Stability of an imploding spherical shell

S. J. Han
Los Alamos National Laboratory, University of California, Los Alamos, New Mexico 87545
(Received 17 June 1991)

It is shown that, in an imploding spherical shell, the surface instabilities are of two different types. The first, which occurs at the outer surface, is the well-known Rayleigh-Taylor instability. The second instability occurs at the inner surface. The characteristics of this instability are similar to those of amplified sound waves in an imploding plasma shell. It is suggested that this instability is driven by the amplified sound waves.

PACS number(s): 52.35.Py, 52.35.Dm, 47.20.−k

I. INTRODUCTION

Few hydrodynamic problems in laser fusion have received more attention in the past than the Rayleigh-Taylor instability in an imploding plasma shell [1–3]. This instability, if it occurs, does not allow the uniform implosion that is critical for high-density compression.

The understanding of the Rayleigh-Taylor instability has not progressed to such a degree that its effect on the implosion dynamics can be derived from a general method such as the energy method or the method of applying the Liouville equation which gives a dispersion relation to examine stability of a system. In the absence of such a general method, it is necessary to appeal directly to the available experimental evidence (i.e., the conservation of mode numbers during an implosion) along with a simplifying assumption, notably self-similar motion, to limit the possible form of self-consistent solutions for the stability analysis.

Most of the previous analyses of the Rayleigh-Taylor instability have been carried out through the use of potential flow in a plane geometry [4,5]. The velocity potential fields are the first-order perturbations with respect to the stationary fluids, which are subject to a constant acceleration. It is characteristic of the velocity potential fields that in the first-order analysis the perturbed fluid elements must move with the given velocity in two orthogonal directions along the interface. The Rayleigh-Taylor instability is, therefore, driven by a nonuniform distribution of fluid elements in velocity space (i.e., the velocity driven instability).

It has been customary, in the potential flow theory, to regard the velocity $\mathbf{v}_1=\nabla\phi_1(x,y,t)$ as the quantity one measures experimentally, and to think of the potential $\phi_1(x,y,t)$ as a convenient, but fictitious, mathematical construction.

The direct inference from Taylor's analysis[4] to a spherical shell implosion has seemed almost axiomatic inasmuch as nearly all stability analyses have been carried out in a plane geometry. However, numerical simulations performed by such methods have rarely been able to implement perturbations consistent with the Rayleigh-Taylor instability.

It has long been clear that the use of the velocity potential fields as initial perturbations offers at best only an awkward way of simulating (in all likelihood incorrect or at best incomplete) the Rayleigh-Taylor instability [6]. Verdon *et al.* [7] were the first to find that the use of a set of Lagrangian displacements, one which arises in a natural way in the numerical simulations of the Rayleigh-Taylor instability, offers a consistent picture of the instability in linear and nonlinear regimes [7]. It should be stressed that the use of Lagrangian displacement vectors is equivalent to the use of velocity perturbations by a flow potential to first order. The need for a more general method led Book and Bernstein [2] to begin the development of a fully self-consistent approach to the problems of stability of a nonstationary system.

Mathematical difficulties in dealing with stability analysis of an imploding spherical shell obscure many questions of critical importance, such as geometrical effects on the stability that arise from questionable Fourier expansions of perturbations. An attempt has, therefore, been made, based on the mathematical techniques of Bernstein *et al.* [8], to find improved methods of obtaining the stability criteria.

A further difference between our approach and previous ones is that it is carried out to apply to a nonstationary system of arbitrary time dependence, rather than just to those which are, on the average, stationary in time [2]. We have obtained the static limits in terms of the familiar growth rate in the hope that they may be of help in understanding the instability problems for a nonstationary system.

Next we turn to the instability of the inner surface of an imploding shell. This instability does not occur to first order in the static limit by any known mechanism, but it occurs in the presence of sound-wave amplifications (SWA) in an imploding shell. Several years ago, Book [9] proposed that amplitudes of sound waves in an imploding spherical shell can be amplified to the order of a shell thickness. More recently, the author [10] has shown that, in an extended model, the amplification of traveling sound waves takes place in a spherical target by adiabatic compression of the fluids that support the sound waves.

A more fruitful approach to the question of SWA is, perhaps, to study the surface instability which is observable and can be offered as evidence for the presence of

SWA in an imploding shell [11–13].

The instability at the inner surface is uniquely different from all other hydrodynamic instabilities; it is oscillatory at early times; the lower mode dominates higher modes in contrast to the Rayleigh-Taylor instability; the physical mechanism of the instability resides in the plasma fluid of the shell as in the case of the shear driven instability in an electron beam in crossed fields [14]. The physically different instability at the inner surface and the Rayleigh-Taylor instability are independent and they do not interact with each other.

The condition for the occurrence of the Rayleigh-Taylor instability is well known, but the physical mechanism by which the inner-surface instability evolves from stable modes to unstable modes is not understood. A possibility of occurrence of the second type of instability was first proposed by the author and Suydam [13] in our stability analysis of an imploding cylindrical plasma shell. However, it was not possible to see the physical mechanism because of shortcomings in the model. To overcome this problem, we have developed a model that removes the massless free surface at the inner surface.

We shall devote the remainder of this paper to the study of instabilities of an imploding spherical plasma shell and illustrating them in three-dimensional analyses. In Sec. II we present a detailed derivation of the basic equations that can be used in the stability analysis for the Rayleigh-Taylor instability and the instability at the inner surface. In Sec. III we solve the differential equation in the displacement vector ξ to establish the stability criteria. The physical interpretation of the instability is then discussed in Sec. IV. We show there that the Rayleigh-Taylor instability and the instability at the inner surface are independent in the linear regime. As a definite illustration of our stability criteria, we present a detailed analysis of the Rayleigh-Taylor instability for specific mode numbers. As a final illustration of the mechanism for the inner-shell instability we discuss in Sec. V the result of numerical simulations of SWA.

