Perturbed ladder-operator method: An algebraic recursive solution of the perturbed Coulomb eigenequation

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The perturbed ladder-operator method is applied to the solution of the Coulomb eigenequation perturbed by an anharmonic potential, i.e., with a total potential $U(x)=-m(m+1)/x^2-2q/x+b_1x+b_2x^2+b_3x^3+\cdots$. This method is an extension of the Schrödinger-Infeld-Hull factorization method within the perturbative scheme. The introduction of specific basis functions for the finite-difference solution of the factorazibility condition, together with the use of the symmetry properties of the Bernoulli polynomials, allows a straightforward determination of analytical expressions of the perturbed Coulomb ladder functions and eigenvalues. Some illustrative examples showing the capabilities of the method are given; particularly, analytical expressions of the linear, quadratic, and cubic Stark shifts are quickly derived.

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I. INTRODUCTION

The Schrödinger-Infeld-Hull factorization method [1,2] is known to be an elegant and powerful method for analytically solving some linear second-order differential equations of fundamental interest in quantum mechanics. Let us recall that, for instance, the hydrogenic radial functions as well as the Dirac hydrogenic functions, the spherical harmonic functions, the harmonic-oscillator and Morse-oscillator diatomic vibration functions, and more generally the confluent and Gauss hypergeometric functions are (or are amenable to) solutions of factorizable equations [2]. Particularly, when an eigenequation is factorizable, the expression of the eigenvalue in terms of the quantum numbers directly follows from knowledge of the factorization function, closed-form expressions of the eigenfunctions involving orthogonal polynomials are known [3], and closed-form expressions of many matrix elements are obtainable by means of an algebraic recursive procedure [4,5]. In fact, there are six fundamental types of exactly factorizable equations (noted types A to F, within the Infeld-Hull nomenclature). Nevertheless, in most cases, the mathematical description of physical phenomena leads to the solution of equations that are not exactly factorizable but can be viewed as "perturbed factorizable" equations: one can extract from the given physical model potential an unperturbed part leading to a factorizable equation. In this case the original range of applicability of the exact factorization method can be extended within the perturbation scheme [2,6–9].

In a recent paper [10] (hereafter referred to as paper I), an algebraic recursive procedure has been proposed for analytically solving wave equations that can be viewed as "perturbed factorizable" equations. The efficiency of this procedure mainly relies on the use of basis functions $y_s(x)$ that satisfy selective ladderlike properties for expanding the perturbation; it also relies on the use of Newton's expansions for the quantum-number dependence of the required perturbed ladder and factorization functions. In paper I types A to D "perturbed factorizations" have been considered in detail and several examples have been worked out. This procedure is valid for all factorization types and can also be applied to the solution of perturbed type-E or type-F eigenequations. However, as already pointed out in paper I, it is worthwhile to examine separately these cases and, possibly, when tackling the determination of the perturbed ladder and factorization functions, to take advantage of their expected symmetry properties in the quantum number.

In the present paper, special attention is paid to the solution of a wave equation with a Coulomb potential (factorizable type F) with a perturbation expanded in terms of basis functions $y_s(x)=x^s$. Analytical solutions of such an eigenequation are known to play a central role in atomic and molecular structure calculations, particularly when studying atoms in external fields, and also they can serve for the analytical solution of fundamental model problems such as the (static or exponential cosine) screened Kepler problem [6]. After a necessary and brief recall of the exact and perturbed factorization schemes, the solution of the factorizability condition is revisited in order to investigate the possible interest of introducing specific basis functions when carrying out the finite-difference solution of the factorizability condition (Sec. II). Focusing on type F and making use of the symmetry properties of the Bernoulli polynomials, analytical expressions of the perturbed ladder functions and eigenvalues are derived (Sec. III). Illustrative examples are worked out (Sec. IV).

II. PERTURBED LADDER-OPERATOR METHOD

In order to set up the definitions and notations, it is first necessary to briefly recall the main features of the exact and perturbed factorization schemes.

A. Exact factorization

After exact or approximate separation of variables, many problems of current interest in quantum mechanics lead to the solution of eigenequations of the Sturm-Liouville type. By an appropriate transformation of variable and function, these equations can be reduced to the standard form

$$\left[\frac{d^2}{dx^2}+U(x,m)+\Lambda_j\right]\Psi_{jm}(x)=0 \qquad (2.1)$$

associated with the boundary conditions ($x_1 \le x \le x_2$)

$$|\Psi(x_1)|^2=|\Psi(x_2)|^2=0, \quad \int_{x_1}^{x_2}|\Psi(x)|^2dx=1 , \qquad (2.2)$$

where $m=m_0+1, m_0+2, \ldots$ is a quantum number which takes successive discrete values labeling the eigenfunctions.

Such an equation (2.1) is factorizable when it can be replaced by each of the following two difference-differential equations:

$$\begin{aligned} H_{m+1}^{-}H_{m+1}^{+}\Psi_{jm}&=[\Lambda_j-L(m+1)]\Psi_{jm} , \\ H_m^{+}H_m^{-}\Psi_{jm}&=[\Lambda_j-L(m)]\Psi_{jm} , \end{aligned} \qquad (2.3)$$

where $L(m)$ is the factorization function, which does not depend on x, and $H_m^{\pm}$ are mutually adjoint ladder operators: $H_m^{\pm}=K(x,m)\mp d/dx$. Owing the mutual adjointness of the ladder operators H_m^{+} and H_m^{-}, the necessary condition for the existence of quadratically integrable solutions of Eq. (2.1), i.e., the quantization condition, is $\epsilon(j-m)=v=\text{integer}\ge 0$, where $\epsilon=+1$ (or $\epsilon=-1$) according to whether $L(m)$ is an increasing (or decreasing) function of m.

The interest in and advantages of the factorization method are well known [2].

(i) Closed-form expressions of the eigenvalues are readily obtainable from the knowledge of the factorization function $L(m)$,

$$\Lambda_j=L\left[j+\frac{\epsilon}{2}+\frac{1}{2}\right] . \qquad (2.4)$$

(ii) The normalized eigenfunctions are solutions to the following pair of difference-differential equations

$$\begin{aligned} \left[K(x,m)+\frac{d}{dx}\right]\Psi_{jm}&=\mathcal{N}_j(m)\Psi_{jm-1} , \\ \left[K(x,m+1)-\frac{d}{dx}\right]\Psi_{jm}&=\mathcal{N}_j(m+1)\Psi_{jm+1} , \end{aligned} \qquad (2.5)$$

with $\mathcal{N}_j(m)=[\Lambda_j-L(m)]^{1/2}$. These "ladder" equations allow the determination of any $\Psi_{jm}(x)$ function from the knowledge of any one of them, particularly from the knowledge of the normalized "key" function $\Psi_{jj}(x)$, which is a solution of the first-order differential equation

$$\left[K\left[x,j+\frac{\epsilon}{2}+\frac{1}{2}\right]-\epsilon\frac{d}{dx}\right]\Psi_{jj}=0 . \qquad (2.6)$$

From the comparison of Eqs. (2.3) and (2.1), it is easily shown that the necessary and sufficient condition to be satisfied by $K(x,m)$ and $L(m)$ allowing the factorization of Eq. (2.1) is

$$\begin{aligned} [K(x,m+1)]^2+\frac{d}{dx}K(x,m+1)+L(m+1)&=-U(x,m), \\ [K(x,m)]^2-\frac{d}{dx}K(x,m)+L(m)&=-U(x,m) . \end{aligned} \qquad (2.7)$$

There are six fundamental types of potential functions $U^{(0)}(x,m)$ (denoted types A to F, within the Infeld-Hull nomenclature) leading to factorizable equations. Moreover, as pointed out by Infeld and Hull [2], when direct factorization is not possible solely because of the inadequate m dependence of the potential function $U(x,m)$ under consideration, one can resort to "artificial" factorization, i.e., one can consider $U(x,m)$ as "embedded" in a new potential function $u(x,m;\mu)$, which depends on a supplementary "artificial" parameter μ such that $u(x,m;\mu)$ can be identified in m with a factorizing potential $U^{(0)}(x,m)$ and that $u(x,m;\mu=m)=U(x,m)$. Then, Eq. (2.1) is factorized using $u(x,m;\mu)$ and the eigenvalues $\Lambda_j(\mu)=L[j+(\epsilon/2)+\frac{1}{2};\mu]$ are determined as well as the eigenfunctions $\Psi_{jm}(x;\mu)$, both depending on the parameter μ. At the end of the ladder procedure (2.5), one merely sets $\mu=m$ and obtains the required solutions $\Lambda_j(m)=\Lambda_j(\mu=m)$ and $\Psi_{jm}(x)=\Psi_{jm}(x;\mu=m)$. This "artificial" or "embedded" factorization device is widely used all along the "perturbed factorization" scheme.

B. Perturbed factorization

Let us now consider an eigenequation (2.1) where the potential function $U(x,m)$ does not belong to any of the six Infeld-Hull factorization types, and let us assume that this potential function, as well as the associated ladder and factorization functions $K(x,m)$ and $L(m)$ to be found, can be expanded in a perturbation series with a parameter η,

$$\begin{aligned} U(x,m)&=U^{(0)}(x,m)+\eta U^{(1)}(x,m)+\eta^2U^{(2)}(x,m)+\cdots , \\ K(x,m)&=K^{(0)}(x,m)+\eta K^{(1)}(x,m) \\ &\quad+\eta^2K^{(2)}(x,m)+\cdots , \\ L(m)&=L^{(0)}(m)+\eta L^{(1)}(m)+\eta^2L^{(2)}(m)+\cdots , \end{aligned} \qquad (2.8)$$

where $K^{(0)}(x,m)$ and $L^{(0)}(m)$ are the ladder and factorization functions allowing an exact factorization of Eq. (2.1) with $U^{(0)}(x,m)$.

In order to satisfy the factorizability condition (2.7) with $U(x,m)$ at any order N of the perturbation, the required perturbed potential ladder and factorization functions have to be solutions of the following differential-difference equations

$$\begin{aligned} \sum_{\nu=0}^{N}K^{(\nu)}(x,m+1)K^{(N-\nu)}(x,m+1)+\frac{d}{dx}K^{(N)}(x,m+1) \\ +L^{(N)}(m+1)=-U^{(N)}(x,m) , \\ \sum_{\nu=0}^{N}K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)-\frac{d}{dx}K^{(N)}(x,m)+L^{(N)}(m) \\ =-U^{(N)}(x,m) , \end{aligned} \qquad (2.9)$$

These equations will be solved recursively; i.e., when considering the determination of $K^{(N)}(x,m)$ and $U^{(N)}(x,m)$, it is assumed that all the $K^{(\nu)}(x,m)$, for $\nu=1,2,\ldots,N-1$, have already been found. Their finite-difference aspect determines the m dependence of the functions while their differential aspect determines their x dependence.

In the same way as in paper I, after choosing suitable expansion basis functions $Y_s(x)$ and $y_s(x)$ for the perturbed ladder and factorization functions $K^{(N)}(x,m)$ and $U^{(N)}(x,m)$, the solution of Eqs. (2.9) will be worked out by means of finite-difference calculus. It is then convenient to write again Eqs. (2.9),

$$2\Delta[K^{(0)}(x,m)K^{(N)}(x,m)]+\frac{d}{dx}[K^{(N)}(x,m+1)+K^{(N)}(x,m)]=-\Delta L^{(N)}(m)-\Delta\sum_{\nu=1}^{N-1}K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)\ , \quad (2.10)$$

$$U^{(N)}(x,m)=\left[\frac{d}{dx}-2K^{(0)}(x,m)\right]K^{(N)}(x,m)-L^{(N)}(m)-\sum_{\nu=1}^{N-1}K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)\ , \quad (2.11)$$

where $\Delta F(m)=F(m+1)-F(m)$ is the usual first difference Δ operator in m.

Equation (2.10) is used to determine the perturbed ladder and factorization functions $K^{(N)}(x,m)$ and $L^{(N)}(m)$. Once they are known, the associated potential function $U^{(N)}(x,m)$ is given by Eq. (2.11) and finally, one obtains the total required "factorizing" potential function $U(x,m)$ of eigenequation (2.1) up to the Nth order of the perturbation. Thus, one can solve physico-mathematical problems with a potential function $V(x,m)$, such as

$$V(x,m)=U^{(0)}(x,m)+\eta V^{(1)}(x)+\eta^2 V^{(2)}(x)+\cdots\ , \quad (2.12)$$

where the $V^{(\nu)}(x)$ have the same dependence on x as the $U^{(\nu)}(x,m)$ and, in most cases, do not depend on m [11].

