm{ST}}\cos(\phi_1-\phi_2)] , \tag{2.33}$$

where the stochastic parameter A_{ST} and η_{ST} are given by the following formulas:

$$A_{\mathrm{ST}}=\frac{\langle I_s^2\rangle+\langle I_i^2\rangle+2\langle I_sI_i\rangle}{[\langle I_s\rangle+\langle I_i\rangle]^2} , \tag{2.34}$$

$$\eta_{\mathrm{ST}}=\frac{2\langle I_sI_i\rangle}{\langle I_s^2\rangle+\langle I_i^2\rangle+2\langle I_sI_i\rangle} . \tag{2.35}$$

The parameter η_{ST} has the interpretation of the second-order fringe visibility of the interference pattern in a ST:

$$\eta_{\mathrm{ST}}=\frac{g^{(2)}_{\max}-g^{(2)}_{\min}}{g^{(2)}_{\max}+g^{(2)}_{\min}} . \tag{2.36}$$

From the formula (2.35) and the classical inequalities (1.1) and (1.2) it follows that the visibility is always less than or equal to $\frac{1}{2}$:

$$\eta_{\mathrm{ST}}\leq\tfrac{1}{2}. \tag{2.37}$$

This stochastic result should be contrasted with the quantum-mechanical prediction (2.15) leading to the quantum visibility always equal to 1:

$$\eta_{\mathrm{QM}}=1 . \tag{2.38}$$

From this fact it follows that, for example, if a photon has been detected at ϕ_1, the detection of the second photon for $\phi_2=\phi_1+\pi$ is impossible with certainty: $p(1|1)=0$. The stochastic description predicts that in this case there is a finite probability of photodetection: $p(1|1)=\frac{1}{2}(1-\eta_{\mathrm{ST}})$. This result is very similar to the effect of photon antibunching. Quantum mechanically single photons are antibunched, while classically the joint probability of photodetection can never be equal to zero.

D. Stochastic versus local description of correlation

There is no general reason to believe that the ST of field correlations is equivalent to the LHV theory of such correlations. In fact, we have pointed out that a test of Bell's inequality requires a series of experiments involving different locations of the photodetectors. On the other hand, the ST can be tested performing a single correlation experiment. For example, a test of the maximum visibility for ϕ_1 and $\phi_2=\phi_1+\pi$ will be sufficient in order to verify the quantum versus stochastic prediction. From this simple argument it follows that a LHV theory is less vulnerable to various experimental tests. This means also that it is possible to have a range of parameters for which a ST cannot hold while the LHV description is still possible. Because of this, extreme care is required before making definite conclusions about the ST or the LHV failure to describe two-photon correlations.

We have seen that the stochastic visibility η_{ST} has been constrained by the inequality (2.37). This inequality has been a consequence of the classical inequalities (1.1) and (1.2). If we take the expression (2.33) at its face value, we can assume that a LHV theory can predict that

$$p_{\mathrm{LHV}}(\phi_1,\phi_2)=\tfrac{1}{4}[1+\eta_{\mathrm{LHV}}\cos(\phi_1-\phi_2)] . \tag{2.39}$$

It is well known that a maximum violation of Bell's inequality (1.4) is reached for the following angles: angle

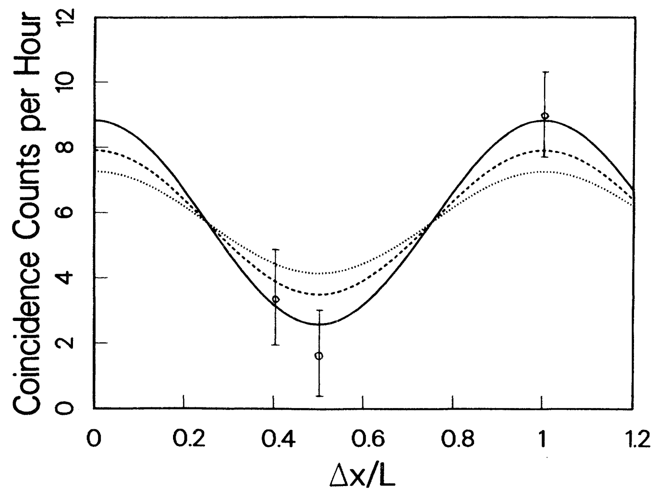

FIG. 1. Joint-probability distribution as a function of relative separation Δx on the screen $[\phi_1-\phi_2=(2\pi\Delta x/L)]$. The solid curve represents quantum mechanics ($\eta=1$), the dotted line represents ST with $\eta=\frac{1}{2}$, the dashed line represents LHV theory with $\eta=\sqrt{2}/2$, and the superimposed experimental points are from Ref. [1].

$(\mathbf{a},\mathbf{b}')=3\pi/4$ and angle$(\mathbf{a},\mathbf{b})=$angle$(\alpha',\mathbf{b}')=$angle$(\mathbf{a}',\mathbf{b})=\pi/4$. For this particular position (orientation) of the photodetectors Bell's inequality imposed on the expression (2.39) leads to the following inequality for the visibility $\eta_{\rm LHV}$:

$$|\eta_{\rm LHV}|\leq\frac{\sqrt{2}}{2}\ . \tag{2.40}$$

We see that this condition is less restrictive than the corresponding semiclassical inequality (2.37). If the visibility is given in the range $0.5\leq|\eta_{\rm LHV}|\leq\sqrt{2}/2$ then the ST of two-photon interferences are not allowed, while a LHV theory of photon correlation is still possible.