II. BASIC EQUATIONS

In this section we shall present a detailed derivation of the basic equations. Although this has been discussed previously [10], a number of its more interesting aspects have evidently been left untouched.

As in the previous analysis [10], the equations that describe an imploding spherical plasma shell are the following:

$$\rho \frac{d}{dt}\mathbf{v} = -\nabla p, \tag{1}$$

$$\frac{d}{dt}\rho + \rho \nabla \cdot \mathbf{v} = 0, \tag{2}$$

$$\frac{d}{dt}(p/\rho^{\gamma}) = 0, \tag{3}$$

where $\gamma > 1$. Here the notations are standard and the plasma fluid is assumed to be initially isentropic.

The stability analysis of an imploding plasma shell in the Lagrangian representation has a number of appealing properties [6] as we shall see. One of the most important of these is that it is extremely simple to incorporate the stability problem in the Lagrangian codes. It can also help in visualizing the growth of an instability in numerical simulations, but we may have to find appropriate boundary conditions for the Rayleigh-Taylor instability.

In the following, we assume that, in the Lagrangian representation, all quantities are functions of $\mathbf{r}_0$, the initial position of a fluid element, and of t, the time. We may further require that the position of a fluid element is given to the first order in the displacement vector ξ by

$$\mathbf{r} = \mathbf{R}(\mathbf{r}_0, t) + \xi(\mathbf{r}_0, t), \tag{4}$$

where $\mathbf{R}(\mathbf{r}_0, t)$ describes the unperturbed trajectory of a fluid element at t.

By using the displacement vector introduced in Eq. (4) we may easily derive from the continuity equation Eq. (2) the perturbed density [13]

$$\rho(\mathbf{R}+\xi) = \rho_0(1 - \nabla_R \cdot \xi) = \rho_0 + \rho_1. \tag{5}$$

Evidently if we limit our calculations to the Rayleigh-Taylor instability, which is the velocity-driven instability, the plasma density must remain unperturbed by the displacement vectors. This condition can be satisfied by the requirement that $\nabla \cdot \xi = 0$. This is the only constraint needed in order to obtain, in the first order of ξ, the stability criteria for the instability. It may be helpful to note that the condition $\nabla \cdot \xi = 0$ is essentially equivalent to boundary conditions of reflecting surfaces since the condition does not allow any form of waves in the fluid except at the surfaces. Thus this condition leads to standing waves at the inner and outer surfaces.

There are certain problems for which it is advantageous to derive the above equation in quite a different manner from that shown in Ref. [13]. An example is the stability analysis of a relativistic electron beam penetrating into plasmas in which the equation of motion for a single electron in the beam can be solved to first order. In that case, the perturbed electron beam density can be derived from the equation of continuity and the equation of motion [15]. The result depends on the linear procedure with respect to the displacement vector and is quite different from Eq. (5), which was derived in a heuristic manner [13].

For the present hydrodynamic problem, it seems to be natural to use Eq. (5) for the perturbed density since the unperturbed quantities are not time independent and solutions of the form $e^{i\omega t}$ for the perturbations are not allowed. It is exactly because of the time-dependent character of the unperturbed motion that the concept of growth rate of an instability in an imploding shell is no longer applicable to a nonstationary system.

By substituting Eq. (5) into the equation of state Eq. (3), we can easily express the perturbed pressure in terms of the displacement vector. The result can be written in the form

$$p(\mathbf{R}+\xi) = p(\mathbf{R})(1 - \gamma \nabla_R \cdot \xi). \tag{6}$$

In order to solve the equation of motion for a fluid element, we must first find the variation of the velocity due to the displacement ξ, which is given by

$$\mathbf{v}(\mathbf{R}+\xi)=\mathbf{v}(\mathbf{R})+\frac{d}{dt}\xi=\mathbf{v}(\mathbf{R})+\frac{\partial}{\partial t}\xi+\mathbf{v}(\mathbf{R})\cdot\nabla_R\xi. \quad (7)$$

An interesting feature of perturbations like Eq. (7) is that a proper set of Lagrangian displacement vectors can introduce the velocity perturbations consistent with the Rayleigh-Taylor instability. The perturbed velocity term in Eq. (7) is of particular interest in the stability analysis because of an explicit dependence on ξ in the perturbation expansions. The similarity of Eq. (7) to the first-order potential flow is obvious when we note that if we choose $\xi=(\xi_x(x,y,t),\xi_y(x,y,t))$ and set $\mathbf{v}(\mathbf{R})=0$ in a plane geometry, $\mathbf{v}_1=\nabla\phi_1(x,y,t)$ is equal to the perturbed term in Eq. (7). It would have taken a good deal of physical insight to see that a potential flow approach is essentially equivalent to the use of Lagrangian displacement vectors [7].

In order to obtain the equation of motion to first order, we need to expand the differential operator with respect to $\mathbf{r}$ to first order in ξ, which is given by

$$\nabla=\nabla_R-\nabla_R\xi\cdot\nabla_R. \quad (8)$$

This is obtained by the usual chain rule of differentiation.

In the earlier paper [10], we discussed the self-consistent solution of Eqs. (1–3) by introducing Sedov's hypothesis of uniform self-similar motion [16]. In the Lagrangian representation it can be written as

$$R(r_0,t)=r_0 f(t), \quad (9)$$

where R is the radial position of a fluid element at t and r_0 is its initial position.

