Amplitude-squared squeezing in the two-photon Jaynes-Cummings model

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Amplitude-squared squeezing is investigated for the two-photon Jaynes-Cummings model with the field initially in a squeezed vacuum or a more general squeezed state. For small values of the coherence parameter and large values of the squeeze parameter, the amplitude-squared variances show threefold oscillatory behavior on the long time scale. It is found that a ground-state atom (as against an excited one) leads to considerable enhancement of the input amplitude-squared squeezing. The behavior of photon number squared and its relationship to the amplitude-squared squeezing is commented upon.

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I. INTRODUCTION

The generation [1,2] of the squeezed states of the electromagnetic field in the laboratory has given impetus to the search for systems that can produce strong squeezing. It has been shown [3] that with an initially coherent field interacting with a two-level atom some amount of squeezing can be generated. Under similar conditions larger amounts of squeezing can be obtained [4] in processes where the interaction between the atom and the field proceeds through the exchange of more than one photon. For example, it has been pointed out recently [5] in the context of a two-photon Jaynes-Cummings model (JCM) that the output squeezing can be increased by increasing detuning. The interaction of a squeezed field with an atom has also been discussed by a number of authors [6]. These investigations have generally concerned themselves with the so-called normal-order (or second-order) squeezing.

Some time ago Hong and Mandel [7] generalized the second-order squeezing to higher orders. A different kind of generalization, known as the amplitude-squared squeezing, has been introduced by Hillery [8], whose approach we adopt for the present discussion. More specifically, we look at the problem of amplitude-squared squeezing (AS squeezing, for short) in the interaction of a field with an atom which involves the emission or absorption of two photons. Motivated by the knowledge that a large degree of AS squeezing can be obtained with the field initially in a coherent state [9], we examine the consequences of having a squeezed vacuum or a more general squeezed state field as input. We also inquire into the role played by the initial state of the atom, ground versus excited. This question is of some interest in view of a recently published result where it is found [9] that the AS squeezing is more effective when the atom starts out from the excited state.

II. DESCRIPTION OF THE MODEL AND DENSITY MATRIX ELEMENTS

The Hamiltonian for the two-photon Jaynes-Cummings model in the rotating-wave approximation is given by

$$H=\hbar\omega a^\dagger a+\tfrac{1}{2}\hbar\omega_0\sigma_3+\hbar g(a^{\dagger 2}\sigma_+ +\sigma_- a^2)\ , \tag{1}$$

where a and $a^\dagger$ are field annihilation and creation operators and ω and ω_0 are the frequencies of the cavity mode and atomic transition, respectively. σ_3 and $\sigma_\pm$ are the Pauli spin matrices. For simplicity, we have taken the case of exact resonance, $\omega_0=2\omega$. The states $|\pm\rangle$ denote the excited and ground states of the atom.

A. Atom initially in its ground state

For the field initially in a general squeezed state and the atom in the ground state, the state vector of the system at $t=0$ is

$$|\psi(0)\rangle=\sum_{n=0}^{\infty} Q_n|-;n\rangle\ , \tag{2}$$

where

$$Q_n=(n!\mu)^{-1/2}(\nu e^{i\phi}/2\mu)^{n/2}H_n(z)\times e^{[-(1/2)|\beta|^2+(\nu e^{-i\phi}/2\mu)\beta^2]}\ , \tag{3}$$

and

$$\beta=\alpha\mu+\alpha^*\nu e^{i\phi}\ ,\quad \mu=\cosh r,\quad \nu=\sinh r,\quad z=\beta(2\mu\nu e^{i\phi})^{-1/2} \tag{4}$$

Here r is the input squeeze parameter and α is a dimensionless intensity parameter $\alpha=|\alpha|e^{i\theta}$. The direction of coherent excitation is θ and ϕ is the squeeze direction relative to that of coherent excitation.