In order to match $V(x,m)$ with the factorizing potential $U(x,m)$, one has to resort to the "artificial" factorization process. The following condition must hold, for any value of x:

$$V^{(\nu)}(x)=U^{(\nu)}(x;m=\mu)\ . \quad (2.13)$$

Finally, one can factorize an eigenequation (2.1) with a given potential function $V(x,m)$ by determining the associated perturbed ladder and factorization functions, which are solutions of the difference-differential equation (2.10) and, as a consequence of Eqs. (2.11) and (2.13), which satisfy the following condition:

$$\left[\frac{d}{dx}-2K^{(0)}(x,\mu)\right]K^{(N)}(x,\mu)-L^{(N)}(\mu)=V^{(N)}(x)+\sum_{\nu=1}^{N-1}K^{(\nu)}(x,\mu)K^{(N-\nu)}(x,\mu)\ . \quad (2.14)$$

Once the perturbed ladder and factorization functions $K^{(\nu)}(x,m;\mu)$ and $L^{(\nu)}(m;\mu)$, both depending on the artificial parameter μ, have been found, the perturbed problem (up to the Nth order) may be handled in the same way as the exact factorizable (unperturbed) problem.

(i) The total perturbed eigenvalue and associated ladder function are [see Eqs. (2.4) and (2.8)]

$$\Lambda_j(m)=L^{(0)}\left[j+\frac{\epsilon}{2}+\frac{1}{2}\right]+\sum_{\nu=1}^{N}\eta^\nu L^{(\nu)}\left[m=j+\frac{\epsilon}{2}+\frac{1}{2};\mu=m\right]\ , \quad (2.15)$$

$$K(x,m;\mu)=K^{(0)}(x,m)+\sum_{\nu=1}^{N}\eta^\nu K^{(\nu)}(x,m;\mu)\ ,$$

where $\epsilon=+1$ (or $\epsilon=-1$) according to whether the unperturbed factorization function $L^{(0)}(m)$ is an increasing (or decreasing) function of m.

(ii) The ladder equations (2.5) and Eq. (2.6) hold with $K(x,m;\mu)$ for the determination of the perturbed eigenfunctions $\Psi_{jm}(x;\mu)$. Once the ladder process is achieved, one sets $\mu=m$ and obtains the required $\Psi_{jm}(x;m)$ perturbed eigenfunctions. One can also use an alternative procedure that provides the perturbed eigenfunctions as linear combinations of the unperturbed eigenfunctions [8,9].

Let us remark that, when handling the Nth order of the perturbation instead of handling the first-order $N=1$, the main difference is that one has to carry the term $\mathcal{W}^{(N)}(x,m;\mu)=\sum_{\nu=1}^{N-1}K^{(\nu)}K^{(N-\nu)}$, which both in Eqs. (2.11) and (2.14) plays the role of an additional perturbing potential.

C. Finite-difference solution of the "perturbed" factorizability condition

Let us first consider the x dependence of the difference-differential equation (2.10) and assume that, at reach order N of the perturbation, the perturbed ladder function can be written

$$K^{(N)}(x,m)=\sum_{s=0}^{S_N}\gamma_s^{(N)}(m)Y_s(x)\ . \quad (2.16)$$

As pointed out in paper I, the $Y_s(x)$ basis set, specific to each factorization type, is to be chosen so that all terms appearing in Eq. (2.10) can be expanded on a common basis set $y_s(x)$. Namely, it is assumed that one can find suitable associated basis sets $Y_s(x)$ and $y_s(x)$ such that

$$2K^{(0)}(x,m)Y_s(x)=A_s(m)y_s(x)+B_s(m)y_{s+1}(x)\ ,$$
$$\frac{dY_s}{dx}=\alpha_s y_s(x)+\beta_s y_{s+1}(x)\ , \tag{2.17}$$

$$Y_s(x)Y_t(x)=\sum_r h(s,t,r)y_r(x)\ , \tag{2.18}$$

and that, as a consequence of this last equation, one can set

$$\sum_{\nu=1}^{N-1} K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)=\sum_s W_s^{(N)}(m)y_s(x)\ , \tag{2.19}$$

where the $W_s^{(N)}(m)$ functions originate from the preceding orders of the perturbation.

When conditions (2.17)–(2.19) have been fulfilled, after substituting for $K^{(N)}$ and $\sum_{\nu=1}^{N-1}K^{(\nu)}K^{(N-\nu)}$ from Eqs. (2.16) and (2.19) into Eq. (2.10), and by equating the coefficients of $y_s(x)$ in both sides one obtains the following finite-difference equations, allowing the determination of the $\gamma_s^{(N)}(m)$ and $L^{(N)}(m)$ functions,

$$[B_{s-1}(m+1)+\beta_{s-1}]\gamma_{s-1}^{(N)}(m+1)-[B_{s-1}(m)-\beta_{s-1}]\gamma_{s-1}^{(N)}(m)$$
$$=-[A_s(m+1)+\alpha_s]\gamma_s^{(N)}(m+1)+[A_s(m)-\alpha_s]\gamma_s^{(N)}(m)-\Delta W_s^{(N)}(m)\ , \tag{2.20}$$

$$\Delta L^{(N)}(m)=-[A_0(m+1)+\alpha_0]\gamma_0^{(N)}(m+1)+[A_0(m)-\alpha_0]\gamma_0^{(N)}(m)-\Delta W_0^{(N)}(m)\ . \tag{2.21}$$

The finite-difference equations (2.20) can be solved recursively, the integer s descending stepwise from $s=S_N+1$ down to 1. For each value of s, one obtains the general solution (see paper I)

$$\gamma_s^{(N)}(m)=Q_s(m)[k_s^{(N)}+F_s^{(N)}(m)]\ , \tag{2.22}$$

where $k_s^{(N)}$ is an arbitrary summation constant and

$$Q_s(m)=\prod_{j=1}^{m-1}\frac{[B_s(j)-\beta_s]}{[B_s(j+1)+\beta_s]}\ , \tag{2.23}$$

$$F_s^{(N)}(m)=\Delta^{-1}[R_{s+1}^{(N)}(m)/Q_s(m+1)]\ , \tag{2.24}$$

$$R_s^{(N)}(m)=\frac{-[A_s(m+1)+\alpha_s]\gamma_s^{(N)}(m+1)+[A_s(m)-\alpha_s]\gamma_s^{(N)}(m)-\Delta W_s^{(N)}(m)}{B_{s-1}(m+1)+\beta_{s-1}}\ .$$

Once the $F_s^{(N)}(m)$ functions have been obtained, the required perturbed ladder function $K^{(N)}(x,m)$ is completely defined by Eqs. (2.16) and (2.22). The associated factorizing perturbed potential $U^{(N)}(x,m)$ is given by Eq. (2.11) and, as well as $K^{(N)}(x,m)$, it depends on the arbitrary constants $k_u^{(N)}$.

Let us now consider the perturbed factorization of eigenequation (2.1) with a given $V(x,m)$ physical model potential (2.12). One has to determine the expressions of the $k_u^{(N)}$ constants in terms of the data specific to that problem by matching $V^{(N)}(x)$ with the factorizing perturbation $U^{(N)}(x,m;k_u^{(N)})$. From expression (2.11), it is easily seen that $U^{(N)}(x,m;k_u^{(N)})$ can be written as a finite expansion on the $y_s(x)$ basis set. Consequently, in order to match $V^{(N)}(x)$ with $U^{(N)}(x,m;k_u^{(N)})$, one has first to expand the $V^{(N)}(x)$ on the $y_s(x)$ basis and to set

$$V^{(N)}(x)=\sum_{s=1}^{S_N+1} b_s^{(N)}y_s(x)\ , \tag{2.25}$$

where the $b_s^{(N)}$ constants are specific to the physical model potential under consideration.

Hence, using the artificial factorization device, the determination of the $K^{(N)}(x,m;\mu;b_i^{(\nu)})$ ladder function associated with $V^{(N)}(x)$ amounts to the determination of the $K^{(N)}(x,m;k_u^{(N)})$ function satisfying condition (2.14). Introducing the expressions (2.16)–(2.19) and (2.25) of the perturbed ladder and potential functions into the condition (2.14) and equation the coefficients of $y_s(x)$ in both sides, one obtains the following relations to be satisfied by the $\gamma_s^{(N)}(m)$ and $L^{(N)}(m)$ functions:

$$[A_s(\mu)-\alpha_s]\gamma_s^{(N)}(\mu)+[B_{s-1}(\mu)-\beta_{s-1}]\gamma_{s-1}^{(N)}(\mu)$$
$$=-b_s^{(N)}-W_s^{(N)}(\mu)\ , \tag{2.26}$$

$$L^{(N)}(\mu)=-[A_0(\mu)-\alpha_0]\gamma_0^{(N)}(\mu)-W_0^{(N)}(\mu)\ . \tag{2.27}$$

As pointed out in paper I, since the $F_s(m)$ functions involved in the expression of $\gamma_s^{(N)}(m)$ are defined within an additive arbitrary summation constant, one can impose the vanishing conditions $F_s^{(N)}(m=\mu)=0$. As a consequence, $\gamma_s^{(N)}(\mu)=Q_s(\mu)k_s^{(N)}$ [see Eq. (2.22)], and by means of the two-term recursive equations (2.26), one obtains the following closed-form expressions of the arbitrary constants $k_s^{(N)}$ in terms of μ and of the $b_u^{(N)}$ expansion coefficients of the $V^{(N)}(x)$ potential (see paper I),

$$k_s^{(N)}=\sum_{u=s+1}^{S_N+1} C_{us}(\mu)[b_u^{(N)}+W_u^{(N)}(\mu)]\ , \tag{2.28}$$

where

$$C_{us}(\mu)=-\prod_{t=s+1}^{u-1}[A_t(\mu)-\alpha_t]\Big/Q_s(\mu)\prod_{t=s}^{u-1}[B_t(\mu)-\beta_t]\ .$$

Briefly stated, for each factorization type and suitable basis sets $[Y_s(x),y_s(x)]$, satisfying the ladderlike conditions (2.17), the required expression of the ladder function $K^{(N)}(x,m;\mu;b_i^{(N)})$, associated with a given perturbation $V^{(N)}(x)$, is completely defined by means of Eqs. (2.16), (2.22), and (2.28): its determination amounts to the solution of the following set of finite-difference equations [see Eq. (2.24) after some rearrangements]:

$$\Delta F_{s-1}(N)=-\frac{[\Delta(A_sQ_s)+2\alpha_sQ_s(m)+\alpha_s\Delta Q_s][k_s^{(N)}+F_s^{(N)}(m)]+[A_s(m+1)+\alpha_s]Q_s(m+1)\Delta F_s^{(N)}+\Delta W_s^{(N)}(m)}{Q_{s-1}(m+1)[B_{s-1}(m+1)+\beta_{s-1}]}\ , \tag{2.29}$$

with the associated vanishing condition $F_s^{(N)}(m=\mu)=0$.

These equations will be solved recursively, starting from $s=S_N$ down to $s=1$. Hence, the associated factorization function $L^{(N)}(m;\mu;b_i^{(N)})$ is the solution of the finite-difference equation (2.21) satisfying condition (2.27), and a closed-form expression of the perturbed eigenvalue associated with each given perturbation $V^{(N)}(x)$ is $\Lambda_j^{(N)}(m)=L^{(N)}(m=j;\mu=m;b_i^{(N)})$. Finally, the total eigenvalue $\Lambda_j(m)$ associated with the given physical model potential $V(x,m)$ directly follows from Eq. (2.15). Let us now apply these general results to the solution of the generalized central-field eigenequation.

III. PERTURBED TYPE-F FACTORIZATION

Let us consider the following eigenequation $(0\le x<\infty)$:

$$\left[\frac{d^2}{dx^2}-\frac{m(m+1)}{x^2}-\frac{2q}{x}+V(x)+\Lambda\right]\Psi_{jm}(x)=0\ , \tag{3.1}$$

where $V(x)=\eta V^{(1)}(x)+\eta^2V^{(2)}(x)+\cdots$ is a perturbation.