The simplest of the physical effects contained in Eq. (9) is the self-similar motion of the isentropic fluids under consideration. This hypothesis is a well-known and extremely powerful method of obtaining self-consistent solutions for hydrodynamic equations, but it prescribes the time-dependent external pressure. This is because the conservation of mass and Eq. (9) determine the time-dependent density profile, which in turn determines the time-dependent pressure profile through the equation of state Eq. (3). This constraint implies that the adiabatic compression by the external pressure in the form of a shock wave by an ablation process must match the prescribed pressure profile; this may be a difficult task. The dynamical behavior of an imploding shell is thus governed by the three basic hydrodynamic equations (1–3) and Eq. (9). It is reasonable to assume, however, that by varying the pulse profile of a laser beam, we may approximately satisfy this requirement.

From the above discussion and using Eqs. (2), (3), and (9), the time-dependent density and pressure profiles can be obtained as

$$\rho(r_0,t)=\rho_0(r_0)f(t)^{-3} \quad (10)$$

and

$$p(r_0,t)=p_0(r_0)f(t)^{-3\gamma}. \quad (11)$$

It is then clear from the equation of motion Eq. (1) that for the unperturbed motion we may write

$$\ddot{f}(t)f(t)^{(3\gamma-2)}=-\frac{1}{\rho_0 r}\frac{d}{dr}\rho_0^\gamma=-\frac{1}{t_c^2}=-1, \quad (12)$$

where the subscript in r is dropped.

Here we shall first study a density profile that is symmetric with respect to the inner and outer surfaces, one that does not introduce a mathematical singularity in the first-order equation and removes the unphysical massless surface. We may then form the density profile with finite density at the inner and outer surfaces with a prescribed pressure profile. The most general density profile that satisfies Eq. (12) takes the form

$$\rho_0(r)=[\rho_+^{\gamma-1}(r^2-r_-^2)/(r_+^2-r_-^2) +\rho_-^{\gamma-1}(r_+^2-r^2)/(r_+^2-r_-^2)]^{1/(\gamma-1)}, \quad (13)$$

where r_+ and r_- are the outer and inner radii of the shell, $\rho_0(r_+)=\rho_+$, and $\rho_0(r_-)=\rho_-$. If we substitute Eq. (13) into Eq. (12), we find that ρ_+ and ρ_- must satisfy the following relation, which is the condition appropriate to the modeling of an imploding plasma shell:

$$\frac{(\gamma-1)}{2}(r_+^2-r_-^2)/(c_+^2-c_-^2)=t_c^2=1, \quad (14)$$

where $c_+^2=\gamma p_+/\rho_+=a\gamma\rho_+^{\gamma-1}$ and $c_-^2=\gamma p_-/\rho_-=a\gamma\rho_-^{\gamma-1}$. Note that Eqs. (13) and (14) imply that the pressure profile remains unchanged in space during the entire implosion process; only the time-dependent part $f(t)$ varies.

For the time-dependent part that is the solution of Eq. (12) one may easily obtain the first integral as

$$\dot{f}(t)=-[(2/\alpha)(f^{-\alpha}-1)]^{1/2}, \quad (15)$$

where $\alpha=3(\gamma-1)$ and the initial conditions $f(0)=1$ and $\dot{f}(0)=0$ are used.

For $\gamma=\frac{5}{3}$, the solution of this equation is easily found and is given by

$$f(t)=(1-t^2)^{1/2}. \quad (16)$$

It is worth noting that $\dot{f}(t)$ approaches infinity as $f(t)\to 0$ for all γ values. In this sense it follows that in the limit of very small $f(t)$, which corresponds to a completion of an implosion, the model breaks down. Thus, the assumption we have made in defining the model, that the self-similar motion is a reasonable description of an imploding plasma shell, is a strong one. However, it can be fairly well fulfilled in much of the imploding process provided that a laser pulse is designed accordingly.

In order to study the stability of the prescribed unperturbed motion we obtain the first order equation in ξ. This calculation is tedious although it is straightforward. The result may be written

$$f^{(3\gamma-1)}\ddot{\xi}=\frac{(\gamma-1)}{2}(r^2-r_1^2)\nabla\sigma+(\gamma-1)\sigma\mathbf{r} +\mathbf{r}\times\omega+(\mathbf{r}\cdot\nabla)\xi, \quad (17)$$

or

$$f^{3(\gamma-1)}\ddot{\xi}=\nabla\left[\frac{(\gamma-1)}{2}(r^2-r_1^2)\nabla\cdot\xi+\xi\cdot\mathbf{r}\right]-\xi\cdot\nabla\mathbf{r}, \tag{18}$$

where $\sigma=\nabla\cdot\xi$, $\omega=\nabla\times\xi$, and

$$r_1^2=(c_+^2r_-^2-c_-^2r_+^2)/(c_+^2-c_-^2).$$

The linear analysis we have presented in this section has been intended to provide a self-contained presentation for the stability analysis in the following sections.

III. SOLUTIONS OF THE FIRST-ORDER EQUATION

In the preceding section we have derived the differential equation linear in ξ to examine stability of an imploding spherical shell. For surface instabilities, it becomes possible to reduce the equation to a considerably simpler form. To be consistent with Taylor's analysis [4], we look for a solution with the boundary conditions that at the surface the perturbed fluid elements may be treated as incompressible and irrotational, $\nabla\cdot\xi=0$ and $\nabla\times\xi=0$. We shall, therefore, seek a solution in the form $\xi=\nabla\chi$, where the potential function χ is the solution of Laplace's equation $\nabla^2\chi=0$.

This form is one that clearly shows similarities between the potential flow calculations and the corresponding ones using the Lagrangian displacement vectors. The use of the potential function χ offers insights into the reason why the calculations based on the displacement vectors are the same as in the potential flow approach, even though the former calculations appear to be far more complex. Viewed in mathematical terms, the use of the Lagrangian displacement vectors in this way makes it possible to study a velocity-driven instability like the Rayleigh-Taylor instability for a nonstationary system. We shall see that the stability criteria are obtained quite conveniently through use of Eq. (17) and vector spherical harmonics.