The density matrix elements of the field at $t>0$ are given by

$$[\rho_{n,m}(T)]_{\rm gr}=e^{-i(n-m)\omega t}\{\rho_{n+2,m+2}(0)\ \sin[\sqrt{(n+1)(n+2)}T]\sin[\sqrt{(m+1)(m+2)}T]+\rho_{n,m}(0)\ \cos[\sqrt{n(n-1)}T]\ \cos[\sqrt{m(m-1)}T]\}\ , \tag{5}$$

where

$$T=gt\ ,$$

$$\rho_{n,m}(0)=Q_n Q_m^* \ . \tag{6}$$

For the field initially in a squeezed vacuum state, the corresponding density matrix elements of the field can be written by going to the limit $\alpha=0$:

$$[\rho_{2n,2m}(T)]_{\rm gr}=e^{-i2(n-m)\omega t}\rho_{2n,2m}(0)\{\cos[\sqrt{2n(2n-1)}T]\cos[\sqrt{2m(2m-1)}T] + (\tfrac{1}{2}\tanh^2 r)\sqrt{(2n+1)(2m+1)/(n+1)(m+1)} \times \sin[\sqrt{(2n+1)(2n+2)}T]\sin[\sqrt{2m+1)(2m+2)}T]\}\ , \tag{7}$$

where $[\rho_{2n,2m}(0)]_{\rm sq\ vac}=Q_n Q_m^*$ and Q_n for this case is given by

$$Q_n=(\cosh r)^{-1/2}[\sqrt{(2n)!}/n!](\tfrac{1}{2}e^{i\phi}\tanh r)^n \ . \tag{8}$$

From these density matrix elements of the field we can calculate expectation values of the square of the average photon number at $t>0$ as

$$\langle \hat{n}^2(T)\rangle = \sum_{n=0}^{\infty} n^2 \rho_{n,n}(T) \ . \tag{9}$$

For the general squeezed field input

$$\langle \hat{n}^2(T)\rangle_{\rm gr} = \sum_{n=0}^{\infty} \rho_{n,n}(0) \times\{n^2-4(n-1)\sin^2[\sqrt{n(n-1)}T]\}\ , \tag{10}$$

and for the squeezed vacuum input

$$\langle \hat{n}^2(T)\rangle_{\rm gr} = \sum_{n=0}^{\infty} 4n^2 \rho_{2n,2n}(0) \times\{(\tfrac{1}{2}\tanh^2 r)[(2n+1)/(n+1)] \times\sin^2[\sqrt{(2n+1)(2n+2)}T] + \cos^2[\sqrt{2n(2n-1)}T]\}\ . \tag{11}$$

B. Atom Initially in its excited state

For the field initially in a general squeezed state and the atom in its excited state, the density matrix elements of the field at $t>0$ read

$$[\rho_{n,m}(T)]_{\rm exc}=e^{-i(n-m)\omega t}\{\rho_{n-2,m-2}(0)\sin[\sqrt{n(n-1)}T]\sin[\sqrt{m(m-1)}T] + \rho_{n,m}(0)\cos[\sqrt{(n+1)(n+2)}T]\cos[\sqrt{(m+1)(m+2)}T]\}\ . \tag{12}$$

The expression for the square of the average number of photons in this case reads

$$\langle \hat{n}^2(T)\rangle_{\rm exc} = \sum_{n=0}^{\infty} \rho_{n,n}(0)\{n^2+4(n+1)\sin^2[\sqrt{(n+1)(n+2)}T]\}\ . \tag{13}$$