In the experiment of Ghosh and Mandel [1] the joint probability for the detection of the photons at two points has been measured. It has been shown that the experimental data are in good agreement with quantum mechanics and violate the predictions of the ST. In order to demonstrate the violation of Bell's inequality four joint probabilities for the detection at four different locations would be required. These particular measurements have not been performed so far. But in order to exhibit the differences between the ST, the LHV theory, and quantum mechanics we have plotted in Fig. 1 the joint-probability distribution (2.39) for $\eta=1$ (quantum mechanics), $\eta=\frac{1}{2}$ (ST), and $\eta=\sqrt{2}/2$ (possible LHV prediction). On this figure we have superimposed the experimental results of Ghosh and Mandel. It is quite clear that the experimental points are in very good agreement with quantum mechanics. While it is clear that ST theory is violated, the experimental evidence for or against the LHV description is much weaker.

We conclude this section by noting that an experiment with interference coincidences measured at several locations can actually rule out the LHV description of correlations in the interference of two photon. The ST can be tested using, in principle, only one location of the detectors. The experimental data from the interference of two photons demonstrate the failure of the ST.

III. INTERFERENCE WITH BEAM SPLITTER AND POLARIZATION

Hong, Ou, and Mandel [2] have demonstrated the violation of Bell's inequality in an experiment involving mixed signal and idler photons. Correlation measurements of this mixed signal have been performed as a function of two linear polarizer settings. The mixing of the idler and signal photons produced in a down-conversion process has been obtained with the help of a dielectric beam splitter (BS). The action of the BS can be described in the Heisenberg picture [17,18] (as a transformation of the electric fields due to the mirrors of the BS) or in the Schrödinger picture where the electric fields are unchanged and only the states of the field are transformed [19]. The Schrödinger picture of the BS transformation has no classical analogy because it deals directly with the quantum states of the electromagnetic field.

We start the discussion of this section describing the action of the BS using the Schrödinger picture.

A. BS in the Schrödinger picture

In this section we still consider the two-photon interference produced by a parametric down-conversion. The difference between this section and the previous one is that the interference experiment involves the beam splitter and two polarizers. Let us consider in the following the experimental configuration of Hong, Ou, and Mandel [2]. Linearly polarized signal photons and idler photons are produced by a parametric down-conversion process. Instead of letting them interfere directly the idler photon passes through a $\pi/2$ polarization rotator. Signal photon and idler photon are then incident from opposite sides onto a beam splitter. The light beam after the BS will consist of mixed signal and idler photons. Then they pass through linear polarizers set at adjustable angles θ_1 and θ_2 and finally fall on the photodetectors. The coincidental counting rate of the two detectors located at positions ϕ_1 and ϕ_2 provides a measure of the joint probability $p(\theta_1,\phi_1;\theta_2,\phi_2)$ of detecting two photons for various settings θ_1 and θ_2 of two linear polarizers.

Let us assume that before the beam splitter the signal and idler photons are x and y polarized, respectively. Let a_x, a_y, b_x and b_y denote the annihilation operators of the x and y polarized signal and idler photons, respectively. The corresponding number states of the system we shall denote by $|n_x,n_y;m_x,m_y\rangle=|n_x,n_y\rangle\otimes|m_x,m_y\rangle$, where the numbers n_x and n_y correspond to the signal photons and the numbers m_x and m_y correspond to the idler photons. Before the BS the state of the field is

$$|\psi_0\rangle=|1,0;0,1\rangle\ . \tag{3.1}$$

With the BS and polarizers the joint probability $p(\theta_1,\phi_1;\theta_2,\phi_2)$ is

$$p_{12}(\theta_1,\phi_1;\theta_2,\phi_2)=\langle\psi_0|U^\dagger\hat{E}_1^{(-)}\hat{E}_2^{(-)}\hat{E}_2^{(+)}\hat{E}_1^{(+)}U|\psi_0\rangle\ , \tag{3.2}$$

where the signal and idler positive-frequency parts of the electric-field operator projected on polarizers with orientations θ_1 and θ_2, respectively, are

$$\hat{E}_1^{(+)}=(\cos\theta_1)\hat{a}_x+(\sin\theta_1)\hat{a}_y\ , \tag{3.3a}$$

$$\hat{E}_2^{(+)}=(\cos\theta_2)\hat{b}_2+(\sin\theta_2)\hat{b}_y\ . \tag{3.3b}$$

The unitary operator U describes the action of the BS and contains in its definition optical-path phases ϕ_1 and ϕ_2. The action of the BS can be described in the Heisenberg picture, leaving the state $|\psi_0\rangle$ unchanged and rotating the electric fields with the help of the U operator. In the Schrödinger picture the operators are unchanged and only the state $|\psi_0\rangle$ is modified by the action of the BS. In the following we choose to work in the Schrödinger picture. For a perfect 50/50 BS we obtain

$$\begin{aligned}|\psi_{\rm BS}\rangle&=U|\psi_0\rangle\\&=\tfrac{1}{2}(|1,0;0,1\rangle e^{i(\phi_2-\phi_1)}+|0,1;1,0\rangle\\&\quad+i|1,1;0,0\rangle e^{-i\phi_1}-i|0,0;1,1\rangle e^{i\phi_2})\ .\end{aligned} \tag{3.4}$$

Equation (3.4) tells us that the BS maps an initial photon state $|\psi_0\rangle$ into a state $|\psi_{\rm BS}\rangle$ being a coherent superposition of four different two-photon states.