A solution of Laplace's equation can be found explicitly in given coordinates by separation of variables and is given in spherical coordinates as

$$\chi=\Sigma_{l,m}[Q_+^l(t)r^l+Q_-^l(t)r^{-(l+1)}]Y_{l,m}(\theta,\phi)\ . \tag{19}$$

Since very little is discussed about the potential function of χ, some insight may be gained by examining the form it takes on in one of the completely solvable problems. Near the origin which corresponds to the initial position of the inner surface, the second term diverges as $r^{-(l+1)}$. The potential is therefore much too strongly singular for higher modes at the origin to permit a short wave perturbation. Indeed for the instability at the inner surface, as we shall see, the lower modes dominate the higher modes in agreement with SWA [10].

It may be worth noting, for its mathematical interest, that as we have seen the Rayleigh-Taylor instability is a velocity-driven instability which was introduced through use of a potential function. In velocity potential fields [4], fluid elements are moved by the perturbed velocity along the tangential plane at the interface. The perturbed velocity which corresponds to the gradient of flow potential may be constructed immediately from a potential field. In practice, of course, we may not often know the form of a potential field that gives the desired velocity perturbations, and so the potential flow approach has a limited use for the study of the Rayleigh-Taylor instability. In the Lagrangian representation, however, the perturbations can be constructed from Eq. (19). The three orthogonal displacement vectors, two of which give the tangential fluid motion, and the third of which gives the radial perturbation at the surface of a sphere, are required to examine stability of an imploding spherical shell; they will be introduced in a manner to be consistent with Taylor's analysis [4].

As indicated in the Introduction, the use of potential fields in describing the perturbations shows similarities between the calculations based on the Lagrangian displacement vectors and the corresponding ones with a potential flow. While these similarities make applications of the displacement vectors particularly clear, they must not be interpreted as indicating that a potential flow approach, which is useful only for a static problem in plane geometry, is any sort of adequate substitute for a nonstationary system.

The time-dependent functions $Q_\pm^l(t)$ defined in Eq. (19) must clearly satisfy Eq. (17), just as the spatial part does. In view of Eqs. (17) and (19), we may introduce a set of three orthogonal vector spherical harmonics which permits us to write the displacement vector as [17,18]

$$\xi=\Sigma_{l,m}[\xi_1^{l,m}\mathbf{a}_1+\xi_2^{l,m}\mathbf{a}_2+\xi_3^{l,m}\mathbf{a}_3], \tag{20}$$

where

$$\mathbf{a}_1=\mathbf{e}_rY_{l,m}(\theta,\phi),$$

$$\mathbf{a}_2=r\nabla Y_{l,m}(\theta,\phi),$$

and

$$\mathbf{a}_3=\mathbf{r}\times\nabla Y_{l,m}(\theta,\phi)\ .$$

The constraint implicit in Eq. (19), that the expansion must depend analytically upon the time t, is what renders this expansion unique. The virtue of an expansion scheme in which the coefficients $Q_\pm^l(t)$ are uniquely determined by Eq. (17) is self-evident. Substitution of Eq. (20) into Eq. (17) together with Eq. (19) yields

$$f^{(3\gamma-1)}(t)\ddot{Q}_\pm^l(t)+[\tfrac{3}{2}\mp(l+\tfrac{1}{2})]Q_\pm^l(t)=0. \tag{21}$$

Setting aside for the moment the general solutions of Eq. (21), we may ask how the solutions of Eq. (21) can be related to the stability of a stationary spherical plasma shell subject to a constant acceleration. The answer for this simple case may be seen without performing any detailed calculations.

If we define the effective acceleration of the shell as

$$g(t)=\ddot{R}(t)=r\ddot{f}(t)=-rf^{-3\gamma+2},$$

then Eq. (21) can be rewritten in a more familiar form,

$$\ddot{Q}_\pm^l(t)-\left[\pm\left[l+\frac{1}{2}\right]-\frac{3}{2}\right]\frac{|g(t)|}{R(t)}Q_\pm^l(t)=0. \tag{22}$$

For $t\ll1$ we may take $R(t)$ and $g(t)$ as constants. The equation is then reduced to an eigenvalue equation with implicit boundary conditions $\nabla\cdot\xi=0$ at the surfaces. The solutions are given by a simple form $ae^{i\omega_\pm t}$ with the eigenvalues $\omega_\pm^2=[\mp(l+\frac{1}{2})+\frac{3}{2}](g/R)$. If any $\omega_\pm^2(l)$ is negative, the corresponding $\omega_\pm(l)$ is imaginary and the system is unstable to the perturbations.

It is clear from this simple analysis that in the static limit the Rayleigh-Taylor instability occurs at the outer surface, but it is only for the mode number $l\geq 2$. This value corresponds to the lowest bound for which the geometrical explanation can be found. The growth rate of the Rayleigh-Taylor instability can be written explicitly

$$\gamma_{R-T}^2=[(l+\tfrac{1}{2})-\tfrac{3}{2}](g/R), \tag{23}$$

where $l\geq 2$.

It is important to note that the mode number l is a conserved quantity and that the wave vector $k_\theta=l/R$ is also conserved in the eigenvalue equation. The growth rate shows that the larger the mode number, the faster the instability grows; the short wave grows faster than the long wave. Furthermore, we notice that the shell is stable to the Rayleigh-Taylor instability for the mode $l=0$, and marginally stable for the $l=1$ mode in agreement with anticipation.

In contrast, the inner surface remains stable for all mode numbers, as we might have expected from the analysis of SWA [10]. Thus we see that Eq. (22) correctly describes the stability of a spherical shell, just as one might have seen in a correct stability analysis of a stationary shell.