For the squeezed vacuum input, the corresponding field density matrix elements are

$$[\rho_{2n,2m}(T)]_{\rm exc}=e^{-i2(n-m)\omega t}\rho_{2n,2m}(0) \times\{(\tfrac{1}{2}\tanh^2 r)^{-1}\sqrt{nm/(2n-1)(2m-1)}\sin[\sqrt{2n(2n-1)}T]\sin[\sqrt{2m(2m-1)}T] + \cos[\sqrt{(2n+1)(2n+2)}T]\cos[\sqrt{(2m+1)(2m+2)}T]\}\ , \tag{14}$$

and the average of the square photon number is given by

$$\langle \hat{n}^2(T)\rangle_{\rm exc} = \sum_{n=0}^{\infty} \rho_{2n,2n}(0)\{\cos^2[\sqrt{(2n+1)(2n+2)}T]+(\tfrac{1}{2}\tanh^2 r)^{-1}[n/(2n-1)]\sin^2[\sqrt{2n(2n-1)}T]\}\ . \tag{15}$$

III. THE AMPLITUDE-SQUARED SQUEEZING

The quadrature operators of the AS squeezing are given by

$$X_1=(A^2+A^{\dagger 2})/2\ ,\quad X_2=(A^2-A^{\dagger 2})/2i\ , \tag{16}$$

where A and $A^\dagger$ are slowly varying annihilation and creation operators of the field and are defined by

$$A = e^{i\omega t}a\ ,\quad A^\dagger = e^{-i\omega t}a^\dagger\ . \tag{17}$$

The commutation relation satisfied by the quadrature operators is given by

$$[X_1, X_2] = i(2n+1)\ ,\quad \hat{n} = a^\dagger a\ , \tag{18}$$

and the uncertainty relation involving variances of the quadrature operators reads

$$\Delta X_1 \Delta X_2 \geq \langle \hat{n} \rangle + \tfrac{1}{2}\ . \tag{19}$$

For the field in a coherent state, it can be checked that the equality sign holds:

$$\Delta X_i^2 = \langle \hat{n} \rangle + \tfrac{1}{2}\ ,\quad i = 1 \text{ or } 2\ . \tag{20}$$

A state is AS squeezed in X_i if its variance satisfies

$$\Delta X_i^2 < (\langle \hat{n} \rangle + \tfrac{1}{2})\ . \tag{21}$$

We introduce the AS squeezing functions given by

$$\begin{aligned} S_i(T) &= \Delta X_i^2 + \langle \hat{n} \rangle + \tfrac{1}{2} \\ &= \tfrac{1}{4}[\pm(\langle A^4 \rangle + \langle A^{\dagger 4} \rangle) + 2\langle A^{\dagger 2} A^2 \rangle \\ &\qquad \pm(\langle A^{\dagger 2} \rangle \pm \langle A^2 \rangle)^2]\ . \end{aligned} \tag{22}$$

The variable X_i will be AS squeezed whenever $S_i < 0$. Note, however, that it is bounded below by $-(\langle \hat{n} \rangle + \frac{1}{2})$.

IV. RESULTS AND DISCUSSION

Using the expressions of the relevant density matrix elements, we have computed the S_i and $\langle \hat{n}^2(T) \rangle$ for various field and atomic states. For simplicity, we take $\theta = 0$ and $\phi = 0$ in our computation.

A. Squeezed vacuum input

We first consider the time development of $S_i(T)$ for the atom initially in the ground state. Figure 1 shows AS squeezing behavior of S_i for small input squeezing ($r = 0.3$). We notice that the oscillations of AS squeezing are not random, but trace out a specific pattern that is repeated after more or less regular intervals of time. The minima in S_i of AS squeezing become deeper with the passage of time. The initial squeezing of about 15% in S_2 (in-phase quadrature remaining unsqueezed initially) can grow to about 24% in either of them in the post interaction period. On the long time scale the envelope depicting the overall oscillatory behavior resembles a wave of large wavelength.