A. Exact type-F factorization

When $V(x)=0$, this eigenequation (3.1) reduces to an exact Infeld-Hull type-F factorizable equation with the following factorizing potential, ladder, and factorization functions [2]:

$$U^{(0)}(x,m)=-\frac{m(m+1)}{x^2}-\frac{2q}{x}\ ,$$
$$K^{(0)}(x,m)=\frac{m}{x}+\frac{q}{m}\ . \tag{3.2}$$
$$L^{(0)}(m)=-\frac{q^2}{m^2}\ .$$

Since $L^{(0)}(m)$ is an increasing function of m, the class parameter is $\epsilon=+1$, and the unperturbed eigenvalue and quantization condition are

$$\Lambda_j^{(0)}=-\frac{q^2}{(j+1)^2},\quad j-m=v \tag{3.3}$$

where v is a non-negative integer.

The unperturbed eigenfunctions are [3]

$$\Psi_{jm}^{(0)}(x)=N_{jm}x^{m+1}\exp\left[\frac{qx}{j+1}\right]L_v^{2m+1}\left[-\frac{2qx}{j+1}\right]\ , \tag{3.4}$$

where $L_v^{2m+1}(\ )$ is an associated Laguerre polynomial of degree v and N_{jm} is a normalization constant.

As expected, when setting $m=1$, $j=n-1$, $q=-Z$, $R_{nl}(r)=(1/r)\Psi_{n-1,l}(r)$, $E=\frac{1}{2}\Lambda$, one finds again the familiar hydrogenic results.

B. Determination of the perturbed ladder function

In order to apply the perturbed factorization device, let us choose the associated basis functions $Y_s(x)=x^{s+1}$, $y_s(x)=x^s$ and set

$$K^{(N)}(x,m)=\sum_{s=0}^{S_N}\gamma_s^{(N)}(m)x^{s+1}\ , \tag{3.5}$$

$$V^{(N)}(x)=\sum_{s=1}^{S_N+1}b_s^{(N)}x^s\ . \tag{3.6}$$

It is easily checked that the "ladderlike" properties (2.17)–(2.19) are fulfilled with the following expressions:

$$A_s(m)=2m,\quad B_s(m)=\frac{2q}{m},\quad \alpha_s=s+1,\quad \beta_s=0\ , \tag{3.7}$$

$$\sum_{\nu=1}^{N-1}K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)=\sum_{s=2}^{S_N}W_s^{(N)}(m)x^s\ , \tag{3.8}$$

where

$$W_s^{(N)}(m)=\sum_{\nu=1}^{N-1}\sum_{t=0}^{s-2}\gamma_t^{(\nu)}(m)\gamma_{s-2-t}^{(N-\nu)}(m)\ .$$

At the first order $N=1$ of the perturbation, $W_s^{(1)}(m)=0$ and the upper bound S_1 that is involved in $K^{(1)}(x,m)$ can be arbitrarily chosen. At the higher orders $N>1$, the highest power of x is already fixed as data following from the preceding orders, and the relation $S_N=S_\nu+S_{N-\nu}+2$ must hold for $\nu=1$ to $N-1$: the value of S_N depends on S_1 and N. One finds the necessary condition to be fulfilled,

$$S_N=NS_1+2(N-1)\ . \tag{3.9}$$

Using Eqs. (2.23) and (2.22), one gets

$$Q_s(m)=m \ , \tag{3.10}$$

$$\gamma_s^{(N)}(m)=m\,[k_s^{(N)}+F_s^{(N)}(m)] \ , \tag{3.11}$$

and, in terms of the artificial parameter μ and of the $b_s^{(N)}$ expansion coefficients of $V^{(N)}(x)$ we have [see Eq. (2.28)]

$$k_s^{(N)}=\sum_{u=s+1}^{S_N+1}\left[-\frac{1}{2q}\right]^{u-s}\mu^{u-s-1}(u-2\mu)_{u-s-1}\times[b_u^{(N)}+W_u^{(N)}(\mu)] \ , \tag{3.12}$$

where $(m)_u=m(m-1)\cdots(m-u+1)$ is a generalized factorial [12].

Finally, the determination of the required $\gamma_s^{(N)}(m)$ functions amounts to the solution of the following set of finite-difference equations [see Eq. (2.29)]:

$$2q\Delta F_{s-1}^{(N)}=-(2m+s+3)(m+1)\Delta F_s^{(N)} -(2m+1)(s+3)[k_s^{(N)}+F_s^{(N)}(m)]-\Delta W_s^{(N)} \tag{3.13}$$

These equations hold at any order N of the perturbation.

Before tackling the determination of the $F_s^{(N)}(m)$ functions, let us recall that the required perturbed ladder and factorization functions are expected to satisfy the following m-parity relationships [6,13]:

$$K^{(N)}(x,-m)=-K^{(N)}(x,m) \ , \quad L^{(N)}(-m)=L^{(N)}(m) \ . \tag{3.14}$$

As a consequence, the required $F_s^{(N)}(m)$ functions have to be even functions of m [see Eq. (3.11)]. Since, also the vanishing condition $F_s^{(N)}(m=\mu)=0$ must be fulfilled, one sets

$$F_s^{(N)}(m)=Z_s^{(N)}(m)-Z_s^{(N)}(\mu) \ , \tag{3.15}$$

$$Z_s^{(N)}(m)=\sum_{k=1}^{S_N-s}C_s^{(N)}(k)m^{2k} \ . \tag{3.16}$$

Hence, when introducing the unified notation $k_s^{(N)}-Z_s^{(N)}(\mu)=C_s^{(N)}(0)$, the required perturbed ladder function $K^{(N)}(x,m)$ is [see Eqs. (3.5) and (3.11)]

$$K^{(N)}(x,m)=\sum_{s=0}^{S_N}x^{s+1}\sum_{k=0}^{S_N-s}C_s^{(N)}(k)m^{2k+1} \ . \tag{3.17}$$

Let us now consider the determination of the $C_s^{(N)}(k)$ coefficients. From Eq. (3.13), after some rearrangements, one obtains (see Appendix A)

$$\begin{aligned}2qZ_{s-1}^{(N)}(m)=&-(s+3)[k_s^{(N)}-Z_s^{(N)}(m)]m^2\\&-2m^2Z_s^{(N)}(m)\\&-(s+1)[mZ_s^{(N)}(m)+2\Delta^{-1}mZ_s^{(N)}(m)]\\&-W_s^{(N)}(m) \ ,\end{aligned} \tag{3.18}$$

where, as a consequence of Eqs. (3.8) and (3.17), the increments $W_s^{(N)}(m)$ that originated from the preceding orders of the perturbation can be written

$$W_s^{(N)}(m)=\sum_{k=1}^{S_N+1-s}w_s^{(N)}(k)m^{2k} \ , \tag{3.19}$$

with

$$w_s^{(N)}(k)=\sum_{\nu=1}^{N-1}\sum_{t=0}^{s-2}\sum_{u=0}^{s-2}C_t^{(\nu)}(u)C_{s-2-t}^{(N-\nu)}(k-1-u) \ .$$

Relation (3.18) will serve to determine the $C_{s-1}^{(N)}(k)$ coefficients in terms of the $C_s^{(N)}(t)$ coefficients.

Indeed, let us recall that (see Appendix B)

$$\Delta^{-1}m^k=k!\varphi_{k+1}(m)+\mathcal{C}_k \ , \tag{3.20}$$

where $\mathcal{C}_k$ is an arbitrary summation constant and $\varphi_k(m)$ is a Bernoulli polynomial of degree k,

$$\varphi_k(m)=\frac{1}{k!}\sum_{u=0}^{k}\begin{bmatrix}k\\u\end{bmatrix}B_u m^{k-u} \ ,$$

and remind the reader that except for $B_1=-\frac{1}{2}$, all the Bernoulli numbers B_u with odd subscript u are zero. Then, when the additive arbitrary summation constant $\mathcal{C}_k$ is conveniently chosen so that definition (3.16) holds for $Z_{s-1}(m)$, one can write

$$\begin{aligned}m^{2k+1}+2\Delta^{-1}m^{2k+1}&=m^{2k+1}+2(2k+1)!\varphi_{2k+2}(m)-\frac{1}{k+1}B_{2k+2}\\&=\frac{1}{k+1}\sum_{t=1}^{k+1}\begin{bmatrix}2k+2\\2t\end{bmatrix}B_{2(k+1-t)}m^{2t} \ .\end{aligned} \tag{3.21}$$

Consequently, from Eqs. (3.18) and (3.16), one obtains, after some rearrangements,

$$C_{s-1}^{(N)}(1)=-\frac{1}{2q}\left[w_s^{(N)}(1)+(s+3)[k_s^{(N)}-Z_s^{(N)}(\mu)]+(s+1)\sum_{u=1}^{S_N-s}a_{u1}C_s^{(N)}(u)\right] \tag{3.22}$$

and for $k\geq 2$

$$C_{s-1}^{(N)}(k)=-\frac{1}{2q}\left[w_s^{(N)}(k)+\frac{s+2k+1}{k}C_s^{(N)}(k-1)+(s+1)\sum_{u=k}^{S_N-s}a_{u1}C_s^{(N)}(u)\right] , \tag{3.23}$$

where the a_{uk} coefficients are the following "modified" Bernoulli numbers with even index:

$$a_{uk}=\frac{1}{u+1}\begin{bmatrix}2u+2\\2k\end{bmatrix}B_{2(u+1-k)}, \quad u\geq k\geq 1 \ . \tag{3.24}$$

Starting from $s=S_N$ down to $s=1$, these relations allow a recursive determination of the $C_s^{(N)}(k)$ $(k=1,S_N-s)$, in terms of the quantities $[k_s^{(N)}-Z_s^{(N)}(\mu)]=C_s^{(N)}(0)$, i.e., via the expression (3.12) of $k_s^{(N)}$, in terms of μ and of the expansion coefficients

$b_u^{(N)}$ of the given perturbation $V^{(N)}(x)$ [see Eq. (3.6)]. However, it is rewarding to work out alternative and more efficient recursive relations, allowing a straightforward determination of the $C_s^{(N)}(k)$ in terms of μ and of the $b_u^{(N)}$.

Since the eigenequation (3.1) depends on m via the product $m(m+1)$, and since, within the artificial factorization scheme, the m dependence of the final results arises when setting $\mu=m$, it is expected that, when expressed in terms of μ and of the $b_u^{(N)}$, the required $C_s^{(N)}(k)$ coefficients will depend on μ but only via the product $\lambda=\mu(\mu+1)$. In order to put in evidence this expected λ dependence of the $C_s^{(N)}(k)$ coefficients, let us remember that Bernoulli polynomials of even order satisfy the symmetry property $\varphi_{2n}(m)=\varphi_{2n}(1-m)$ and, therefore, can also be viewed as polynomials of degree n of the product $m(m-1)$ (see Appendix C). After noting that $\varphi_{2n}(m+1)=\varphi_{2n}(m)+\Delta\varphi_{2n}=\varphi_{2n}(m)+m^{2n-1}/(2n-1)!$, the expression (3.21) can be alternatively written

$$\frac{1}{k+1}\sum_{t=1}^{k+1}\begin{bmatrix}2k+2\\2t\end{bmatrix}B_{2(k+1-t)}\mu^{2t} =2(2k+1)!\varphi_{2k+2}(\mu+1)-\mu^{2k+1}-\frac{1}{k+1}B_{2k+2}\ . \qquad (3.25)$$

Hence, using the expression (3.18) of $Z_{s-1}^{(N)}(m)$, one gets

$$2qZ_{s-1}^{(N)}(\mu)=-(s+3)\mu^2k_s^{(N)}+(s+1)\mu^2Z_s^{(N)}(\mu)+(s+1)\mu Z_s^{(N)}(\mu)-(s+1)\sum_{k=1}^{S_N-s}C_s^{(N)}(k)\mathcal{P}_{k+1}(\lambda)-W_s^{(N)}(\mu)\ , \qquad (3.26)$$

where

$$\mathcal{P}_k(\lambda)=2(2k-1)!\varphi_{2k}(\mu+1)-\frac{1}{k}B_{2k}\ . \qquad (3.27)$$

On the other hand, using the closed-form expression (3.12) of the $k_s^{(N)}$ constants, one can write

$$k_{s-1}^{(N)}=-\frac{1}{2q}[b_s^{(N)}+W_s^{(N)}(\mu)+\mu(2\mu-s-1)k_s^{(N)}]\ . \qquad (3.28)$$