B. Angular momentum description of the interference

Following the procedure described in Sec. II A, we introduce an angular momentum description of the two-photon interference with BS and polarization.

The intensity operator $\hat{I}(\theta)$ of the signal field can be expressed as

$$\hat{I}(\theta_1)=\hat{E}_1^{(-)}\hat{E}_1^{(+)}=\frac{\hat{N}_a}{2}+\hat{J}_a(\theta_1)\ , \tag{3.5}$$

where

$$\hat{J}_a(\theta_1)=\hat{\mathbf{J}}_a\cdot\mathbf{n}_1 \tag{3.6a}$$

is a projection of an angular momentum on the unit vector

$$\mathbf{n}_1=(\sin2\theta_1,0,\cos2\theta_1)\ . \tag{3.6b}$$

In Schwinger's representation we have

$$N_a=\hat{a}_x^\dagger\hat{a}_x+\hat{a}_y^\dagger\hat{a}_y\ , \tag{3.7a}$$

$$\hat{J}_{a+}=\hat{a}_x^\dagger\hat{a}_y,\quad \hat{J}_{a-}=\hat{a}_y^\dagger\hat{a}_x\ , \tag{3.7b}$$

and

$$\hat{J}_{ax}=\tfrac{1}{2}(\hat{a}_x^\dagger\hat{a}_x-\hat{a}_y^\dagger\hat{a}_y)\ . \tag{3.7c}$$

Similarly, for the intensity operator $I(\theta_2)$ of the idler field we obtain

$$\hat{I}(\theta_2)=\hat{E}_2^{(-)}\hat{E}_2^{(+)}=\frac{\hat{N}_b}{2}+\hat{J}_b(\theta_2)\ , \tag{3.8}$$

where $\hat{N}_b$ and $\hat{J}_b$ are the corresponding angular momentum operators composed of the b-boson operators.

In terms of angular momentum states the BS wave function (3.4) has the following form:

$$\begin{aligned}|\psi_{\rm BS}\rangle=&\tfrac{1}{2}(|+\rangle_a\otimes|-\rangle_b e^{i(\phi_2-\phi_1)}+|-\rangle_a\otimes|+\rangle_b\\&+i|0\rangle_a\otimes|1_b\rangle e^{-i\phi_1}-i|1_a\rangle\otimes|0\rangle_b e^{i\phi_2})\ .\end{aligned} \tag{3.9}$$

where $|+\rangle,|-\rangle$ are states of spin-$\frac{1}{2}$ with $m=\frac{1}{2}$ and $m=-\frac{1}{2}$, respectively. The state $|0\rangle$ corresponds to spin 1 and $m=0$. In terms of this angular momentum state we obtain the following formula for the wave function after the action of the BS:

$$|\psi_{\rm BS}\rangle=\frac{1}{\sqrt{2}}(|A\rangle+i|B\rangle)\ , \tag{3.10}$$

where the state $|A\rangle$ involves only spin-$\frac{1}{2}$ states:

$$|A\rangle=\frac{1}{\sqrt{2}}(|+\rangle_a\otimes|-\rangle_b e^{i(\phi_2-\phi_1)}+|-\rangle_a\otimes|+\rangle_b)\ , \tag{3.11a}$$

and where the state $|B\rangle$ involves only spin-1 states (1_a and 1_b are unity operators):

$$|B\rangle=\frac{i}{\sqrt{2}}(|0\rangle_a\otimes|1_b\rangle e^{i\phi_1}-|1_a\rangle\otimes|0\rangle_b e^{i\phi_2})\ . \tag{3.11b}$$

From these definitions we obtain

$$\langle A|B\rangle=0,\quad \langle A|A\rangle=\langle B|B\rangle=1\ . \tag{3.12}$$