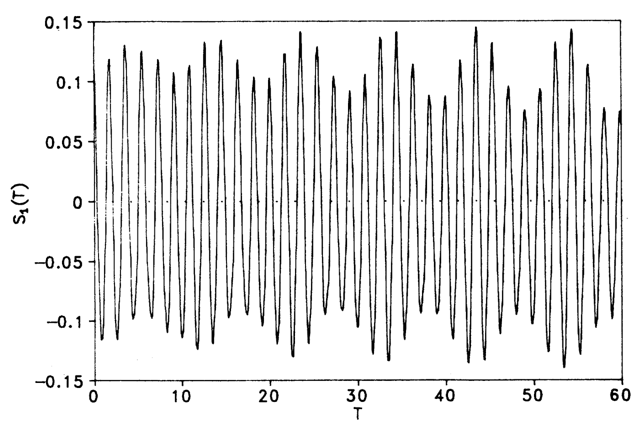

FIG. 1. $S_1(T)$ with the field initially in a squeezed vacuum state and the atom in the ground state for $r = 0.3$ ($\bar{n} = 0.09$). S_2 shows similar behavior.

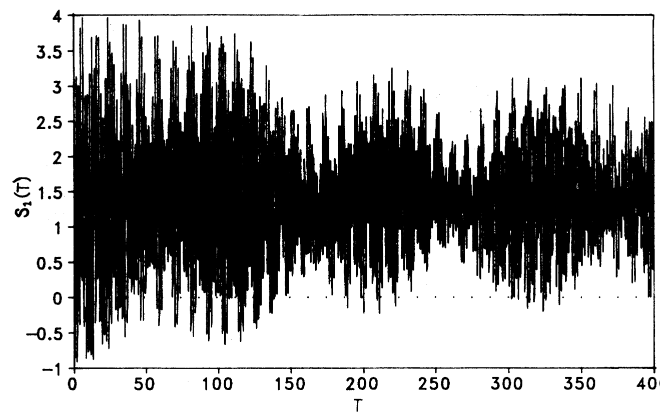

FIG. 2. Same as Fig. 1 but $r = 0.9$ ($\bar{n} = 1.05$).

Figure 2 displays the fact that as input squeezing is increased by taking $r = 0.9$, the maximum degree of AS squeezing also increases though with a reduced time duration. Here with the passage of time the minima of S_i shift upward in contrast to the case of low input squeezing, and the AS squeezing eventually disappears. Nonetheless, for the initial period of time, $T, \leq 15$, we see a slight increase in the value of AS squeezing (enhanced value of 73% compared to the initial value of 67%). For $S_i(T)$ beyond $T = 60$, we recognize that the above-mentioned behavior is repeated for $60 \leq T \leq 140$ with decreased amount of AS squeezing. The $S_i(T)$ are seen to exhibit threefold oscillations consisting of (i) "elementa-

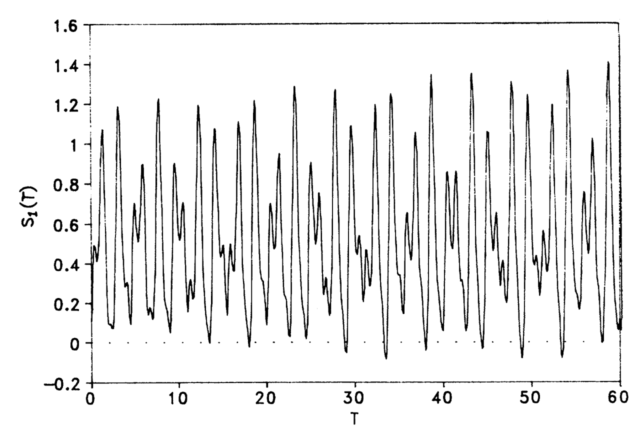

FIG. 3. Same as Fig. 1 but the atom in the excited state.