Let us return to Eq. (21). The general solution is readily found for the prescribed motion by using Eq. (15) and is given in terms of two linearly independent hypergeometric functions [19],

$$Q_\pm^l(t)=c_1\mathcal{F}_\pm(l,\gamma,t)+c_2\mathcal{G}_\pm(l,\gamma,t). \tag{24}$$

Here

$$\mathcal{F}_\pm(l,\gamma,t)={}_2F_1(a,b,c;1-f(t)^{-\alpha}) \tag{25}$$

and

$$\mathcal{G}_\pm(l,\gamma,t)=(1-f^{-\alpha}){}_2F_1(a+\tfrac{1}{2},b+\tfrac{1}{2},c+1;1-f(t)^{-\alpha}), \tag{26}$$

where

$$a=\frac{1}{4}+\frac{(2+id_\pm)}{4\alpha},$$

$$b=\frac{1}{4}+\frac{(2-id_\pm)}{4\alpha},$$

$$c=\tfrac{1}{2},$$

$$\alpha=3(\gamma-1),$$

and

$$d_\pm=\{8\alpha[\tfrac{3}{2}\mp(l+\tfrac{1}{2})]-(\alpha+2)^2\}^{1/2}.$$

Since our boundary conditions are appropriate for reflecting surfaces which lead to standing waves, we would find standing waves at the surfaces, but the wave vector $k_\theta=l/R(t)$ is time dependent. We then face the following question: how can this wave be expressed in terms of conventional standing waves? This question may not have a definite answer unless the eikonal treatment [20] of surface waves is carried out; but, it is clear that since the wave vector depends on time explicitly the solution does not have a spherical wave expansion.

Now we come to important differences in stability analyses for a nonstationary system. The differences between the conventional waves and the present surface waves have two causes: one is the time-dependent wave vector by the conservation of mode number l during the implosion, and the other is that the medium which supports the waves is being compressed and is accelerated by the time-varying external pressure.

To incorporate the proper geometrical effects, we calculate the asymptotic limits, augumented by $k_\theta=l/R(t)\propto 1/f(t)$. The mode number l will be assumed to be a conserved quantum number during the implosion. In the limit $t\rightarrow 1$, calculations leads to the simple results,

$$\lim_{t\rightarrow 1}[\mathcal{F}_\pm(l,\gamma,t)/f(t)]\cong a_\pm^l f^{(\alpha-2+id_\pm)/4}+\text{c.c.}, \tag{27}$$

$$\lim_{t\rightarrow 1}[\mathcal{G}_\pm(l,\gamma,t)/f(t)]\cong b_\pm^l f^{(\alpha-2+id_\pm)/4}+\text{c.c.}, \tag{28}$$

where

$$a_\pm^l=\Gamma\left[\frac{1}{2}\right]\Gamma\left[\frac{-id_\pm}{2\alpha}\right]\left[\Gamma\left[\frac{1}{4}+\frac{(2-id_\pm)}{4\alpha}\right]\Gamma\left[\frac{1}{4}-\frac{(2+id_\pm)}{4\alpha}\right]\right]^{-1} \tag{29}$$

and

$$b_\pm^l=i\Gamma\left[\frac{3}{2}\right]\Gamma\left[\frac{-id_\pm}{2\alpha}\right]\left[\Gamma\left[\frac{3}{4}+\frac{(2-id_\pm)}{4\alpha}\right]\Gamma\left[\frac{3}{4}-\frac{(2+id_\pm)}{4\alpha}\right]\right]^{-1}. \tag{30}$$

This prescription for handling the geometrical convergence effect ensures only the consistency for a nonstationary system; the simple form of the asymptotic limit is at best incomplete and somewhat arbitrary. Had we started this calculation by incorporating the geometrical effect in Eq. (19) as a direct product and making the observation for a conservation of the mode number, we should have come to the same result. The virtue of the asymptotic limits is that they permit discussion of general features of instabilities in a nonstationary system. We should therefore regard Eqs. (27) and (28) as no more than a crude, simple model.

It may be natural at this point to ask the meaning of the stability criteria. The results given in Eqs. (27) and (28) suggest that there is, in fact, a substantial ambiguity involved in speaking of the instability. We must notice, however, that the structure of the asymptotic form with the geometrical effects was dictated by experience with the Rayleigh-Taylor instability in the static limit. Although the role of a wave vector is ambiguous in this analysis, the geometrical convergence effects such as conservation of both mass and the mode number l are self-evident. One of the essential respects in which the present stability analysis differs from the conventional approach for a stationary system is that the time-dependent displacement vector is found by solving the equation of the displacements as the initial and boundary-value problem.

To underscore the correctness of Eqs. (27) and (28), we will analyze the Rayleigh-Taylor instability in some detail, and show that the geometrical convergence effects are properly incorporated in the stability criteria.

IV. PHYSICAL INTERPRETATION OF THE FIRST-ORDER ANALYSIS

In the preceding sections we have derived the stability criteria for the Rayleigh-Taylor instability and the instability at the inner surface of an imploding shell. It has been customary, in discussions of the instabilities, to regard $\xi_0 e^{\gamma t}$ as the exponentially growing amplitude of a perturbation, and to take γ as the growth rate, but this has little bearing on the present analysis except for the static limits.

In this section we shall show how the Rayleigh-Taylor instability, which is located at the outer surface and which is independent of the internal dynamics of an imploding shell, can be related within the framework of a compressible fluid model to the classical Rayleigh-Taylor instability. Such an analysis may indeed help to convince of the correctness of the present analysis, but it still leaves open serious questions of consistency, and risks overlooking an important, new physical mechanism. The need for a more consistent, clear picture has led us to develop the Lagrangian code which, as we have seen in Ref. [10], has clarified many of the unsettling issues.