Let us express the initial state (3.1) in terms of the angular momentum states following the definitions (3.7). As a result we obtain

$$|\psi_0\rangle=|+\rangle_a\otimes|-\rangle_b\ , \tag{3.13}$$

i.e., an uncorrelated product of spin-$\frac{1}{2}$ up and down states. The BS transformation U maps this uncorrelated state into a correlated state given by the formula (3.10). From the definitions (3.10) and (3.11) we see that the vectors $|\psi_0\rangle$ and $|A\rangle$ are spanned by spin-$\frac{1}{2}$ states, while the vector $|B\rangle$ consists of spin-1 states. Because of the superselection rules [see also Eq. (3.12)] we can ignore in all next considerations the component $|B\rangle$ of the transformed state. This component will never contribute to any observable quantities. It is now well established that the correct quantum state produced by weakly driven parametric down-conversion contains a vacuum component $|0_a,0_b\rangle$ in its definition (3.9). It is easy to check that this component in the subspace of angular momentum (3.11) never contributes to the quantum expectation values. As a result of this step the action of the BS is the mapping: $|\psi_0\rangle\to|A\rangle$, or in a more explicit way,

$$\begin{aligned}U:|+\rangle_a\otimes|-\rangle_b\to&\frac{1}{\sqrt{2}}(|+\rangle_a\otimes|-\rangle_b e^{i(\phi_2-\phi_1)}\\&+|-\rangle_a\otimes|+\rangle_b)\ .\end{aligned} \tag{3.14}$$

From this remarkable relation we conclude that the BS transforms a product state into an entangled state which has a form very similar to the well-known EPR-correlated spin state. In fact, for $\phi_2=\phi_1+\pi$ the BS state is identical to the spin singled state. From these discussions we conclude that an interference of two photons mixed by a 50/50 BS is mathematically completely equivalent to the EPR correlations of entangled spin-$\frac{1}{2}$ states.

The joint probability defined by Eq. (3.2) can be written in the form of $p_{12}(\Omega_1,\Omega_2)$ where $\Omega_1=(2\theta_1,\phi_1)$ and $\Omega_2=(2\theta_2,\phi_2)$ are two arbitrary directions characterized by their solid angles. If the BS is removed the directions are characterized just by the two angles θ_1 and θ_2 and for the uncorrelated state (3.13) we obtain

$$p_{12}(\theta_1,\theta_2)=\langle\psi_0|\hat{I}_1\hat{I}_2|\psi_0\rangle=\langle\hat{I}_1\rangle\langle\hat{I}_2\rangle\ , \tag{3.15}$$

where

$$\langle\hat{I}_1\rangle=\cos^2\theta_1,\quad \langle\hat{I}_2\rangle=\cos^2\theta_2\ . \tag{3.16}$$

It is well known that in classical wave optics the Malus law predicts an attenuation of light intensity through a linear polarizer. Equations (3.16) are just examples of the Malus law. Because the initial state of the field is uncorrelated, if the BS is removed, the joint detection is just

$$p_{12}(\theta_1,\theta_2)=\cos^2\theta_1\cos^2\theta_2\ . \tag{3.17}$$

If the BS is included, the joint probability of photodetection becomes

$$\begin{aligned}p_{12}(\Omega_1,\Omega_2)&=\langle A|\hat{E}_1^{(-)}\hat{E}_2^{(-)}\hat{E}_2^{(+)}\hat{E}_1^{(+)}|A\rangle\\&=\langle A|{:}\hat{I}(\theta_1)\hat{I}(\theta_2){:}|A\rangle\ ,\end{aligned} \tag{3.18}$$

where $\hat{E}_1^{(+)}$ and $\hat{E}_2^{(+)}$ are given by Eqs. (3.3). Because the signal and the idler boson operators commute the normal ordering of the intensities can be disregarded. As a result of this we have

$$p_{12}(\Omega_1,\Omega_2)=\langle A|\hat{I}(\theta_1)\hat{I}(\theta_2)|A\rangle\ , \tag{3.19}$$

where the intensities are given by the angular momentum formulas (3.5) and (3.8). For the BS state (3.14) this correlation is

$$p_{12}(\Omega_1,\Omega_2)=\tfrac{1}{4}[1+\cos\alpha-2(\cos2\theta_1)(\cos2\theta_2)]\ , \tag{3.20}$$

where

$$\cos\alpha=\cos2\theta_1\cos2\theta_2+\sin2\theta_1\sin2\theta_2\cos(\phi_1-\phi_2)\ . \tag{3.21}$$

Equation (3.21) defines α, which is the relative spherical angle between the two directions Ω_1 and Ω_2.

The spherical cosine function (3.21) is rotationally invariant, while the probability distribution function (3.20) is not. This property follows from the fact that the BS state $|A\rangle$ from the point of view of angular momentum has no well-defined magnetic number m for arbitrary ϕ_1 and ϕ_2. If $\phi_2=\phi_1+\pi$ this state becomes the EPR spin-singlet state and then the probability distribution is rotationally invariant:

$$p_{12}(\theta_1,\phi_1;\theta_2,\phi_1+\pi)=\tfrac{1}{2}\sin^2(\theta_1-\theta_2)\ . \tag{3.22}$$

In this case the problem of the interference of the two photons is identical to the problem of EPR correlations of the spin-singlet state.