Then, collecting together the above expressions of $Z_{s-1}^{(N)}(\mu)$ and of $k_{s-1}^{(N)}$, and keeping in mind that $\lambda=\mu(\mu+1)=\mathcal{P}_1(\lambda)$, one obtains, in addition to the recurrence relations (3.22) and (3.23), the required expression of the $C_{s-1}^{(N)}(0)=k_{s-1}^{(N)}-Z_{s-1}^{(N)}(\mu)$ coefficients in terms of the $C_s^{(N)}(k)$:

$$C_{s-1}^{(N)}(0)=-\frac{1}{2q}\left[b_s^{(N)}-(s+1)\sum_{t=0}^{S_N-s}C_s^{(N)}(t)\mathcal{P}_{t+1}(\lambda)\right]\ . \qquad (3.29)$$

Now, let us remark that the expansion coefficients $w_s^{(N)}(k)$ of the increments $W_s^{(N)}(m)$ that originated from the preceding orders of the perturbation play a role quite comparable with the expansion coefficients $b_s^{(N)}$ of the perturbation $V^{(N)}(x)$. Thus, let us introduce the unified notation $b_s^{(N)}=b_s^{(N)}(0)$, $w_s^{(N)}(k)=b_s^{(N)}(k)$, and $a_{t0}=-\mathcal{P}_{t+1}(\lambda)$. Relations (3.22), (3.23), and (3.29) reduce to the unique recurrence formula

$$C_{s-1}^{(N)}(k)=-\frac{1}{2q}\left[b_s^{(N)}(k)+\frac{s+2k+1}{k}C_s^{(N)}(k-1)+(s+1)\sum_{t=k}^{S_N-s}a_{tk}C_s^{(N)}(t)\right]\ . \qquad (3.30)$$

Using this recurrence formula successively for $s=S_N+1,S_N,S_N-1,\ldots$, it is easily inferred that one can write (see Appendix D)

$$C_{S_N-\sigma}^{(N)}(k)=\sum_{j=0}^{\max(k,\sigma-1)}\sum_{u=j}^{u_m}\left[-\frac{1}{2q}\right]^{\sigma+1-u}d_j(\sigma-u,k,S_N-u)b_{S_N+1-u}^{(N)}(j)\ , \qquad (3.31)$$

where $u_m=\sigma-|j-k|$ and the $d_j(\sigma,k,s)$ satisfy the following recurrence formula:

$$d_j(\sigma+1,k,s)=\frac{s-\sigma+2k+1}{k}d_j(\sigma,k-1,s)+(s-\sigma+1)\sum_{t=k}^{\sigma+j}a_{tk}d_j(\sigma,t,s)\ . \qquad (3.32)$$

Let us emphasize that the $d_j(\sigma,k,s)$ do not depend on the order of the perturbation and on the particular perturbed type-F problem under consideration. Therefore, using the starting value $d_j(0,j,s)=1$ and $d_j(0,k,s)=0$ for $k<j$, closed-form expressions of the $d_j(\sigma,k,s)$ are easily obtainable, once and for all, by means of the recurrence formula (3.32) (see Appendix D).

C. Determination of the perturbed eigenvalues

As pointed out in Sec. II, the perturbed factorization functions $L^{(N)}(m;\mu;b_i^{(N)})$ associated with a given perturbation $V^{(N)}(x)$, are the solutions of the finite-difference equation (2.21), which satisfy condition (2.27). Using the above expressions of $A_s(m)$, α_s, and $\gamma_s(m)$ [see Eqs. (3.7), (3.11), (3.15), and (3.16)], one gets

$$\Delta L^{(N)}=-(m+1)(2m+3)[k_0^{(N)}+F_0^{(N)}(m+1)]+m(2m-1)[k_0^{(N)}+F_0^{(N)}(m)]\ , \qquad (3.33)$$

with the associated condition to be fulfilled,

$$L^{(N)}(m=\mu)=-\mu(2\mu-1)k_0^{(N)} . \tag{3.34}$$

Using again expressions (3.21) and (3.25) together with some rearrangements and setting $\lambda=\mu(\mu+1)$, one obtains the following expression (see Appendix E):

$$L^{(N)}(m;\mu;b_i^{(N)})=\sum_{k=0}^{S_N} C_0^{(N)}(k)T_{k+1}(m,\lambda) , \tag{3.35}$$

where

$$T_{k+1}(m,\lambda)=\mathcal{P}_{k+1}(\lambda)-\frac{2k+3}{k+1}m^{2k+2}-\sum_{t=1}^{k} a_{kt}m^{2t} . \tag{3.36}$$

Hence, using the artificial factorization scheme, the expression of the perturbed type-F eigenvalue associated with $V^{(N)}(x)$ is $\Lambda_j^{(N)}(m)=L^{(N)}(m=j+1;\mu=m;b_i^{(N)})$, i.e.,

$$\Lambda_j^{(N)}(m)=\sum_{k=0}^{S_N} C_0^{(N)}(k)T_{k+1}(j+1,m(m+1)) . \tag{3.37}$$

Now, introducing in Eq. (3.37) the expression (3.31) of the $C_{S_N-\sigma}^{(N)}(k)$ and making some rearrangements, one gets

$$\Lambda_j^{(N)}(m)=\sum_{u=0}^{S_N}\sum_{t=0}^{S_N-u} b_{t+1}^{(N)}(u)\mathcal{T}_{t+1}(u) , \tag{3.38}$$

where

$$\mathcal{T}_{t+1}(u)=\left[-\frac{1}{2q}\right]^{t+1}\sum_{k=0}^{t+1} d_u(t,k,t)\times T_{k+1}(j+1,m(m+1)) . \tag{3.39}$$

Since closed-form expressions of the $T_{k+1}(n,\lambda)$ polynomials are known (see Appendix C), closed-form expressions of the $\mathcal{T}_{t+1}(u)$, in terms of $j+1$ and $m(m+1)$ can be made available, once and for all, up to any required values of u and t. Finally, at each order N of the perturbation, the determination of an analytical expression of the perturbed type-F eigenvalue $\Lambda_j^{(N)}(m)$, associated with the perturbation $V^{(N)}(x)$, simply amounts to the computation of the $b_t^{(N)}(u\neq 0)$ coefficients. In fact, at each order N of the perturbation, each $b_t^{(N)}(u)$ is just the coefficient of $x^t m^{2u}$ in the expansion of the additional perturbative pseudo-potential

$$\mathcal{W}^{(N)}(x,m;\mu)=\sum_{\nu=1}^{N-1} K^{(\nu)}K^{(N-\nu)}$$

generated from the preceding orders of the perturbation [see Eqs. (3.8) and (3.19), and keep in mind that we have set $b_t^{(N)}(u)=w_t^{(N)}(u)$].

Let us note that, as a by-product of the method, one obtains a closed-form expression of the diagonal integral $\langle jm|x^s|jm\rangle$ between the unperturbed (type-F) eigenfunctions $\Psi_{jm}^{(0)}(x)$ [see Eq. (3.4)]. Indeed, when comparing the first-order ($N=1$) expression of $\Lambda_j^{(1)}(m)$ with its alternative expression within the classical Rayleigh-Schrödinger framework, it follows that the coefficient of $b_{S_N+1}^{(1)}(0)$ is merely the integral $\langle jm|x^{S_N+1}|jm\rangle$. Consequently, we have $\langle jm|x^s|jm\rangle=-\mathcal{T}_s(0)$, i.e.,

$$\langle jm|x^s|jm\rangle=-\left[-\frac{1}{2q}\right]^{s}\sum_{k=0}^{s-1} d_0(s-1,k,s-1)\times T_{k+1}(j+1,m(m+1)) . \tag{3.40}$$

Particularly, setting $j+1=n$, $m=1$, and $q=-Z$ in this expression, one gets the following closed-form expression of the hydrogenic radial integrals:

$$\langle nl|r^s|nl\rangle=-\left[\frac{1}{2Z}\right]^{s}\sum_{k=0}^{s-1} d_0(s-1,k,s-1)T_{k+1}(n,\lambda) \tag{3.41}$$

where $\lambda=l(l+1)$, the $T_{k+1}(n,\lambda)$ polynomials are given by Eq. (3.36), and the $d_0(s,k,s)$ coefficients obey the recurrence formula (3.32). For instance, picking up, in Appendix D and C, the required expressions of $d_0(s,k,s)$ and $T_k(n,\lambda)$, one finds again the well-known expressions of the hydrogenic radial integrals [14]

$$\begin{aligned}\langle nl|r|nl\rangle&=\frac{1}{2Z}(3n^2-\lambda) ,\\ \langle nl|r^2|nl\rangle&=\frac{n^2}{4Z^2}(10n^2-6\lambda+2) ,\\ \langle nl|r^3|nl\rangle&=\frac{n^2}{8Z^3}(35n^4+25n^2-30n^2\lambda+3\lambda^2-6\lambda) ,\end{aligned} \tag{3.42}$$

$$\langle nl|r^4|nl\rangle=\frac{n^4}{8Z^4}(63n^4+105n^2-70n^2\lambda+15\lambda^2-50\lambda+12) ,$$

$$\langle nl|r^5|nl\rangle=\frac{n^4}{16Z^5}[231n^6+105n^4(7-3\lambda)+21n^2(14-25\lambda+5\lambda^2)-5\lambda(\lambda-2)(\lambda-6)] .$$

In the same way, one gets (see Appendix D)

$$\begin{aligned}\mathcal{T}_2(1)&=-\left[\frac{1}{2Z}\right]^2(7n^6+5n^4-3n^2\lambda^2) ,\\ \mathcal{T}_3(1)&=-\left[\frac{1}{2Z}\right]^3[21n^8+35n^6+4n^4-15n^4\lambda^2+2n^2\lambda^2(\lambda-2)] ,\\ \mathcal{T}_4(1)&=-\left[\frac{1}{2Z}\right]^4[66n^{10}+210n^8+84n^6+18n^2-70n^6\lambda^2+20n^4\lambda^2(\lambda-3)] ,\\ \mathcal{T}_2(2)&=-\left[\frac{1}{2Z}\right]^2[6n^8+7n^6-n^4-n^2\lambda^2(2\lambda-1)] ,\end{aligned} \tag{3.43}$$

$$\mathcal{T}_3(2)=-\left[\frac{1}{2Z}\right]^3[\tfrac{33}{2}n^{10}+42n^8-14n^6-2n^4-5n^4\lambda^2(2\lambda-1)+n^2\lambda^2(\tfrac{3}{2}\lambda^2-4\lambda+2)]\ ,$$

$$\mathcal{T}_2(3)=-\left[\frac{1}{2Z}\right]^2[\tfrac{11}{2}n^{10}+9n^8-7n^6+n^4-n^2\lambda^2(\tfrac{3}{2}\lambda^2-2\lambda+1)]\ .$$

D. Determination of the perturbed eigenfunctions

Let us now consider the determination of the perturbed wave functions up to any order of the perturbation and apply the usual factorization scheme. The total ladder function is given by Eq. (2.8) and, within the artificial factorization scheme, can be written

$$K(x,m;\mu)=K^{(0)}(x,m)+K_N(x,m;\mu)\ , \tag{3.44}$$

where

$$K^{(0)}(x,m)=\frac{m}{x}+\frac{q}{m}\ ,$$
$$K_N(x,m;\mu)=\eta K^{(1)}(x,m;\mu)+\eta^2K^{(2)}(x,m;\mu)+\cdots .$$

The "perturbed key" function $(m=j)$ is the solution of the first-order differential equation [see Eq. (2.6)]

$$\left[K^{(0)}(x,j+1)+K_N(x,j+1;\mu)-\frac{d}{dx}\right]\Psi_{jj}(x;\mu)=0\ , \tag{3.45}$$

and one finds

$$\Psi_{jj}(x;\mu)\approx\Psi_{jj}^{(0)}(x)\exp\left[\int K_N(x,j+1;\mu)dx\right]\ , \tag{3.46}$$

where $\Psi_{jj}^{(0)}(x)=N_jx^{j+1}\exp[qx/(j+1)]$ is the zeroth-order normalized key function, which is the solution of the first-order differential equation

$$\left[\frac{j+1}{x}+\frac{q}{j+1}-\frac{d}{dx}\right]\Psi_{jj}^{(0)}(x)=0\ . \tag{3.47}$$

Starting from the perturbed key eigenfunction, the complete set of perturbed eigenfunctions can be generated by successive application of the ladder operation [see Eq. (2.5)]

$$\left[K(x,m;\mu)+\frac{d}{dx}\right]\Psi_{jm}(x;\mu)=[L(j+1;\mu)-L(m;\mu)]^{1/2}\Psi_{jm-1}(x;\mu)\ , \tag{3.48}$$

where $L(m;\mu)$ is the total factorization function [see Eq. (2.8)]. Once the $\Psi_{jm}(x;\mu)$ function is obtained, the artificial parameter μ has to be set to its actual value $\mu=m$. For instance, applying Eq. (3.48) once with $m=j$ gives

$$\Psi_{jj-1}(x;\mu)\approx\left[\frac{q(2j+1)^{1/2}}{j(j+1)}\Psi_{jj-1}^{(0)}+[K_N(j)+K_N(j+1)]\Psi_{jj}^{(0)}\right]\exp\left[\int K_N(j+1)dx\right]\ , \tag{3.49}$$

where

$$\Psi_{jj-1}^{(0)}(x)=N_j(2j+1)^{1/2}\left[\frac{j(j+1)}{q}+x\right]x^j\exp\left[\frac{qx}{j+1}\right]$$

is the zeroth-order normalized eigenfunction and the shortened notation $K_N(j+1)=K_N(x,j+1;\mu)$ is used.