Let us first examine once more the stability at early times. In the short-wavelength limit, the growth rate of the Rayleigh-Taylor instability is given by Eq. (23),

$$\gamma_{RT}^2 = |g_0|\frac{l}{R} = |g_0|k_\theta . \tag{31}$$

This is the growth rate of the classical Rayleigh-Taylor instability of static media; it is independent of density profile, thickness of a shell, and γ. We also notice that the perturbations on the inner and outer surfaces behave independently since the mode $Q_-^l(t)$ is oscillatory. Furthermore, one can show from Eq. (15) of Ref. [10] that sound waves are oscillatory at early times. This analysis suggests that while the Rayleigh-Taylor instability is independent of the internal dynamics of fluids, the instability at the inner surface may not be.

Let us focus on the specific modes and first consider the $l=0$ mode. For $\gamma < \frac{5}{3}$, the limits of $\mathcal{F}_+/f(t)$ and $\mathcal{G}_+/f(t)$ are finite. Hence the $l=0$ mode at the outer surface is stable, which one would have normally expected since the mode corresponds to a uniform radial perturbation. We see, therefore, that there is no inconsistency in the analysis. On the other hand, if $\gamma < \frac{5}{3}$, $F_-/f(t)$ and $G_-/f(t)$ diverge asymptotically, which signals instability. This analysis shows that the two surface waves do not interact with each other or the fluid does not mediate the interaction between the waves in the linear regime.

A more significant property of the $Q_+^l(t)$ is that for the $l=1$ mode both $F_+/f(t)$ and $G_+/f(t)$ weakly diverge in the asymptotic limit and thus the Rayleigh-Taylor instability, which is marginally stable in the static limit, occurs for the mode number $l=1$. It is the unique exception in that the mode number $l=1$ corresponds to the displacement of the center of a shell without perturbing the surface. In the static limit, the instability grows linearly in time and thus it tends to move the shell away from the center of an imploding device.

The asymptotic limits tend to enhance the degree of divergence when the geometrical correction is made to the calculations of $Q_\pm^l(t)$. By itself, Eq. (24) is inadequate to explain the $l=1$ mode, which was marginally stable in the static limit, and becomes stable in the asymptotic limit. This is not surprising, since the conservation of mode number was not explicitly incorporated in the analysis and the time-dependent wave vector does not play any precise role in an implosion dynamics; Eq. (21) is no longer an eigenvalue equation. The specific mode $l=1$ we have discussed is, of course, an elementary one, but should serve to illustrate some of the points noted earlier regarding the consistency with the classical Rayleigh-Taylor instability. It is worth noting, in particular, that the asymptotic limits of the quotient $Q_\pm^l(t)/f(t)$ predict the correct stability criteria which play much the same role in these calculations as the stability criteria in the analyses based on the energy method; they are only indicative of the instability of a system.

It may be of interest to note that while the Rayleigh-Taylor instability is weakly unstable for the $l=1$ mode, the inner surface, as is shown later, remains unstable for the same mode. We then face the question: if the $l=1$ mode corresponds to a displacement of a shell as a whole, how does the inner surface become unstable? The fact that the stability of the inner surface has this property should not be a surprise in view of the way the displacement vectors are determined by Eq. (4). The displacement of the outer surface with the $l=1$ mode does not, therefore, imply the absence of a local perturbation in fluids since the magnitude of the displacement vectors are different at different positions in the shell.

For more general cases, we may examine the asymptotic limits Eqs. (27) and (28) for $l \geq 2$. The limits behave as $f^{-\kappa}$ when $f(t)\to 0$, where $\kappa=(\alpha-2+id_+)/4$. A detailed study of the limits shows that the larger the values of l and γ, the faster the mode grows, which is consistent with the classical Rayleigh-Taylor instability.

Although this consistency is extremely helpful in the analysis, and offers insights into the nature of the Rayleigh-Taylor instability, one must not lose sight of the fact that the present analysis for a nonstationary system is quite different from that of a stationary case. The asymptotic limit only signals an instability just as the stability criteria one obtains by using the energy principle [8].

The time-dependent behavior of the Rayleigh-Taylor instability in a compressible fluid model can be understood by noting that the effective acceleration increases in time as $f^{(-3\gamma+2)}$. Note also that, in a converging geometry, the wave vector $k_\theta=l/R(t)$ increases as $f^{-1}(t)$, since the mode number l is a conserved quantity. But this effect has been incorporated in the stability criteria by taking the limit of $Q^1_{\pm}(t)/R(t)$. We remind the reader that the growth rate depends on the wave vector, not the mode number at any given instant.

Next we consider, as above, the asymptotic limits of both $\mathcal{F}_-/f$ and $\mathcal{G}_-/f$ of $Q^l_-(t)$ modes. For $l \geq 1$ and $\gamma < 5/3$, the limits again diverge; the degree of the divergence is not as severe as that of the Rayleigh-Taylor instability. We now pose the question: Why does the special value of γ determine the stability and not other values? How does the initially stable mode evolve to an unstable mode? To see the physical significance of the γ value, we note that when $\gamma < \frac{5}{3}$, sound waves are amplified during the course of an implosion. Furthermore, the divergence factors in Eqs. (10) and (11) are the same as in the case of sound-wave amplifications [see Eqs. (20) and (21) of Ref. [10]]. Moreover, Fig. 1 of this paper and Fig. 2 of Ref. [10] show a remarkable similarity between the coefficients of amplification factors $|a^l_-(\gamma)|$ and $|a(l,\gamma,\mu_n)|$. In both cases, the instabilities grow faster for the smaller mode number l. The same similarity persists between $b^l_-(\gamma)$ and $b(l,\gamma,\mu_n)$.

This identical behavior suggests that the instability of the inner surface may be considered to be due to internal sound waves which act as the source of the surface wave, and that the dynamics of this instability lies in the plasma fluid of the shell just as for the shear driven instability of the electron beam in crossed fields [14]. Note also that the absence of amplified sound waves implies a stable inner surface; they are not really separable.