For the state $|\psi_{BS}\rangle=|A\rangle$, the average of intensities is found to be

$$\langle\hat{I}_1\rangle=\langle\hat{I}_2\rangle=\tfrac{1}{2}\ . \tag{3.23}$$

Then the normalized second-order coherence function $g^{(2)}$ is

$$\begin{aligned}g^{(2)}(\Omega_1,\Omega_2)&=\frac{\langle\hat{I}_1\hat{I}_2\rangle}{\langle\hat{I}_1\rangle\langle\hat{I}_2\rangle}\\&=[1+\cos\alpha-2(\cos2\theta_1)(\cos2\theta_2)]\ .\end{aligned} \tag{3.24}$$

From these equations we see that we have two choices for the correlation experiment. Either fixing ϕ_1,ϕ_2 (i.e., positions of the photodetectors) and changing θ_1,θ_2 (the polarizers) or fixing θ_1,θ_2 and changing ϕ_1,ϕ_2.

If ϕ_1,ϕ_2 are fixed and equal to $\phi_2=\phi_1+\pi$ the state $|A\rangle$ becomes a singlet spin state for which we have

$$g^{(2)}=2\sin^2(\theta_1-\theta_2)\ . \tag{3.25}$$

In this case the coherence function is rotationally invariant and for $\theta_1=\theta_2$ the joint photodetection is equal to zero as in the photon-antibunching effect.

If $\phi_2=\phi_1$ the state $|A\rangle$ becomes a triplet spin state for which we have

$$g^{(2)}=2\sin^2(\theta_1+\theta_2)\ , \tag{3.26}$$

i.e., a coherence function which is not rotationally invariant. If the orientations θ_1 and θ_2 are fixed and equal to $\theta_1=\theta_2=\pi/4$, we have

$$g^{(2)}=1+\cos(\phi_1-\phi_2)\ . \tag{3.27}$$

Note that this result is closely related to the case that we have investigated in Sec. II.

C. Nonlocal description of photon correlations with the BS

In the subspace of spin-$\frac{1}{2}$ up and down states the intensity operators (3.5) and (3.8) become spin projection operators $\hat{P}(\theta_1)$ and $\hat{P}(\theta_2)$. Following the procedure described in Sec. II B we obtain

$$p(\Omega_1,\Omega_2)=\int d\lambda_1\int d\lambda_2\lambda_1\lambda_2 p(\Omega_1,\lambda_1;\Omega_2,\lambda_2)\ , \tag{3.28}$$

where the distribution function $p(\Omega_1,\lambda_1;\Omega_2,\lambda_2)$ is

$$\begin{aligned}p(\Omega_1,\lambda_1;\Omega_2,\lambda_2)&\\&=\tfrac{1}{2}p[\delta(\lambda_1)\delta(\lambda_2)+\delta(\lambda_1-1)\delta(\lambda_2-1)]\\&\quad+\tfrac{1}{2}(1-p)[\delta(\lambda_1)\delta(\lambda_2-1)+\delta(\lambda_1-1)\delta(\lambda_2)]\end{aligned} \tag{3.29a}$$

and

$$p=\tfrac{1}{2}[1+\cos\alpha-2(\cos2\theta_1)(\cos2\theta_2)]. \tag{3.29b}$$

This distribution function is nonlocal in the sense already discussed by us in Sec. II B. From this nonlocal joint probability we can derive the following formulas for the conditional probabilities of yes (1) and no (0) photodetections:

$$p(1|1)=p(0|0)=p\ , \tag{3.30a}$$

$$p(0|1)=p(1|0)=1-p \tag{3.30b}$$

For the singlet state, and when $\theta_1-\theta_2=n\pi$, we have

$$p(1|1)=p(0|0)=0\ . \tag{3.31}$$

It means that for opposite orientations of the polarizers one can predict with certainty that no correlations of digits $1\to 1$ and $0\to 0$ (yes-yes and no-no) can occur at the photodetectors. It means that the two sequences of random numbers $\lambda_1=(1,0,1,0,\ldots)$ and $\lambda_2=(0,1,0,1,\ldots)$ have to be perfectly anticorrelated. This situation is very similar to the photon-antibunching effect. The same conclusion holds for the triplet states when $\theta_1+\theta_2=n\pi$.

Because of the nonlocal character, the probability distribution (3.29a) and the resulting distributions (3.30) or (3.28) violate Bell's inequality (1.4). In this case the removal of a polarizer is equivalent to a summation over the projection operators $\hat{P}(\theta_1)$ or $\hat{P}(\theta_2)$ according to the formula (2.17). As a result of this step we obtain

$$p(\Omega_1,\infty)=p(\infty,\Omega_2)=\tfrac{1}{2}\ . \tag{3.32}$$

From this discussion we conclude that the interference of two photons with polarizers and the BS is mathematically equivalent to the theory of quantum correlations of entangled spin state $|A\rangle$, which depending on the position of the photomultipliers can become a spin-singlet state or a spin-triplet state.

For photon polarizations the entangled state can be regarded in cascade-photon experiments and the violation of Bell's inequality has been observed for different orientations of the polarizers [20].