The ladder process (3.48) can be pursued until the determination of the required $\Psi_{jm}(x;\mu=m)$ function. Note that, when dealing with functions far from the key, an alternative procedure providing the perturbed functions as linear combinations of the unperturbed functions can also be used [9].

IV. ILLUSTRATIVE APPLICATIONS

Since the main purpose of this paper is to present the method rather than to give new results or extensive tables, we give only two illustrative test examples and a short application.

A. First example

Let us first consider the solution of eigenequation (3.1) up to the second order of the perturbation and, setting $b_1^{(2)}(0)=b_2^{(2)}(0)=0$, let us assume that the perturbation is

$$V^{(1)}(x)=b_1x+b_2x^2\ ,\qquad V^{(2)}(x)=b_3x^3+b_4x^4+b_5x^5\ . \tag{4.1}$$

The associated perturbed eigenvalues are [set respectively $S_1=1$ and $S_2=4$ in Eq. (3.38)].

$$\Lambda_j^{(1)}(m)=b_1\mathcal{T}_1(0)+b_2\mathcal{T}_2(0)\ ,$$
$$\Lambda_j^{(2)}(m)=b_3\mathcal{T}_3(0)+b_4\mathcal{T}_4(0)+b_5\mathcal{T}_5(0)+\sum_{u=2}^{4}b_u^{(2)}(1)\mathcal{T}_u(1)+\sum_{u=2}^{3}b_u^{(2)}(2)\mathcal{T}_u(2)+b_2^{(2)}(3)\mathcal{T}_2(3)\ . \tag{4.2}$$

Since closed-form expressions of the required $\mathcal{T}_t(u)$ are known [see Eqs. (3.42) and (3.43) and remember that, particularly, $\mathcal{T}_t(0)=\langle x^t\rangle$], one needs only the expressions of the $b_t^{(2)}(u\neq 0)$ in terms of the b_u and of λ. Hence, let us calculate the additive potential function $\mathcal{W}_s^{(2)}(x,m)=[K^{(1)}(x,m)]^2$ generated at the second order of the perturbation. The first-order ladder function is [see Eq. (3.17)]

$$K^{(1)}(x,m)=[C_0^{(1)}(0)m+C_0^{(1)}(1)m^3]x+C_1^{(1)}(0)mx^2 , \tag{4.3}$$

where [set $S_N=1$ in Eqs. (D1)]

$$C_1^{(1)}(0)=-\frac{1}{2q}b_2 ,$$

$$C_0^{(1)}(0)=-\frac{1}{2q^2}\lambda b_2-\frac{1}{2q}b_1 ,$$

$$C_0^{(1)}(1)=\frac{1}{q^2}b_2 .$$

One gets

$$(K^{(1)})^2=\{[C_0^{(1)}(0)]^2m^2+2C_0^{(1)}(0)C_0^{(1)}(1)m^4+[C_0^{(1)}(1)]^2m^6\}x^2 \\ +2C_1^{(1)}(0)[C_0^{(1)}(0)m^2+C_0^{(1)}(1)m^4]x^3+[C_1^{(1)}(0)]^2m^2x^4 , \tag{4.4}$$

and, consequently, picking up the coefficients $b_t(u)$ of x^tm^{2u}, one gets

$$b_2^{(2)}(1)=\frac{1}{4q^4}\lambda^2b_2^2+\frac{1}{2q^3}\lambda b_1b_2+\frac{1}{4q^2}b_1^2,\quad b_2^{(2)}(2)=-\frac{1}{q^4}\lambda b_2^2-\frac{1}{q^3}b_1b_2,\quad b_2^{(2)}(3)=\frac{1}{q^4}b_2^2 , \\ b_3^{(2)}(1)=\frac{1}{2q^3}\lambda b_2^2+\frac{1}{2q^2}b_1b_2,\quad b_3^{(2)}(2)=-\frac{1}{q^3}b_2^2,\quad b_4^{(2)}(1)=\frac{1}{4q^2}b_2^2 . \tag{4.5}$$

Then, using Eq. (4.2), one obtains

$$\Lambda_j^{(2)}(m)=b_3\mathcal{T}_3(0)+b_4\mathcal{T}_4(0)+b_5\mathcal{T}_5(0)+\frac{1}{4q^2}b_1^2\mathcal{T}_2(1)+\frac{1}{2q^3}\lambda b_1b_2[\lambda\mathcal{T}_2(1)+q\mathcal{T}_3(1)-2\mathcal{T}_2(2)] \\ +\frac{1}{4q^4}b_2^2[\lambda^2\mathcal{T}_2(1)+2q\lambda\mathcal{T}_3(1)+q^2\mathcal{T}_4(1)-4\lambda\mathcal{T}_2(2)-4q\mathcal{T}_3(2)+4\mathcal{T}_2(3)] . \tag{4.6}$$

Finally, introducing the expressions (3.43) of the $\mathcal{T}_s(u)$ and setting $j+1=n$, $\lambda=l(l+1)$, $q=-Z$, and $x=r$, the perturbed Coulomb energies are found to be

$$2E_{nl}^{(1)}=-b_1\langle r\rangle-b_2\langle r^2\rangle ,$$

$$2E_{nl}^{(2)}=-b_3\langle r^3\rangle-b_4\langle r^4\rangle-b_5\langle r^5\rangle-\frac{1}{16Z^4}b_1^2n^2(7n^4+5n^2-3\lambda^2)-\frac{1}{16Z^5}b_1b_2n^4[45n^4+7n^2(9-2\lambda)-5\lambda(2+3\lambda)] \\ -\frac{1}{32Z^6}b_2^2n^6[143n^4+15n^2(21-6\lambda)+7(4-18\lambda-3\lambda^2)] . \tag{4.7}$$

B. Second example

Let us now consider the solution of eigenequation (3.1) up to the third order $N=3$ of the perturbation and, in order to avoid writing down too many cumbersome expressions, let us choose the low value $S_1=0$ and, therefore, $S_2=2$, $S_3=4$ [see Eq. (3.9)], and setting $b_1^{(2)}=b_1^{(3)}=b_2^{(3)}=b_3^{(3)}=0$, assume that the perturbed potentials are

$$V^{(1)}(x)=b_1x\ ,$$
$$V^{(2)}(x)=b_2x^2+b_3x^3\ , \tag{4.8}$$
$$V^{(3)}(x)=b_4x^4+b_5x^5\ .$$

The associated perturbed eigenvalues are [see Eq. (3.38)]

$$\Lambda_j^{(1)}(m)=b_1\mathcal{T}_1(0)\ ,$$
$$\Lambda_j^{(2)}(m)=b_2\mathcal{T}_2(0)+b_3\mathcal{T}_3(0)+b_2^{(2)}(1)\mathcal{T}_2(1)\ , \tag{4.9}$$
$$\Lambda_j^{(3)}(m)=b_4\mathcal{T}_4(0)+b_5\mathcal{T}_5(0)+\sum_{u=2}^{4}b_u^{(3)}(1)\mathcal{T}_u(1)+\sum_{u=2}^{3}b_u^{(3)}(2)\mathcal{T}_u(2)+b_2^{(3)}(3)\mathcal{T}_2(3)\ .$$

One has to calculate the additive potentials $\mathcal{W}_s^{(2)}(x,m)=(K^{(1)})^2$ and $\mathcal{W}_s^{(3)}(x,m)=2K^{(1)}K^{(2)}$ generated at each other ($N=2$ and 3) of the perturbation. The perturbed ladder functions are given by Eq. (3.17) and we have

$$K^{(1)}(x,m)=C_0^{(1)}(0)mx\ ,$$
$$K^{(2)}(x,m)=[C_0^{(2)}(0)m+C_0^{(2)}(1)m^3+C_0^{(2)}(2)m^5]x+[C_1^{(2)}(0)m+C_1^{(2)}(1)m^3]x^2+C_2^{(2)}(2)mx^3\ , \tag{4.10}$$

where [see Eqs. (3.31) and (D2) or set $S_N=2$ in Eqs. (D1)]

$$C_0^{(1)}(0)=-\frac{1}{2q}b_1\ ,$$
$$C_2^{(2)}(0)=-\frac{1}{2q}b_3\ ,$$
$$C_1^{(2)}(0)=-\frac{3}{4q^2}\lambda b_3-\frac{1}{2q}b_2\ ,$$
$$C_1^{(2)}(1)=\frac{5}{4q^2}b_3-\frac{1}{2q}b_2^{(2)}(1)\ ,$$
$$C_0^{(2)}(0)=-\frac{1}{8q^3}\lambda^2b_3-\frac{1}{2q^2}\lambda b_2-\frac{1}{4q^2}\lambda^2b_2^{(2)}(1)\ ,$$
$$C_0^{(2)}(1)=-\frac{1}{8q^3}(-12\lambda+5)b_3+\frac{1}{q^2}b_2-\frac{1}{4q^2}b_2^{(2)}(1)\ ,$$
$$C_0^{(2)}(2)=-\frac{15}{8q^3}b_3+\frac{3}{4q^2}b_2^{(2)}(1)\ .$$

One finds

$$b_2^{(2)}(1)=\frac{1}{4q^2}b_1^2 \tag{4.11}$$

and

$$b_2^{(3)}(1)=\frac{1}{8q^4}\lambda^2b_1b_3+\frac{1}{2q^3}\lambda b_1b_2+\frac{1}{16q^5}\lambda^2b_1^3\ ,$$
$$b_2^{(3)}(2)=\frac{1}{8q^4}(-12\lambda+5)b_1b_3-\frac{1}{q^3}b_1b_2-\frac{1}{16q^5}b_1^3\ ,$$
$$b_2^{(3)}(3)=\frac{15}{8q^4}b_1b_3-\frac{3}{16q^5}b_1^3\ , \tag{4.12}$$
$$b_3^{(3)}(1)=\frac{3}{4q^3}\lambda b_1b_3+\frac{1}{2q^2}b_1b_2\ ,$$
$$b_3^{(3)}(2)=-\frac{5}{4q^3}b_1b_3+\frac{1}{8q^4}b_1^3\ ,$$
$$b_4^{(3)}(1)=\frac{1}{2q^2}b_1b_3\ ,$$

and we have

$$\Lambda_j^{(2)}(m)=b_2\mathcal{T}_2(0)+b_3\mathcal{T}_3(0)+\frac{1}{4q^2}b_1^2\mathcal{T}_2(1)$$
$$\Lambda_j^{(3)}(m)=b_4\mathcal{T}_4(0)+b_5\mathcal{T}_5(0)+\frac{1}{2q^3}\lambda b_1b_2[\mathcal{T}_2(1)+q\mathcal{T}_3(1)-2\mathcal{T}_2(2)]+\frac{1}{8q^4}b_1b_3[\lambda^2\mathcal{T}_2(1)+6q\lambda\mathcal{T}_3(1)+4q^2\mathcal{T}_4(1)-(12\lambda+5)\mathcal{T}_2(2)-10q\mathcal{T}_3(2)+15\mathcal{T}_2(3)]+\frac{1}{16q^5}b_1^3[\lambda^2\mathcal{T}_2(1)-\mathcal{T}_2(2)+2q\mathcal{T}_3(2)-3\mathcal{T}_2(3)]\ . \tag{4.13}$$

Now using the expressions (3.43) of the pseudointegrals $\mathcal{T}_s(u)$ and setting $j+1=n$, $\lambda=l(l+1)$, $q=-Z$, and $x=r$, one finds the following expressions of the perturbed Coulomb energies:

$$2E_{nl}^{(1)}=-b_1\langle r\rangle\ ,$$

$$2E_{nl}^{(2)}=-b_2\langle r^2\rangle-b_3\langle r^3\rangle-\frac{1}{16Z^4}b_1^2n^2(7n^4+5n^2-3\lambda^2)\ , \tag{4.14}$$

$$2E_{nl}^{(3)}=-b_4\langle r^4\rangle-b_5\langle r^5\rangle-\frac{1}{16Z^5}b_1b_2n^4[45n^4+7n^2(9-2\lambda)-5\lambda(2+3\lambda)]$$
$$-\frac{3}{32Z^6}b_1b_3n^4[77n^6+15n^4(13-3\lambda)+7n^2(4-9\lambda-3\lambda^2)-5\lambda(2-\lambda)]$$
$$-\frac{1}{64Z^7}b_1^3n^4(33n^6+75n^4-7n^2\lambda^2-10\lambda^3)\ .$$

Expressions (4.7) and (4.14) compare well with previous results [6].