It should be stressed that all the discussions of this section are based on the linear stability analysis for an imploding spherical shell. Such an analysis is valid only if the time scale of the growth of instabilities bears an approximate relation to other time scales (e.g., an implosion time) of the system. For example, in the short-wavelength limits, the Rayleigh-Taylor instability will grow fast and quickly reaches a nonlinear regime at about the time when the wavelength becomes roughly equal to the wave amplitude. The initially isentropic fluid then becomes turbulent flow through the nonlinear process and the short waves dissipate in the form of energy during the implosion. In the nonlinear regime, all three waves, the sound wave and the two surface waves, are no longer independent; they may couple together through a three-wave interaction mechanism. Turbulent flow is beyond the scope of the present work and we refer the interested reader to the many available papers on the subject [21].

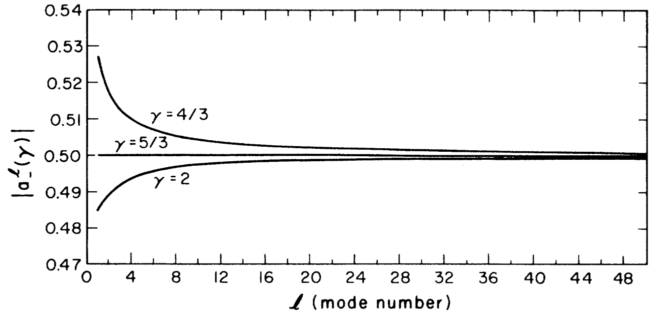

FIG. 1. The absolute values of $a^l_-(\gamma)$ for $\gamma=\frac{4}{3}$, $\frac{5}{3}$, and 2 plotted against the mode number $l=1-50$.

V. DISCUSSION AND CONCLUSION

If the proposed mechanism is correct, why then does the $l=0$ mode at the outer surface remain stable when $\gamma < \frac{5}{3}$ while the same mode at the inner surface becomes unstable? To answer this question and to see if the above is the only mechanism that drives the instability at the inner surface, we have carried out numerical simulations using a one-dimensional Lagrangian code written specifically for this purpose. The code solves the equation for mass conservation, the energy equation, the equation of state, and the equation of motion for the plasma fluid with the prescribed time-dependent external pressure. We have also taken the initial density profile given by the equation for $\rho_0(r)$.

The simulations show that the perturbations give rise to traveling sound waves that propagate toward the inner surface in much the same way as a surface wave in a moving fluid. However, the velocity of the sound wave is so much less than the velocity of the fluid that the reflected sound wave at the inner surface does not reach the outer surface. Because of a pressure gradient across the fluid layer, we do not observe the standing sound waves as in a stationary fluid; the velocity of the sound wave decreases as it propagates toward the inner surface since the fluid, which supports the sound waves, is being adiabatically compressed. The traveling sound waves are thus amplified by superposition and localized near the inner surface. As the sound waves reflect from the inner surface, they perturb the surface, which is an observable instability of the inner surface.

That such a surface instability does not occur in the absence of SWA can also be seen from the case of an exploding cylindrical shell [22,23], in which neither SWA nor a surface instability at the outer surface occurs.

Similar but two dimensional (i.e., r and θ components) calculations were performed for an imploding cylindrical shell [13,23]. The stability criteria were similar to those of the spherical shell implosion problem, but with the critical value of $\gamma_c = 2$. The difference in γ values can be easily understood by noting that for an ionized gas the ratio of specific heats $\gamma = (2+n)/n$, where n is the degree of freedom.

In spherical geometry, we have carried out a fully three-dimensional stability analysis by taking $\xi = (\xi_r, \xi_\theta, \xi_\phi)$, which allows a fluid element to undergo three-dimensional displacements from its equilibrium position. Therefore, γ_c should take two different values: $\gamma_c = \frac{5}{3}$ for the imploding spherical shell in which $n = 3$, and $\gamma_c = 2$ for the imploding cylindrical shell in which $n = 2$. However, this argument begs the question whether the macroscopic description of a plasma can employ the concept derived in a microscopic picture of an ionized gas (e.g., charged particles).

If we define the displacement vector ξ as the ensemble average of displacements of charged particles from their equilibrium positions, then the above argument is correct since the degree of freedom in translational motion of charged particles remains the same in both descriptions of the system. By the equipartition theorem, the ratio of specific heats is determined solely by the degree of freedom for translational motion of charged particles (e.g., an ideal gas) [24]. Note also that if the plasma were magnetized, the above argument would not be valid due to gyromotion of charged particles in an applied magnetic field. Furthermore, in a cylindrical shell, the system is assumed to be translationally invariant with respect to z.

In both analyses, we have been able to obtain analytically tractable solutions for complex hydrodynamic problems by employing the elegant method of Sedov [16], which restricts the underlying implosion dynamics. In particular, the external pressure is self-consistently determined by Sedov's similarity hypothesis and the equation of state. It is thus not surprising to find the same phenomena in both cylindrical and spherical shell implosions, which imply no change in implosion dynamics.

Finally, we turn to an experiment of observing the development of an instability of the type discussed here. A cylindrical shell imploded by an ion beam or a laser beam is ideally suited for a study of this instability. Observation of the instability along the axis of a cylindrical shell is obviously much easier, technically, than simultaneous observations of a spherical shell in various directions by x rays to see the instability [25–27]. Furthermore, the spherical geometry makes it impossible to directly observe the instability. This difficulty stems from the fact that an l number of x-ray sources is required to obtain a spatial resolution for a wave with the wave vector $k_\theta = l/R$ (see Refs. [26] and [27]). Furthermore, quantitative x-ray flash photography cannot resolve the void at the center of a spherical shell, due to quantum effects such as Compton scattering, pair production, and pair annihilation [27].