D. Stochastic description of the interference with the BS

Following the stochastic description in Sec. II C, we can write the electric field at the first detector as a linear superposition of transmitted and reflected signals. With 50/50 BS and polarizers θ_1 and θ_2, we obtain that the stochastic joint probability $p_{\rm ST}(\Omega_1,\Omega_2)$ of photodetection can be expressed as

$$p_{\rm ST}(\Omega_1,\Omega_2)=\tfrac{1}{4}\{(\cos^2\theta_1)(\cos^2\theta_2)\langle I_s^2\rangle+(\sin^2\theta_2)(\sin^2\theta_2)\langle I_i^2\rangle+\langle I_sI_i\rangle[(\cos^2\theta_1)(\sin^2\theta_2)+(\cos^2\theta_2)(\sin^2\theta_1)$$
$$+\tfrac{1}{2}(\sin 2\theta_1)(\sin 2\theta_2)\cos(\phi_1-\phi_2)]\}\ , \tag{3.33}$$

where $I_s=|\mathcal{E}_s|^2$ and $I_i=|\mathcal{E}_i|^2$ are the stochastic intensities of the signal and idler signals.

Let us define the following two parameters:

$$k=\frac{2\langle I_iI_s\rangle}{\langle I_i^2\rangle+\langle I_s^2\rangle}\ , \tag{3.34a}$$

and

$$\chi=\frac{\langle I_i^2\rangle}{\langle I_s^2\rangle}\ . \tag{3.34b}$$

If the idler and the signal fields are the same and if an attenuator with efficiency η is inserted in the path of the idler photons we have $I_i=\eta I_s$ and accordingly $\chi=\eta^2$.

From the classical inequality (1.2) we obtain that for equal beams we have

$$o\le k\le 1\ . \tag{3.35}$$

Let us compare now the stochastic description given by Eq. (3.33) with the quantum-mechanical result if the state is a spin-triplet state ($\phi_2-\phi_1=2\pi$). In this case the quantum probability is given by

$$p_{\rm QM}(\theta_1,\theta_2)=\tfrac{1}{2}\sin^2(\theta_1+\theta_2)\ . \tag{3.36}$$

Let us consider in the following the specific case where $\theta_2=\pi/4$ and $\phi_2-\phi_1=2\pi$. From Eq. (3.33) we obtain in this case

$$p_{\rm ST}\left[\theta_1,\frac{\pi}{4}\right]=\tfrac{1}{16}\langle(I_s+I_i)^2\rangle[1+v_{\rm ST}\sin(2\theta_1-\gamma_{\rm ST})]\ , \tag{3.37}$$

where

$$v_{\rm ST}=\frac{[(\langle I_i^2\rangle-\langle I_s^2\rangle)^2+4\langle I_iI_s\rangle^2]^{1/2}}{\langle(I_s+I_i)^2\rangle}$$
$$=\frac{1}{1+k}\left[1+k^2-\frac{4\chi}{(1+\chi)^2}\right]^{1/2}\ , \tag{3.38a}$$

$$\gamma_{\rm ST}=\tan^{-1}\left[\frac{1}{k}\frac{\chi-1}{\chi+1}\right]\ . \tag{3.38b}$$

From the definitions (3.34) it is clear that the parameters k and χ cannot be independent because of the classical inequalities (1.1)–(1.3). This means that the phase $\gamma_{\rm ST}$ and the amplitude $v_{\rm ST}$ in Eq. (3.37) are related. From the stochastic inequalities (1.1)–(1.3) it follows that

$$\frac{k}{1+k}\le v_{\rm ST}\le\frac{1}{1+k}\ . \tag{3.39}$$

For equal beams we have $\eta=1$, $k=1$ and as a result a visibility $v_{\rm ST}=\frac{1}{2}$. This result is identical to the result given by Eq. (2.37). The only difference between these results is the inequality (3.42) which leads to 50% visibility only if the beams are not attenuated, i.e., $\eta=1$.

E. Stochastic versus local description of correlations in the presence of the BS

Following the discussion presented in Sec. II D, we shall compare the stochastic description of photodetection with polarizers and the BS with a LHV theory based on a local distribution of hidden parameters λ_1 and λ_2 in Eq. (3.28) [8]. Such a local description leads to Bell's inequality (1.4) which is violated by the quantum result (3.29). We shall compare the result of the stochastic description given by Eq. (3.33) with the constraints imposed by Bell's inequality. If we take the expression (3.33) at its face value for a LHV description of the two-photon interference we can assume that a local realism leads to the expression

$$p_{\rm LHV}(\theta_1,\theta_2)=\tfrac{1}{4}N\left[\cos^2\theta_1\cos^2\theta_2+(\sin^2\theta_1-\cos^2\theta_2)\frac{\chi_{\rm LHV}}{1+\chi_{\rm LHV}}+\frac{k_{\rm LHV}}{2}\sin^2(\theta_1+\theta_2)\right], \quad (3.40)$$

where N is a normalization constant. In addition to this expression, we need the following results if the polarizers θ_1 or θ_2 are removed:

$$p_{\rm LHV}(\theta_1,\infty)=\tfrac{1}{4}N\left[\cos^2\theta_1-\frac{\chi_{\rm LHV}}{1+\chi_{\rm LHV}}\cos2\theta_1+\frac{k_{\rm LHV}}{2}\right], \quad (3.41a)$$

$$p_{\rm LHV}(\infty,\theta_2)=\tfrac{1}{4}N\left[\cos^2\theta_2-\frac{\chi_{\rm LHV}}{1+\chi_{\rm LHV}}\cos2\theta_2+\frac{k_{\rm LHV}}{2}\right], \quad (3.41b)$$

$$p_{\rm LHV}(\infty,\infty)=\tfrac{1}{4}N(1+k_{\rm LHV}) . \quad (3.41c)$$

In these expressions the parameters $\chi_{\rm LHV}$ and $k_{\rm LHV}$ are constrained now by the requirements that $p_{\rm LHV}$ are positive everywhere and Bell's inequality (1.4). If the signal and the idler beams have the same intensity we have $\eta=1$ and accordingly $\chi_{\rm LHV}=1$.