The computation can be pursued to higher orders of the perturbation without special difficulty: the critical point is having at one's disposal analytical expressions of the $\langle r^s\rangle$ and $\mathcal{T}_s(u)$ functions up to higher values of s and u, as well as the required expressions of the perturbed ladder functions $K^{(\nu)}(x,m)$ for the computation of the $b_s^{(\nu)}(t)$ coefficients. Note that both determinations amount to the computation of the $d_j(\sigma,k,x)$ by means of the rather easy-to-handle recurrence formula (3.32).

In a previous paper [6], it has been shown how expressions such as Eqs. (4.7) or (4.14) can be used to obtain accurate analytical expressions of the bound-state energies of the screened Kepler problem. In the present paper, in order to illustrate how the perturbed ladder method manages when applied to a particular problem, it is interesting to consider again [15], among other possible interesting applications, the analytical determination of the linear, quadratic and cubic Stark shifts.

C. Stark effect in hydrogen

The Schrödinger equation for the hydrogen atom in a uniform electrostatic field in the negative z direction, with relativistic and spin-orbit effects neglected, is

$$\left[-\tfrac{1}{2}\nabla^2-\frac{1}{r}+fz-\mathcal{E}\right]\Psi=0\ . \tag{4.15}$$

This equation separates in parabolic coordinates

$$x=(\xi\eta)^{1/2}\cos\varphi\ ,$$
$$y=(\xi\eta)^{1/2}\sin\varphi\ , \tag{4.16}$$
$$z=\tfrac{1}{2}(\xi-\eta)\ .$$

Setting $\Psi=(\xi\eta)^{-1/2}F(\eta)G(\eta)e^{iM\varphi}$, one gets the two following coupled equations:

$$\left[\frac{d^2}{d\xi^2}-\frac{M^2-1}{4\xi^2}+\frac{1+\beta}{2\xi}-\tfrac{1}{4}f\xi+\tfrac{1}{2}\mathcal{E}\right]F(\xi)=0\ , \tag{4.17}$$

$$\left[\frac{d^2}{d\eta^2}-\frac{M^2-1}{4\eta^2}+\frac{1-\beta}{2\eta}+\tfrac{1}{4}f\eta+\tfrac{1}{2}\mathcal{E}\right]G(\eta)=0\ , \tag{4.18}$$

where β is a separation constant.

These equations are both perturbed type-F eigenequations with a perturbed potential reduced to $V(x)=b_1x$ and [see Eq. (3.1)]

$$\Lambda=\tfrac{1}{2}\mathcal{E},\quad m(m+1)=\tfrac{1}{4}(M^2-1),\ \text{ i.e., } m=\tfrac{1}{2}|M|-\tfrac{1}{2} \tag{4.19}$$

(note that m is assumed to be positive or zero within the factorization scheme). Equation (4.17) is a standard type-F with

$$q=-\tfrac{1}{4}(1+\beta),\quad b_1=-\tfrac{1}{4}f,\quad j-m=k_1,\quad k_1=0,1,2,\ldots\ . \tag{4.20}$$

Equation (4.18) is a standard type F with

$$q=-\tfrac{1}{4}(1-\beta),\quad b_1=\tfrac{1}{4}f,\quad j-m=k_2,\quad k_2=0,1,2,\ldots\ . \tag{4.21}$$

Setting $b_2=b_3=b_4=b_5=0$ and $Z=-q$ in the results of the preceding sections [see Eqs. (3.3) and (4.14)], one gets

$$\Lambda^{(0)}=-\frac{q^2}{n^2}\ ,$$
$$\Lambda^{(1)}=\frac{b_1}{2q}(3n^2-\lambda)\ ,$$
$$\Lambda^{(2)}=-\frac{b_1^2n^2}{16q^4}(7n^4+5n^2-3\lambda^2)\ , \tag{4.22}$$
$$\Lambda^{(3)}=\frac{b_1^3n^4}{64q^7}(33n^6+75n^4-7n^2\lambda^2-10\lambda^3)\ ,$$

where $n=j+1$ and $\lambda=m(m+1)$.

The problem is to find solutions of Eqs. (4.17) and (4.18) with same $\mathcal{E}$ and β. Both $\mathcal{E}$ and β depend on f and we set

$$\mathcal{E}=\mathcal{E}^{(0)}+f\mathcal{E}^{(1)}+f^2\mathcal{E}^{(2)}+\cdots\ ,$$
$$\beta=\beta^{(0)}+f\beta^{(1)}+f^2\beta^{(2)}+\cdots\ .$$

Now, putting $q=-\tfrac{1}{4}(1+\beta)$, $b_1=\tfrac{1}{4}f$, and $n=n_1=k_1+\tfrac{1}{2}|M|+\tfrac{1}{2}$ in Eqs. (4.22), one obtains the expansion of the eigenvalue $\mathcal{E}$ of Eq. (4.17) in powers series in f, where each $\mathcal{E}^{(\nu)}$ depends on n_1 and on $\beta^{(0)},\beta^{(1)},\ldots,\beta^{(\nu)}$. Then putting $q=-\tfrac{1}{4}(1-\beta)$, $b_1=\tfrac{1}{4}f$, and $n=n_2=k_2+\tfrac{1}{2}|M|+\tfrac{1}{2}$ in the same Eqs. (4.22) one obtains an alternative expansion of $\mathcal{E}$, where the $\mathcal{E}^{(\nu)}$ depend on n_2 and $\beta^{(0)},\beta^{(1)},\ldots,\beta^{(\nu)}$. Finally, eliminating successively $\beta^{(0)},\beta^{(1)},\ldots,\beta^{(\nu)}$ between these alternative expressions of $\mathcal{E}^{(\nu)}$ ($\nu=0$ to 3), one finds the following expressions of the zeroth-order energy and of the linear, quadratic, and cubic Stark shifts

$$\begin{aligned}
\mathcal{E}^{(0)}&=-\frac{1}{2k^2}\,,\\
\mathcal{E}^{(1)}&=-\tfrac{3}{2}k(k_1-k_2)\,,\\
\mathcal{E}^{(2)}&=-\tfrac{1}{16}k^4[17k^2-3(k_1-k_2)^2-9M^2+19]\,,\\
\mathcal{E}^{(3)}&=-\tfrac{3}{32}k^7(k_1-k_2)[23k^2-(k_1-k_2)^2+11M^2+39]\,,
\end{aligned}\tag{4.23}$$

where $k=k_1+k_2+|M|+1$. These expressions are in accordance with previous results [16,17].

Let us now consider the determination of the perturbed wave functions up to the second order ($N=2$) of the perturbation. The total ladder function is given by Eq. (3.44), where

$$K_2(x,m;\mu)=-\frac{b_1mx}{2q}-\frac{b_1^2}{16q^4}\times\{[\mu^2(\mu+1)^2m+m^3-3m^5]x+2qm^3x^2\}\,. \tag{4.24}$$

Using the standard expressions (3.46) and (3.49), one gets, after expanding the exponential term in powers series in b_1, up to second order, the following expressions of the key function ψ_{jj}:

$$\psi_{jj}\approx\psi_{jj}^{(0)}\left[1-\frac{b_1}{4q}(j+1)x^2+b_1^2F(x,j+1;\mu=j)\right], \tag{4.25}$$

where

$$F(x,m;\mu)=-\frac{1}{32q^4}[\mu^2(\mu+1)^2m+m^3-3m^5]x^2-\frac{1}{24q^3}m^3x^3+\frac{1}{32q^2}m^2x^4\,.$$

Then, the determination of the perturbed eigenfunctions can be pursued by means of the ladder operation (3.48)

$$\psi_{jm-1}(x;\mu)\simeq\left[K^{(0)}(x,m)+K_2(x,m;\mu)+\frac{d}{dx}\right]\psi_{jm}(x;\mu)\,. \tag{4.26}$$

Finally, solutions of Eqs. (4.17) and (4.18) are obtained when giving to b_1, q, j, and m their actual respective values

V. CONCLUSION

Summarizing the main features of the perturbed ladder method, we can say that, once the perturbed factorization and ladder functions have been obtained, one finds again, within the perturbation scheme, all the advantages of the original Schrödinger-Infeld-Hull factorization method: analytical expressions of the perturbed eigenvalues in terms of the quantum numbers are readily obtained; the complete set of the perturbed eigenfunctions can be generated by repeated application of the ladder operator on the perturbed key function, which is the solution of a first-order differential equation. Since the perturbed eigenvalues and eigenfunctions are obtained without having to calculate explicitly either the excited unperturbed eigenfunctions or any matrix element, the treatment of high orders N of the perturbation can be carried out with a minimum effort: this is not the case with the usual perturbative methods. One may add that there is interest in the computational point of view of the method, which involves only algebraic recursive manipulations.

When dealing with perturbed type-F factorization, where the unperturbed ladder function is not a linear function of the quantum number, the main difficulty lies in the finite-difference solution of the factorizability condition. The straightforward Infeld-Hull [2] extension of the "unperturbed scheme," i.e., trying to determine the perturbed ladder right from the beginning, and even the use of the general formulas of paper I, lead to rather intricate calculations. In the present paper, it is shown that the consideration of the symmetry properties together with the use of a specific and well-adapted m expansion of the required perturbed ladder and factorization functions greatly simplifies the analytical solution of the x^k-perturbed type-F eigenequation. Moreover, it is found that taking advantage of the interesting finite-difference and symmetry properties of the Bernoulli polynomials allows a straightforward generation of the expected n^2 and $l(l+1)$ dependence of the perturbed Coulomb eigenvalues. Briefly stated, at each order N of the perturbation, the perturbed eigenvalue is obtained as a linear combination of specific type-F functions $\mathcal{T}_s(u)$, which play the same role as the $\langle x^s\rangle$ Coulombic integrals and do not depend on the order N of the perturbation. As soon as one has at one's disposal analytical expressions of the required $\mathcal{T}_s(u)$ functions as well as the analytical expressions of the required diagonal $\langle x^s\rangle$ Coulombic integrals, the determination of the perturbed eigenvalues simply amounts to the computation of the $b_s^{(N)}(u)$ expansion coefficients of an additional perturbed potential generated from the preceding orders of the perturbation: their determination involves only a few algebraic manipulations.

Owing to the present results obtained for perturbed type-F functions, it appears that the capabilities of the perturbed factorization scheme have not yet been completely explored, even for the cases where the unperturbed ladder function is a linear function of the quantum number (A to D factorization types). Particularly, considering again the x^{2k}-perturbed harmonic-oscillator eigenequation and introducing Bernoulli polynomials for the m expansion of the perturbed type-D ladder and factorization functions, it is found that the already known [18] property that the perturbed eigenvalues of order N are polynomials in $(v+\frac{1}{2})$ of degree $N+1$ and with parity $(-1)^{N+1}$, for the case of a quartic anharmonic perturbation, is naturally exhibited. This is under study and will be presented elsewhere.

APPENDIX A: DETERMINATION OF THE INTERMEDIATE LADDER FUNCTION $Z_s(m)$

Let us set $G(m+1)=(2m+s+3)(m+1)$ and use the following expression of the finite difference of a product [12]:

$$\Delta[G(m)F(m)]=G(m+1)\Delta F+F(m)\Delta G \ . \tag{A1}$$

Setting $G(m)=(2m+s+1)m$, $\Delta G=4m+s+3$, one can write

$$(2m+s+3)(m+1)\Delta F=\Delta[(2m+s+1)mF(m)] -(4m+s+3)F(m) \ , \tag{A2}$$

and the finite-difference equation (3.13) can be written again

$$2q\Delta F_{s-1}^{(N)}=-(2m+1)(s+3)k_s^{(N)} -\Delta[m(2m+s+1)F_s^{(N)}(m)] -2m(s+1)F_s^{(N)}(m)-\Delta W_s(N) \ . \tag{A3}$$

Setting $F_s^{(N)}(m)=Z_s^{(N)}(m)-Z_s^{(N)}(\mu)$, one gets

$$2q\Delta Z_{s-1}^{(N)}=-(2m+1)(s+3)k_s^{(N)}-Z_s^{(N)}(\mu)\Delta[m(2m+s+1)]-\Delta[m(2m+s+1)Z_s^{(N)}(m)] -2m(s+1)[Z_s^{(N)}(m)-Z_s^{(N)}(\mu)]-\Delta W_s^{(N)} \ , \tag{A4}$$

and, since, within an arbitrary summation constant, $\Delta^{-1}(2m+1)=m^2$, one finally obtains the expression (3.18) of $2qZ_{s-1}^{(N)}(m)$.