The end view of an imploding cylindrical shell is amenable to direct observation of the inner surface by optical diagnostics [28,29]. Moreover, the end view provides clean data devoid of effects due to ablation processes. Unfortunately, this does not give precise information about the density profile of a shell. By means of Abel's inversion [27], a side view could, in principle, provide the density profile. However, in the presence of ablation processes, the experiment of measuring soft-x-ray emission from the plasma shell would be a difficult task.

In summary, we have demonstrated that in the static limit the stability analysis gives the correct growth rate for the Rayleigh-Taylor instability. This analysis brings to light many similarities between the Rayleigh-Taylor instability for a stationary system and the corresponding calculations for a nonstationary system. The static limits provide insights into the reasons why the stability criteria for a nonstationary system are the same as for a stationary case, even though the two approaches are quite different.

We have shown, as simple illustrations, that the Rayleigh-Taylor instability does not occur for the modes $l = 0, 1$. This demonstrates that the fundamental characteristics of the instability remain the same in both stationary and nonstationary systems. It is intuitively clear then that the convergence effect and the time-dependent acceleration do not affect the nature of the Rayleigh-Taylor instability at any instant during the implosion process.

We would like to emphasize that the instability at the inner surface will occur for any imploding plasma shell that is adiabatically compressed by a slowly increasing external pressure, regardless of the geometry. Furthermore, we find it remarkable that the instability has been observed in the experiment with an aluminum plasma shell [28]; there is evidence for a critical γ value above which the instability does not occur and which should mark the beginning of full ionization of a metal [30]. The stability issue of fusion targets still remains of great interest, and it merits further experimental study on a cylindrical shell which would provide insight into implosion dynamics.

ACKNOWLEDGMENTS

I am grateful to F. H. Harlow for useful discussions and for his help in writing the code. I would also like to thank A. White for his help in solving a number of numerical problems. Finally, I am grateful to F. C. Perry at Sandia National Laboratory for useful discussions on his experiments of cylindrical shell implosions and for providing experimental data. This work was supported by the U.S. Department of Energy.

[1] R. E. Kidder, Nucl. Fusion **16**, 3 (1976).
[2] D. L. Book and I. B. Bernstein, J. Plasma Phys. **23**, 521 (1980).
[3] F. Hattori, H. Takabe, and K. Mima, Phys. Fluids **29**, 1719 (1986).
[4] G. I. Taylor, Proc. R. Soc. London Sec. A **201**, 192 (1950).
[5] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, London, 1961), p. 428.
[6] F. H. Harlow and J. E. Welch, Phys. Fluids **9**, 842 (1966).
[7] C. P. Verdon, R. L. McCrory, R. L. Morse, G. R. Baker, D. I. Meiron, and S. A. Orszag, Phys. Fluids **25**, 1653 (1982).
[8] I. Bernstein, E. Frieman, M. Kruskal, and R. Kulsrud, Proc. R. Soc. London Ser. A **244**, 17 (1958); E. Frieman and M. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960).
[9] D. L. Book, Phys. Rev. Lett. **41**, 1552 (1978).
[10] S. J. Han, Phys. Rev. A **37**, 207 (1988).
[11] S. J. Han, Los Alamos National Laboratory Report No. LA-UR-82-3083 (unpublished).
[12] D. L. Book and S. E. Bodner, Phys. Fluids **30**, 367 (1987).
[13] S. J. Han and B. R. Suydam, Phys. Rev. A **26**, 926 (1982).
[14] R. H. Levy, Phys. Fluids **8**, 1288 (1965).
[15] Steven Weinberg, J. Math. Phys. **8**, 614 (1967). We have carried out the stability analysis of a penetrating electron beam into plasmas with similar hydrodynamic calculations and obtained the same dispersion relations as Weinberg's orbital theory; S. J. Han (unpublished).
[16] L. I. Sedov, Rev. Mod. Phys. **30**, 1077 (1958).
[17] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952), Appendix B.
[18] I. B. Bernstein and D. L. Book, Astrophys. J. **225**, 633 (1978).
[19] *Handbook of Mathematical Functions,* edited by M. A. Abramowitz and I. A. Stegun, Natl. Bur. Stand. (U.S.) Appl. Ser. No. 55 (U.S. GPO, Washington, DC, 1972), Chap. 15.
[20] Steven Weinberg, Phys. Rev. **126**, 1899 (1962).
[21] See, for example, Zhen-Su She and Steven A. Orszag, Phys. Rev. Lett. **66**, 1701 (1991), and references therein; Robert H. Kraichnan, Phys. Rev. Lett. **65**, 575 (1990); D. C. Besnard and F. H. Harlow, Int. J. Multiphase Flow **14**, 679 (1988); D. Besnard, F. H. Harlow, and R. Rauenzahn, Los Alamos National Laboratory Report No. LA-10911-MS.
[22] S. J. Han, Phys. Fluids **25**, 1723 (1982).
[23] S. J. Han (unpublished).
[24] L. D. Landau and E. M. Lifshitz, *Statistical Physics,* 2nd revised English ed. (Pergamon New York, 1977), p. 122.
[25] Y. Nagayama, J. Appl. Phys. **62**, 2702 (1987).
[26] Y. Nagayama, S. Tsuji, and K. Kawahata, Phys. Rev. Lett. **61**, 1839 (1988).
[27] H. Hörman, Z. Phys. **97**, 539 (1935); S. J. Han, Los Alamos National Laboratory Report No. LA-UR-91-1966 (unpublished).
[28] F. C. Perry, in *Shock Waves in Condensed Matter–1981 (Menlo Park),* Proceedings of the Conference on Shock Waves in Condensed Matter, edited by W. J. Nellis, L. Seaman, and R. A. Graham, AIP Conf. Proc. No. **78** (AIP, New York, 1982), p. 639.
[29] B. Amini, Phys. Fluids **28**, 387 (1985).
[30] F. C. Perry (private communication).