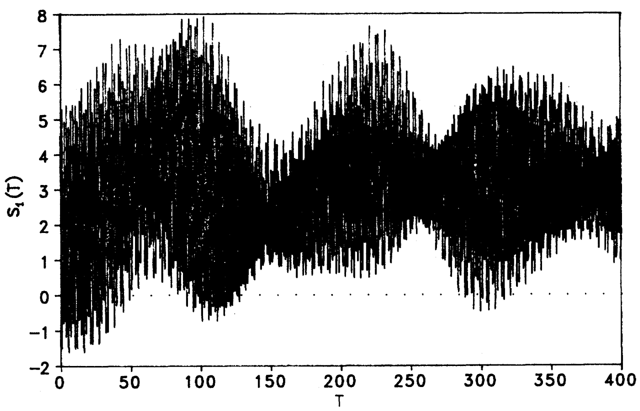

FIG. 4. Same as Fig. 2 but the atom in the excited state.

ry" packets in each of which the AS squeezing function executes several rapid oscillations with some of these packets showing collapses and revivals of AS squeezing and the others remaining above the standard quantum limit (SQL); (ii) "composite" packets, each consisting of several elementary packets; and (iii) collapses and revivals of the composite packets which make their envelope function look like a pulsating wave. Correspondingly, the AS squeezing disappears on three time scales that are related to the time period of oscillations in the elementary packet and revival times of the elementary and composite packets.

Figures 3 and 4 show behavior of the AS squeezing with the atom initially in the excited state. Comparing Figs. 1 and 3 we see that the oscillations here are irregular and remain mostly above the SQL. The AS squeezing in the in-quadrature component disappears almost immediately after the interaction is turned on. However, it returns later in the form of irregular excursions to the region below the SQL, disappearing afterward. Although the in-phase component is initially unsqueezed it becomes squeezed later and, in fact, continues to go below the SQL more often and until much later than the in-quadrature component. Nonetheless, the amount of AS squeezing is negligible.

Figure 4 shows the case for $r=0.9$. The AS squeezing revives periodically up to $T\leq 30$ after which it starts to decrease and vanishes around $T\geq 50$. The AS squeezing reappears in bunches of excursions below the SQL for the time period $80\leq T\leq 130$ and later on S_1 squeezes intermittently during the time interval $280\leq T\leq 320$. The AS squeezing then disappears. Comparing with Fig. 2 we notice that, firstly, the AS squeezing now is less persistent than in the previous case, and, secondly, the elementary packets here pack more densely to form composite ones. In other words, the revival times for the elementary packets are smaller now though the composite envelope function continues to be characterized by more or less the same revival time as in the case of Fig. 2. For this case as well we do not observe any increase over the initial value of AS squeezing.

Thus we see that the squeezing behavior pertaining to the two initial states of the atom is significantly different. The detailed oscillatory behavior in the two cases differs both in terms of the amplitude and frequency. We find, moreover, that for both small and large input squeezing the output AS squeezing is more effective in the case when the atom starts out from the ground rather than the excited state.

Keeping in mind the fact that AS squeezing depends upon the $\langle \hat{n}^2(T)\rangle$, we examine its behavior for the values of r in Figs. 1 and 2 (or Figs. 3 and 4). For $r=0.3$, Fig. 5 shows that the function oscillates periodically whether the atom is in the ground or excited state. The amplitude within which $\langle \hat{n}^2(T)\rangle$ oscillates is obviously larger for the atom initially in the excited state than when it is in the ground state [cf. Eqs. (11) and (15)]. Figure 6 shows that as input squeezing is increased to correspond to $r=0.9$, the oscillations are no longer regular but become chaotic. For $T>20$, $\langle \hat{n}^2(T)\rangle$ begins to oscillate rapidly and the amplitude limits within which it oscillates become smaller.

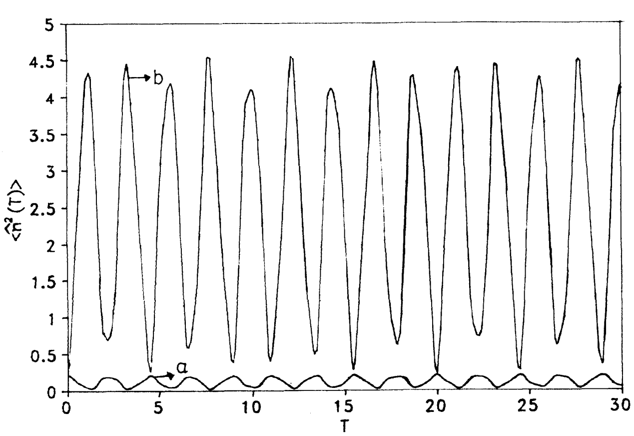

FIG. 5. $\langle \hat{n}^2(T)\rangle$ with the field initially in a squeezed vacuum state, and the atom in the (a) ground state (b) excited state, for $r=0.3$ ($\bar{n}=0.09$).