APPENDIX B: BERNOULLI POLYNOMIALS AND BERNOULLI NUMBERS

The Bernoulli polynomials of the first kind of degree k are [12]

$$\varphi_k(m)=\frac{1}{k!}(m+B)^k=\frac{1}{k!}\sum_{t=0}^{k}\begin{bmatrix}k\\t\end{bmatrix}B_t m^{k-t} \ , \tag{B1}$$

where, in the expansion of $(m+B)^k$, B_t is to be used instead of B^t.

Values of the B_t, i.e., the Bernoulli numbers, can be found in the tables of Ref. [12] or can be calculated recursively by means of the symbolical equation $(1+B)^t-B_t=0$:

$$B_0=1,\ B_2=\tfrac{1}{6},\ B_4=-\tfrac{1}{30},\ B_6=\tfrac{1}{42},\ B_8=-\tfrac{1}{30},\ B_{10}=\tfrac{5}{66},\ B_{12}=-\tfrac{691}{2730},\ldots, \quad B_1=-\tfrac{1}{2},\ B_{2t+1}=0 \text{ for any } t>0 \ . \tag{B2}$$

Particularly,

$$\begin{aligned}
&\varphi_0(m)=1 \ ,\\
&\varphi_1(m)=m-\tfrac{1}{2} \ ,\\
&\varphi_2(m)=\tfrac{1}{2}(m^2-m+\tfrac{1}{6}) \ ,\\
&\varphi_3(m)=\tfrac{1}{6}(m^3-\tfrac{3}{2}m^2+\tfrac{1}{2}m) \ ,\\
&\varphi_4(m)=\tfrac{1}{24}(m^4-2m^3+m^2-\tfrac{1}{30}) \ ,\\
&\varphi_5(m)=\frac{1}{5!}(m^5-\tfrac{5}{2}m^4+\tfrac{5}{3}m^3-\tfrac{1}{6}m) \ ,\\
&\varphi_6(m)=\frac{1}{6!}(m^6-3m^5+\tfrac{5}{2}m^4-\tfrac{1}{2}m^2+\tfrac{1}{42}) \ ,
\end{aligned} \tag{B3}$$

and so on. Note that, for $k>1$,

$$\varphi_k(0)=\varphi_k(1)=B_k/k! \ . \tag{B4}$$

The Bernoulli polynomials satisfy the following difference and differential properties:

$$\Delta\varphi_k=\frac{m^{k-1}}{(k-1)!} \ , \tag{B5}$$

$$\Delta^{-1}m^k=k!\varphi_{k+1}(m)+\mathcal{C}_k \ , \tag{B6}$$

$$\Delta^{-1}\varphi_k(m)=(m-1)\varphi_k(m)-k\varphi_{k+1}(m)+\mathcal{C}_k \ , \tag{B7}$$

$$\frac{d}{dm}\varphi_k(m)=D\varphi_k(m)=\varphi_{k-1}(m) \ , \tag{B8}$$

where $\mathcal{C}_k$ is an arbitrary summation constant. The following symmetry property holds:

$$\varphi_k(1-m)=(-1)^k\varphi_k(m) \ . \tag{B9}$$

The expansion of a function $f(m)$ in terms of Bernoulli polynomials is

$$f(m+u)=\int_u^{u+1}f(t)dt+\sum_{k=1}^{\infty}\varphi_k(m)\Delta D^{k-1}f(m=u) \ . \tag{B10}$$

In particular, one gets

$$m\varphi_k(m)=(k+1)\varphi_{k+1}(m)+\tfrac{1}{2}\varphi_k(m) +\sum_{t=1}^{k-1}\varphi_t(m)\varphi_{k+1-t}(0) \ ,$$

and then, using Eq. (B7),

$$\Delta^{-1}\varphi_k(m)=\varphi_{k+1}(m)-\tfrac{1}{2}\varphi_k(m)+\sum_{t=1}^{k-1}\varphi_t(m)\varphi_{k+1-t}(0) . \quad \text{(B11)}$$

Consequently, one obtains

$$\varphi_k(m)+2\Delta^{-1}\varphi_k(m)=2\varphi_{k+1}(m)+\sum_{t=1}^{k-1}\frac{2B_{k+1-t}}{(k+1-t)!}\varphi_t(m) . \quad \text{(B12)}$$

APPENDIX C: EXPRESSIONS OF THE $\mathcal{P}_k(\lambda)$ AND $T_k(n,\lambda)$ POLYNOMIALS

Since $\varphi_{2k}(1-m)=\varphi_{2k}(m)$, the Bernoulli polynomials $\varphi_{2k}(m)$ of even degree $2k$ can also be viewed as polynomials of degree k of $M=m(m-1)$. Using a Taylor expansion of $\varphi_{2k}(m)$, one can write

$$\varphi_{2k}(m)=\varphi_{2k}(M=0)+\frac{d}{dM}\varphi_{2k}(M=0)M+\frac{1}{2!}\frac{d^2}{dM^2}\varphi_{2k}(M=0)M^2+\cdots . \quad \text{(C1)}$$

Since $d/dM=[1/(2m-1)]d/dm$ and $(d/dm)\varphi_{2k}=\varphi_{2k-1}(m)$, one gets, successively,

$$\frac{d}{dM}\varphi_{2k}=\frac{1}{2m-1}\varphi_{2k-1} ,$$

$$\frac{d^2}{dM^2}\varphi_{2k}=\frac{-2}{(2m-1)^3}\varphi_{2k-1}+\frac{1}{(2m-1)^2}\varphi_{2k-2} ,$$

$$\frac{d^3}{dM^3}\varphi_{2k}=\frac{12}{(2m-1)^5}\varphi_{2k-1}-\frac{6}{(2m-1)^4}\varphi_{2k-2}+\frac{1}{(2m-1)^3}\varphi_{2k-3} ,$$

$$\frac{d^4}{dM^4}\varphi_{2k}=\frac{-120}{(2m-1)^7}\varphi_{2k-1}+\frac{60}{(2m-1)^6}\varphi_{2k-2}-\frac{12}{(2m-1)^5}\varphi_{2k-3}+\frac{1}{(2m-1)^4}\varphi_{2k-4} ,$$

and, more generally,

$$\frac{d^j}{dM^j}\varphi_{2k}=\sum_{u=1}^{j}\frac{h(u,j)}{(2m-1)^{2j-u}}\varphi_{2k-u} , \quad \text{(C2)}$$

where the $h(u,j)$ obey the recurrence relation

$$h(u,j)=h(u-1,j-1)-2(2j-u-2)h(u,j-1) \quad \text{with } h(1,1)=1 .$$

Hence, we have

$$h(j-1,j)=-j(j-1) ,$$

$$h(j-2,j)=(j+1)j(j-1)(j-2)/2! ,$$

$$h(j-3,j)=-(j+2)_6/3! ,$$

and, more generally,

$$h(j-s,j)=(-1)^s(j+s-1)_{2s}/s! . \quad \text{(C3)}$$

Since $\varphi_u(0)=B_u/u!$, one gets [see Eqs. (C2) and (C3), and set $m=0$]

$$\left[\frac{d^j}{dM^j}\varphi_{2k}\right]_{M=0}=(-1)^j\sum_{u=1}^{j}\frac{(2j-u-1)!B_{2k-u}}{(u-1)!(j-u)!(2k-u)!} .$$

For $k>1$, since $B_{2k-u}=0$ for any odd value of u, one can write

$$\left[\frac{d^j}{dM^j}\varphi_{2k}\right]_{M=0}=(-1)^j\sum_{t=1}^{\{j/2\}}\frac{(2j-2t-1)\,B_{2k-2t}}{(2t-1)!(j-2t)!(2k-2t)!} ,$$

where $\{j/2\}$ is the integer part of $j/2$.

For $k=1$, we have $[(d/dM)\varphi_2]_{M=0}=-B_1=\frac{1}{2}$ and $\varphi_2(m)=\frac{1}{12}+\frac{1}{2}M$. For $k>1$, one obtains the following expansion of $\varphi_{2k}(m)$ in terms of $M=m(m-1)$:

$$\varphi_{2k}(m)=\frac{B_{2k}}{(2k)!}+\sum_{j=1}^{k}(-1)^j\frac{M^j}{j!}\sum_{t=1}^{\{j/2\}}\frac{(2j-2t-1)!B_{2k-2t}}{(2t-1)!(j-2t)!(2k-2t)!} . \quad \text{(C4)}$$

Setting $\lambda=m(m+1)$ and $\mathcal{P}_k(\lambda)=2(2k)!\varphi_{2k}(m+1)-(1/k)B_{2k}$, one gets, for $k\geq 1$, the following expression of $\mathcal{P}_{k+1}(\lambda)$:

$$\mathcal{P}_{k+1}(\lambda)=2\sum_{j=2}^{k+1}\frac{(-\lambda)^j}{j}\sum_{t=1}^{\{j/2\}}\begin{bmatrix}2k+1\\2t-1\end{bmatrix}\begin{bmatrix}2j-2t-1\\j-1\end{bmatrix}B_{2k+2-2t} . \quad \text{(C5)}$$

Particularly, we have

$$\mathcal{P}_1(\lambda)=\lambda ,$$
$$\mathcal{P}_2(\lambda)=\tfrac{1}{2}\lambda^2 ,$$
$$\mathcal{P}_3(\lambda)=\tfrac{1}{3}\lambda^3-\tfrac{1}{6}\lambda^2 , \qquad (C6)$$
$$\mathcal{P}_4(\lambda)=\tfrac{1}{4}\lambda^4-\tfrac{1}{3}\lambda^3+\tfrac{1}{6}\lambda^2 ,$$
$$\mathcal{P}_5(\lambda)=\tfrac{1}{5}\lambda^5-\tfrac{1}{2}\lambda^4+\tfrac{3}{5}\lambda^3-\tfrac{3}{10}\lambda^2 .$$

Now, let us consider the polynomials

$$T_{k+1}(n,\lambda)=\mathcal{P}_{k+1}(\lambda)-\frac{2k+3}{k+1}n^{2k+2}-\sum_{t=1}^{k}a_{kt}n^{2t} .$$

Using the expression (3.24) of the a_{kt} coefficients, one gets

$$a_{11}=\tfrac{1}{2},\ a_{21}=-\tfrac{1}{6},\ a_{22}=\tfrac{5}{6} ,$$
$$a_{31}=\tfrac{1}{6},\ a_{32}=-\tfrac{7}{12},\ a_{33}=\tfrac{7}{6} , \qquad (C7)$$
$$a_{41}=-\tfrac{3}{10},\ a_{42}=1,\ a_{43}=-\tfrac{7}{5},\ a_{44}=\tfrac{3}{2},\ldots ,$$

and one obtains

$$T_1(n,\lambda)=-3n^2+\lambda ,$$
$$T_2(n,\lambda)=-\tfrac{5}{2}n^4-\tfrac{1}{2}n^2+\tfrac{1}{2}\lambda^2 ,$$
$$T_3(n,\lambda)=-\tfrac{7}{3}n^6-\tfrac{5}{6}n^4+\tfrac{1}{6}n^2+\tfrac{1}{3}\lambda^3-\tfrac{1}{6}\lambda^2 , \qquad (C8)$$
$$T_4(n,\lambda)=-\tfrac{9}{4}n^8-\tfrac{7}{6}n^6+\tfrac{7}{12}n^4-\tfrac{1}{6}n^2+\tfrac{1}{4}\lambda^4-\tfrac{1}{3}\lambda^3+\tfrac{1}{6}\lambda^2 ,$$
$$T_5(n,\lambda)=-\tfrac{11}{5}n^{10}-\tfrac{3}{2}n^8+\tfrac{7}{5}n^6-n^4-\tfrac{3}{10}n^2+\tfrac{1}{5}\lambda^5-\tfrac{1}{2}\lambda^4 +\tfrac{3}{5}\lambda^3-\tfrac{3}{10}\lambda^2 .$$

The computation of the $T_{k+1}(n,\lambda)$ polynomials can be pursued up to higher values of k without any special difficulty.