If we choose $2\theta_1=\pi/4$, $2\theta_2=\pi/2$, $2\theta_1'=\frac{3}{4}\pi$, and $2\theta_2'=0$, and assume that the location of the detectors is such that $\phi_2=\phi_1+2\pi$, Bell's inequality and the positivity of $p_{\rm LHV}$ will lead to the following condition for the parameter $k_{\rm LHV}$:

$$0\le k_{\rm LHV}\le\frac{1}{2-\sqrt{2}}\simeq1.707 . \quad (3.42)$$

This condition is weaker than the corresponding condition of the ST which predicts in this case $k=1$. So here we see again a range of k for which the stochastic theory fails but the LHV theory is still possible.

F. Attenuation in quantum mechanics, ST, and LHV theory

The difference between quantum and stochastic or LHV description of the interference can be made more pronounced if a quantum attenuator with the intensity transmission coefficient η is placed in the idler beam. If we assume that the attenuator is in a vacuum state, due to the normal ordering of the field operators in the photodetection process we obtain that the quantum result is modified by a trivial scaling factor. For the triplet state $(\phi_2-\phi_1=2\pi)$ in the special case of $\theta_2=\pi/4$, we obtain the following quantum result:

$$p_{\rm QM}\left[\theta_1,\frac{\pi}{4}\right]=\frac{\eta}{4}(1+v_{\rm exp}\sin2\theta_1) , \quad (3.43)$$

where $v_{\rm exp}$ is a geometrical factor that takes into account the reduction of visibility due to an imperfect alignment of the interfering beams.

The attenuation in the ST or the LHV theory is quite different. We still consider the case of $\phi_2-\phi_1=2\pi$ and $\theta_2=\pi/4$ but with the attenuator in the idler beam.

In the ST we have in such a case $I_i=\eta I_s$ and as a result of this

$$p_{\rm ST}\left[\theta_1,\frac{\pi}{4}\right]=\frac{N}{4}[1+v_{\rm ST}v_{\rm exp}\sin(2\theta_1-\gamma_{\rm ST})] , \quad (3.44a)$$

where according to the formulas (3.38) we have

$$v_{\rm ST}=\frac{1+\eta^2}{(1+\eta)^2},\quad \gamma_{\rm ST}=\tan^{-1}\left[\frac{\eta^2-1}{2\eta}\right] . \quad (3.44b)$$

Note that in this case the visibility $v_{\rm ST}$ is reaching its maximum value in the inequality (3.39).

The attenuation of the LHV theory is given by the parameter $\chi_{\rm LHV}=\eta^2$ and the expression (3.40) with $\theta_1=\pi/4$ and $k_{\rm LHV}$ confined by the condition (3.42). We can write this coincidence rate in the following form:

$$p_{\rm LHV}\left[\theta_1,\frac{\pi}{4}\right]=\frac{N}{4}[1+v_{\rm LHV}v_{\rm exp}\sin(2\theta_1-\gamma_{\rm LHV})] , \quad (3.45a)$$

where

$$v_{\rm LHV}=\frac{1}{1+k_{\rm LHV}}\left[1+k_{\rm LHV}^2-\frac{4\chi}{(1+\chi)^2}\right]^{1/2} , \quad (3.45b)$$

and

$$\gamma_{\rm LHV}=\tan^{-1}\left[\frac{1}{k_{\rm LHV}}\frac{\eta^2-1}{\eta^2+1}\right] . \quad (3.45c)$$

We take for $k_{\rm LHV}$ the maximum value allowed by Bell's inequality:

$$k_{\rm LHV}=\left[\frac{\sqrt{2}}{2}+1-\frac{2\eta^2}{1+\eta^2}\right]\Big/(\sqrt{2}-1) . \quad (3.45d)$$

In Ou and Mandel's experiments a 1:1 and a 8:1 neutral density filter have been inserted in the path of the idler photons. This corresponds to $\eta=1$ and $\frac{1}{8}$. In addition,