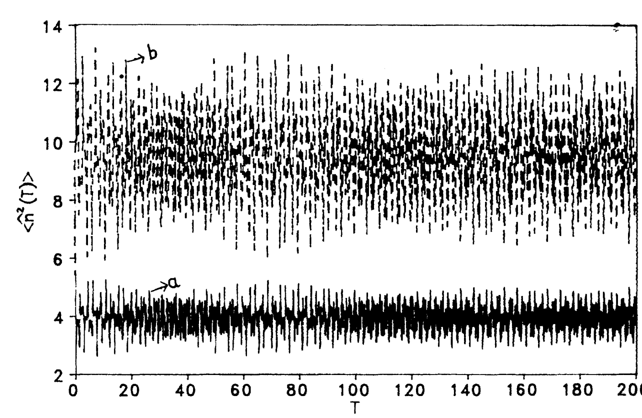

FIG. 6. Same as Fig. 5 but $r=0.9$ ($\bar{n}=1.05$).

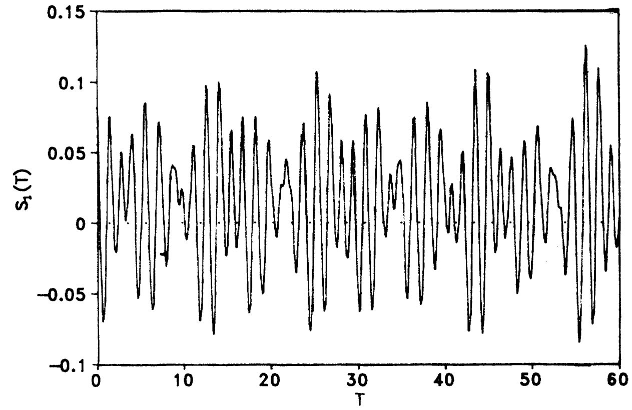

FIG. 7. $S_1(T)$ with the field initially in a general squeezed state and the atom in the ground state, for $r=0.3$ and $\alpha=0.25$ ($\bar{n}=0.15$).

A comparison of Figs. 5(a) and 1 and Figs. 5(b) and 2 reveals that the AS squeezing occurs whenever $\langle \hat{n}^2(T)\rangle$ is close to its minimum value. For $r=0.9$, we notice that at the time after which $\langle \hat{n}^2(T)\rangle$ starts oscillating rapidly the AS squeezing begins to diminish (see also Figs. 3, 4, and 6). There is an interesting statement made by Hillery [8] in the context of a relationship between $\langle \hat{n}^2(T)\rangle$ and AS squeezing (namely, $\langle(\hat{n}+\frac{1}{2})^2\rangle \geq \{1+(\langle\hat{n}\rangle+\frac{1}{2})^2/[S_1(T)+\langle\hat{n}\rangle+\frac{1}{2}]\}$) that "As the amount of AS squeezing increases the square of the number of photons must grow." Our discussion of $\langle \hat{n}^2(T)\rangle$ and AS squeezing leads to the conclusion that the statement must be interpreted to mean that although the inequality holds at every instant of time it implies no direct proportionality relationship between the two quantities.

B. General squeezed input

We first discuss the case when the atom is initially in the ground state. Figure 7 shows the result for $r=0.3$ and $\alpha=0.25$ (i.e., low input intensity and small initial squeezing). The AS squeezing behavior is more or less similar to the case when the input field is taken in a squeezed vacuum state, namely, that revivals of AS squeezing occur. With an input AS squeezing of 6% we get up to a maximum of 13% during the time interval considered. This behavior persists on the long time scale. The present case differs from the squeezed vacuum input case in at least two ways: Firstly, the packets of oscillations seen are sharper, and, secondly, the composite packets are absent. It is instructive to compare our result with that for a coherent input case. It is known [10] that in the latter case the AS squeezing increases with the increase in the initial average photon number up to a point. Here we do not observe such an increase, as is evident from a comparison of Figs. 1 and 7. Figure 8 shows that as the input squeezing is increased, the initially present AS squeezing of about 45% increases to a maximum of about 55% at $T\cong 15$. The collapses and revivals can also be seen clearly over the time interval considered. However, the AS squeezing is permanently revoked beyond $T\geq 250$.

Figures 9(a) and 9(b) show the AS squeezing for a relatively higher input intensity ($\alpha=4$) and $r=0.3, 0.9$, respectively. For low input squeezing, $r=0.3$, with an input of 42% the maximum AS squeezing seen in Fig. 9(a) is only about 49%. The AS squeezing continues to occur periodically even at the long time scale, the time period being approximately 1 g^{-1}. As the input squeezing is increased to correspond to $r=0.9$, the oscillations become smoother and the period increases to 1.28 g^{-1}. A relatively stronger AS squeezing is present now. It is instructive to compare these results with those obtained for the one-photon intensity-dependent coupling JCM (IDC JCM). Both for low and high input squeezing, the AS squeezing in IDC JCM is known [11] to revive periodically with a period very close to the figure determined here. The differences between the two models are marked by the presence in the two-photon JCM AS squeezing of wiggles in the region where S_1 achieves its minimum values in the case of low input squeezing. For the case of high input squeezing even this difference disappears. This is simply a reflection of the fact that the initial aver-

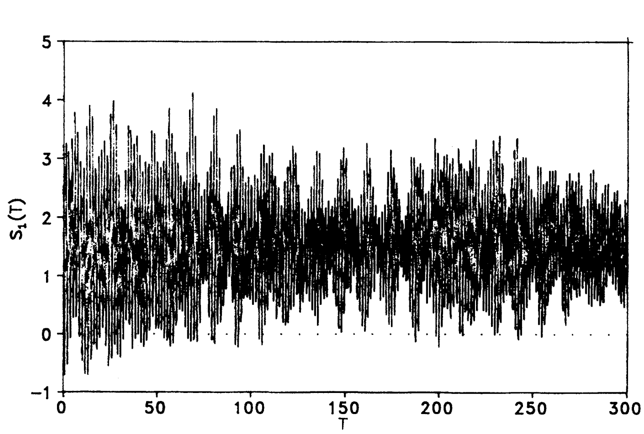

FIG. 8. Same as Fig. 7 but $r=0.9$ and $\alpha=0.25$ ($\bar{n}=1.11$).

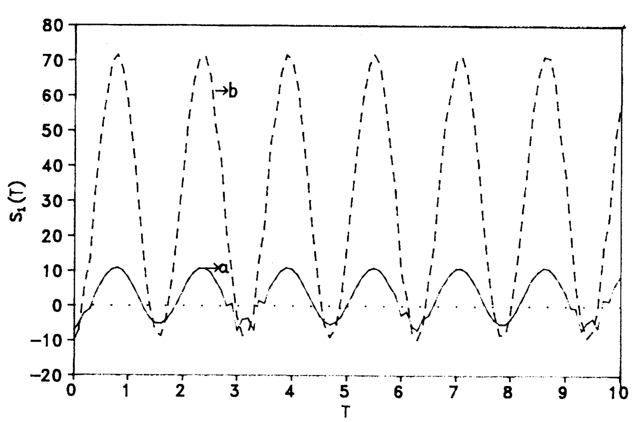

FIG. 9. Same as Fig. 7 but (a) $r=0.3$, $\alpha=4$ ($\bar{n}=16.11$) and (b) $r=0.9$, $\alpha=4$ ($\bar{n}=17.05$). S_2 shows no AS squeezing.