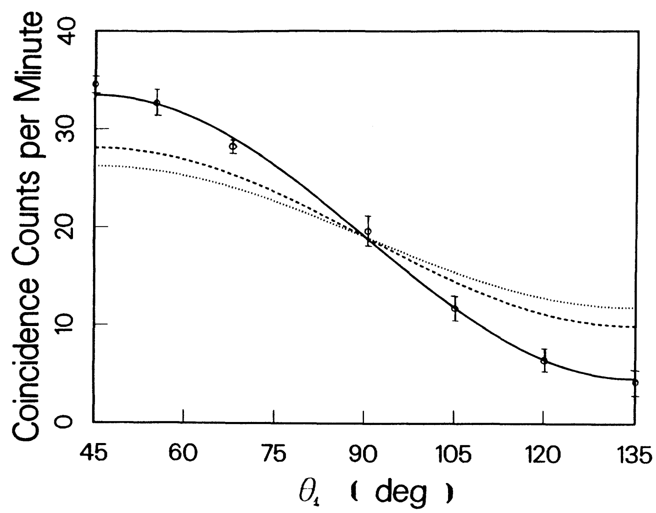

FIG. 2. Joint-probability distribution as a function of the polarizer angle θ_1 with no attenuation of the idler photons ($\eta=1$). The solid curve represents quantum mechanics, the dotted line represents ST, the dashed line reprerents LHV, and the superimposed experimental points are from Ref. [3].

the experimental setup was such that $v_{\rm exp}=0.76$.

In Fig. 2 we have compared the modulations of the coincidence counting rate for quantum, ST, LHV, and experimental predictions of Ou and Mandel for $\eta=1$. In this case according to the formulas (3.44) and (3.45) we have $v_{\rm ST}=\frac{1}{2}$ and $\gamma_{\rm ST}=0$ for the ST and $v_{\rm LHV}=(3-\sqrt{2})^{-1}$ and $\gamma_{\rm LHV}=0$ for the LHV theory.

In Fig. 3 we have the same comparison, but with the idler photons attenuated by $\eta=\frac{1}{8}$. In this case we have $v_{\rm ST}=\frac{65}{81}\cong 0.802$. $\gamma_{\rm ST}=-\tan^{-1}(\frac{63}{16})=-75.75°$ for the ST and $v_{\rm LHV}\simeq 0.824$ and $\gamma_{\rm LHV}=-13.47°$ for the LHV.

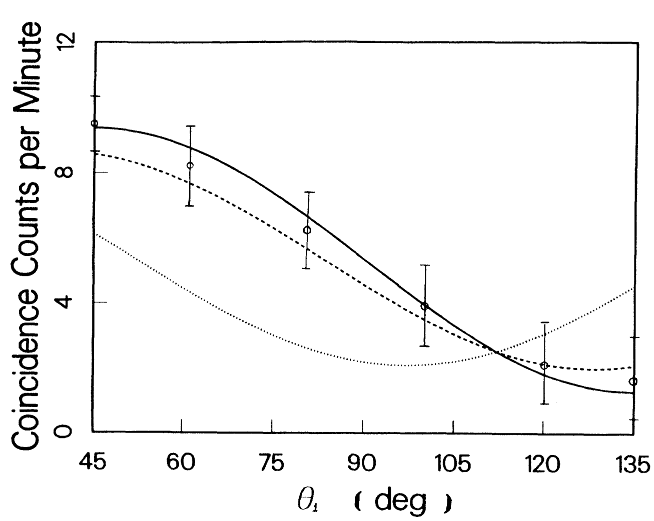

FIG. 3. The same as in Fig. 2, but with an attenuation of the idler photons $\eta=\frac{1}{8}$.

Note that quantum mechanics predicts for all values of η a visibility equal to 1 and no phase shift in the modulation of the interference pattern.

From these figures it is quite clear that the experimental data rule out the ST and the violation of the classical inequalities (1.1)–(1.3) is evident. The LHV theory predictions are much closer to the observed points. While the experimental data does favor the quantum prediction, the violation of local realism associated with the LHV description is less pronounced.

IV. CONCLUSIONS

In this paper, quantum versus stochastic or hidden-variable fluctuations in two-photon interference produced by a parametric down-conversion process have been discussed. We have shown that the nonclassical effects in the two-photon interference can be discussed in the framework of two different descriptions. In the first description a stochastic theory of electromagnetic fluctuations can be used in the discussion of the interference effects. In the second description a theory of hidden-variable fluctuations can be applied to photon correlations. Using the Schwinger-boson representation of the angular momentum we have shown that the correlated idler and signal photons can be described in terms of spin-correlated states. In particular, we have shown that the action of a BS on the two photons in a parametric down-conversion is equivalent to the production of an entangled state which is very similar to the EPR-correlated spin state. We have shown that the ST of two-photon fluctuations is not equivalent to a LHV theory of photon correlations. We have shown that quantum correlations are nonlocal and violate Bell's inequality.

We have performed a comparison of quantum, ST, and LHV predictions with the experimental results and we have concluded that a violation of the ST has been clearly observed, while the violation of the LHV theory is less pronounced. In view of new experiments [21–23] involving correlations of photons in down-conversion processes we hope that a careful distinction of the ST from the LHV theories can play an important role in the investigations of quantum effects in fourth-order interference.

In two-color photon pairs experiments [21] the connection between spin-$\frac{1}{2}$ entangled EPR states and the photon correlations is much more complex due to the finite photon bandwidth. In this case the theory of LHV involves a continuous superposition of different angular momenta corresponding to different states of the photon spectrum. We plan to study this and other related problems in a forthcoming paper.

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*Permanent address: Institute of Theoretical Physics, Warsaw University, Warsaw 00-681, Poland.

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