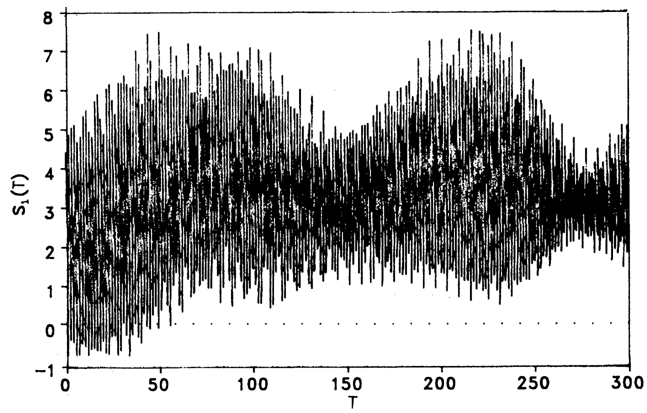

FIG. 10. Same as Fig. 8 but the atom in the excited state.

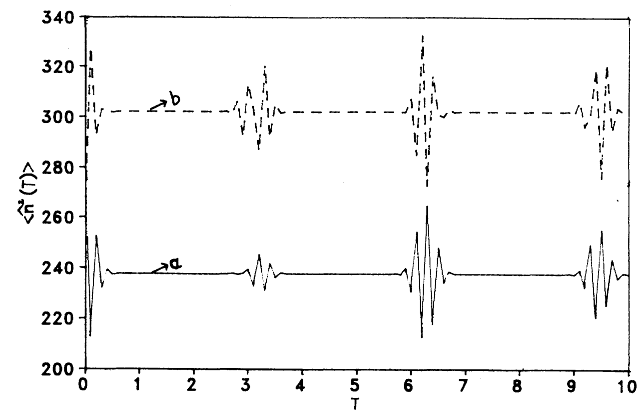

FIG. 11. $\langle \hat{n}^2(T) \rangle$ with the field initially in a general squeezed state and the atom in the (a) ground state, (b) excited state, for $r=0.3$ and $\alpha=4$ ($\bar{n}=16.11$).

age photon number is large ($\bar{n}=16.11$ and 17.05) and the Rabi frequency of this case goes over into that of the IDC JMC case for large input squeezing and intensity.

For the atom initially in the excited state, the post interaction behavior of AS squeezing changes dramatically with changing input intensity and initial squeezing. For low input intensity and squeezing, $\alpha=0.25$ and $r=0.3$ (not shown here), the initially present AS squeezing does not recur and, in fact, is revoked soon after the interaction is turned on (it appears only once, at a level of 3%). We see no AS squeezing at the long time scale. For $r=0.9$, Fig. 10 shows that the initially present AS squeezing revives though with a decreased amount. However, on the long time scale it vanishes in this case as well. Compared to the case of the ground-state atom, the elementary packets of oscillations are more densely spaced. For large initial intensity but small initial squeezing, on the other hand, we find that the initial AS squeezing revives periodically and gets enhanced to a point. This enhancement in the present case is larger than when the atom starts out from the ground state. This is somewhat surprising because one expects to see additional noise due to spontaneous emission. With increased input squeezing, we still obtain revivals of AS squeezing but with a decreased degree. The spontaneous emission does not seem to degrade the AS squeezing.

In the case of low input intensity the oscillatory behavior of $\langle \hat{n}^2(T) \rangle$ for the atom initially in the ground or excited states is more or less similar to the case of squeezed vacuum input. For large initial intensity ($r=0.3$ and $\alpha=4$), on the other hand, it shows collapses and revivals both with a ground- and excited-state atom as is evident from Fig. 11. These collapses and revivals are periodic and we notice that the AS squeezing occurs during revival as well as collapse periods.

ACKNOWLEDGMENT

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