APPENDIX D: DETERMINATION OF THE $\mathcal{T}_s(u)$ FUNCTIONS

Using the recursive formulas (3.30) successively for $s=S_N+1,S_N,S_N-1,\ldots$, one gets

$$C_{S_N}(0)=-\frac{1}{2q}b_{S_N+1}(0) ,$$

$$C_{S_N-1}(0)=-\left[-\frac{1}{2q}\right]^2(S_N+1)\mathcal{P}_1b_{S_N+1}(0)+\left[-\frac{1}{2q}\right]b_{S_N}(0) ,$$
$$C_{S_N-1}(1)=\left[-\frac{1}{2q}\right]^2(S_N+3)b_{S_N+1}(0)+\left[-\frac{1}{2q}\right]b_{S_N}(1) ,$$
$$C_{S_N-2}(0)=\left[-\frac{1}{2q}\right]^3S_N(S_N-1)\mathcal{P}_2b_{S_N+1}(0)-\left[-\frac{1}{2q}\right]^2S_N\mathcal{P}_1b_{S_N}(0)+\left[-\frac{1}{2q}\right]b_{S_N-1}(0)-\left[-\frac{1}{2q}\right]^2S_N\mathcal{P}_2b_{S_N}(1) , \qquad (D1)$$
$$C_{S_N-2}(1)=-\left[-\frac{1}{2q}\right]^3[(S_N+2)(S_N+1)\mathcal{P}_1-\tfrac{1}{2}S_N(S_N+3)]b_{S_N+1}(0) +\left[-\frac{1}{2q}\right]^2(S_N+2)b_{S_N}(0)+\left[-\frac{1}{2q}\right]^2\tfrac{1}{2}S_Nb_{S_N}(1)+\left[-\frac{1}{2q}\right]b_{S_N-1}(1) ,$$
$$C_{S_N-2}(2)=\left[-\frac{1}{2q}\right]^3\frac{(S_N+4)(S_N+3)}{2}b_{S_N+1}(0)+\left[-\frac{1}{2q}\right]^2\frac{(S_N+4)}{2}b_{S_N}(1)+\left[-\frac{1}{2q}\right]b_{S_N-1}(2) ,$$

and so on. More generally, it is easily inferred that expression (3.31) holds for the $C_{S_N-\sigma}(k)$.

Substituting for $C_{S_N-\sigma}(k)$ from Eq. (3.31) into Eq. (3.30) and rearranging the summations, it is found that the $d_j(\sigma,k,s)$ coefficients obey the recurrence formula

$$d_j(\sigma+1-j,k,S_N-j) =\frac{S_N-\sigma+2k+1}{k}d_j(\sigma-j,k-1,S_N-j) +(S_N-\sigma+1)\sum_{t=k}^{\sigma}a_{tk}d_j(\sigma-j,t,S_N-j) ,$$

or, equivalently, formula (3.32).

Particularly, using formula (3.32) together with the expressions (C7) of the a_{tk}, and keeping in mind that $a_{t0}=-\mathcal{P}_{t+1}(\lambda)$, one gets

$$d_0(1,0,s)=-(s+1)\mathcal{P}_1 ,$$
$$d_0(1,1,s)=(s+3) ,$$

$$d_0(2,0,s)=s(s-1)\mathcal{P}_2 ,$$
$$d_0(2,1,s)=-(s+2)(s+1)\mathcal{P}_1+\tfrac{1}{2}s(s+3) ,$$
$$d_0(2,2,s)=\tfrac{1}{2}(s+4)(s+3) ,$$

$$d_0(3,0,s)=-\tfrac{1}{2}(s-1)(s-2)(s-3)\mathcal{P}_3 -\tfrac{1}{2}(s+1)(s-1)(s-2)\mathcal{P}_2 ,$$

$$d_0(3,1,s)=(s+1)s(s-1)\mathcal{P}_2 -\tfrac{1}{2}(s+2)(s+1)(s-1)\mathcal{P}_1 +\tfrac{1}{2}(s+3)(s-1)(s-2) ,$$

$$d_0(3,2,s)=-\tfrac{1}{2}(s+3)(s+2)(s+1)\mathcal{P}_1 +\tfrac{1}{3}(s+3)(2s^2+6s-5) ,$$

$$d_0(3,3,s)=\tfrac{1}{6}(s+5)(s+4)(s+3) ,$$

$$d_0(4,0,s)=\tfrac{1}{6}(s-2)(s-3)(s-4)(s-5)\mathcal{P}_4 +\tfrac{1}{3}(s-2)(s-3)(2s^2-6s-5)\mathcal{P}_3 +\tfrac{1}{6}(s+1)(s-2)(s-3)\mathcal{P}_2 ,$$

$$d_0(4,1,s)=-\tfrac{1}{2}s(s-1)(s-2)(s-3)\mathcal{P}_3 -\tfrac{1}{6}(s+2)(s+1)(s-2)(s-3)\mathcal{P}_1 -\tfrac{1}{3}(s+3)(s-2)(s-3) ,$$

$$d_0(4,2,s)=\tfrac{1}{2}(s+2)(s+1)s(s-1)\mathcal{P}_2 -\tfrac{1}{3}(s+2)(s+1)(2s^2+2s-9)\mathcal{P}_1 +\tfrac{1}{24}(s+3)(s-2)(13s^2+21s-84) ,$$

$$d_0(4,3,s)=-\tfrac{1}{6}(s+4)(s+3)(s+2)(s+1)\mathcal{P}_1 +\tfrac{5}{12}(s+4)(s+3)(s^2+3s-6) ,$$

$$d_0(4,4,s)=\tfrac{1}{24}(s+6)(s+5)(s+4)(s+3) ,$$

$$d_1(1,0,s)=-(s+1)\mathcal{P}_2 ,$$

$$d_1(1,1,s)=\tfrac{1}{2}(s+1) ,$$

$$d_1(1,2,s)=\tfrac{1}{2}(s+5) ,$$

$$d_1(2,0,s)=s(s-1)\mathcal{P}_3 ,$$

$$d_1(2,1,s)=-(s+2)(s+1)\mathcal{P}_2+\tfrac{1}{6}s(s-1) ,$$

$$d_1(2,2,s)=\tfrac{1}{3}(2s^2+10s+3) ,$$

$$d_1(2,3,s)=\tfrac{1}{6}(s+6)(s+5) ,$$

$$d_1(3,0,s)=-\tfrac{1}{2}(s-1)(s-2)(s-3)\mathcal{P}_4 -\tfrac{1}{2}(s+1)(s-1)(s-2)\mathcal{P}_3 ,$$

$$d_1(3,1,s)=(s+1)s(s-1)\mathcal{P}_3 -\tfrac{1}{2}(s+2)(s+1)(s-1)\mathcal{P}_2 -\tfrac{1}{3}(s-1)(s-2) ,$$

$$d_1(3,2,s)=-\tfrac{1}{2}(s+3)(s+2)(s+1)\mathcal{P}_2 +\tfrac{1}{24}(s-1)(13s^2+47s-50) ,$$

$$d_1(3,3,s)=\tfrac{5}{12}(s+5)(s^2+5s-2) ,$$

$$d_1(3,4,s)=\tfrac{1}{24}(s+7)(s+6)(s+5) ,$$

$$d_2(1,0,s)=-(s+1)\mathcal{P}_3 ,$$

$$d_2(1,1,s)=-\tfrac{1}{6}(s+1) ,$$

$$d_2(1,2,s)=\tfrac{5}{6}(s+1) ,$$

$$d_2(1,3,s)=\tfrac{1}{3}(s+7) ,$$

$$d_2(2,0,s)=s(s-1)\mathcal{P}_4 ,$$

$$d_2(2,1,s)=-(s+2)(s+1)\mathcal{P}_3-\tfrac{1}{6}s(s-1) ,$$

$$d_2(2,2,s)=\tfrac{1}{12}(5s^2-13s-4) ,$$

$$d_2(2,3,s)=\tfrac{1}{3}(2s^2+14s+5) ,$$

$$d_2(2,4,s)=\tfrac{1}{12}(s+8)(s+7) ,$$

$$d_3(1,0,s)=-(s+1)\mathcal{P}_4 ,$$

$$d_3(1,1,s)=\tfrac{1}{6}(s+1) ,$$

$$d_3(1,2,s)=-\tfrac{7}{12}(s+1) ,$$

$$d_3(1,3,s)=\tfrac{7}{6}(s+1) ,$$

$$d_3(1,4,s)=\tfrac{1}{4}(s+9) . \qquad \text{(D2)}$$

Finally using these expressions together with the expression (3.39) of $\mathcal{T}_t(s)$ and the expressions (C6) of the $\mathcal{P}_k(\lambda)$, one obtains the required expressions (3.42) and (3.43) of the $\mathcal{T}_t(s)$ in terms of the quantum numbers.

APPENDIX E: DETERMINATION OF THE PERTURBED TYPE-F FACTORIZATION FUNCTION $L^{(N)}(m;\mu{:}b_i^{(N)})$

Keeping in mind that $F(m+1)=F(m)+\Delta F$, the finite-difference equation (3.33) can be written again,

$$\Delta L^{(N)}=-3(2m+1)[k_0^{(N)}+F_0^{(N)}(m)] -(m+1)(2m+3)\Delta F_0^{(N)} . \qquad \text{(E1)}$$

Using again Eq. (A1), one gets

$$\Delta L^{(N)}=-3(2m+1)k_0^{(N)}-\Delta[m(2m+1)F_0^{(N)}(m)] -2mF_0^{(N)}(m) , \qquad \text{(E2)}$$

and, within an additive arbitrary constant,

$$L^{(N)}(m)=-3m^2k_0^{(N)}+2m^2F_0^{(N)}(m) +[mF_0^{(N)}+\Delta^{-1}mF_0^{(N)}(m)] . \qquad \text{(E3)}$$

Introducing expressions (3.15) and (3.16) of $F_0^{(N)}(m)$ and $Z_0^{(N)}(m)$ and using again relation (3.21) together with the condition (3.34) to be fulfilled, one finds

$$L^{(N)}(m;\mu;b_i^{(N)})=[\mu(\mu+1)-3m^2]k_0^{(N)}+(3m^2-\mu^2)Z_0^{(N)}(\mu)-2m^2Z_0^{(N)}(m)$$
$$-\sum_{t=1}^{S_N}\frac{C_0^{(N)}}{t+1}\sum_{j=0}^{t+1}\begin{bmatrix}2t+2\\2j\end{bmatrix}B_{2(t+1-j)}(m^{2j}-\mu^{2j})\ , \tag{E4}$$

or, alternatively, after some rearrangements and using again Eqs. (3.21) and (3.25),

$$L^{(N)}(m;\mu;b_i^{(N)})=[\mu(\mu+1)-3m^2][k_0^{(N)}-Z_0^{(N)}(\mu)]$$
$$-\sum_{t=1}^{S_N}C_0^{(N)}(t)[2(2t+1)!\varphi_{2t+2}(m)-m^{2t+1}+2m^{2t+2}-2(2t+1)!\varphi_{2t+2}(m+1)]\ . \tag{E5}$$

Keeping in mind that

$$2(2t+1)!\varphi_{2t+2}(m)-m^{2t+1}+2m^{2t+2}-\frac{B_{2t+2}}{t+1}=\frac{2t+3}{t+1}m^{2t+2}+\frac{1}{t+1}\sum_{j=1}^{t}\begin{bmatrix}2t+2\\2j\end{bmatrix}B_{2(t+1-j)}m^{2j}\ , \tag{E6}$$

one obtains, after noting that $\mu(\mu+1)=\mathcal{P}_1(\lambda)$, the required compact expression (3.35) of $L^{(N)}(m;\mu;b_i^{(N)})$.

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[11] If $V^{(N)}(x,m)$ would involve m such that it cannot be identified in m with $U^{(N)}(x,m)$, one could introduce an additional artificial parameter μ', then one uses $V^{(N)}(x;m;\mu')$ such that $V^{(N)}(x;m;\mu'=m)$ and sets $\mu'=m$ at the final stage of the computation.

[12] C. Jordan, *Calculus of Finite Differences* (Chelsea, New York, 1965).

[13] These properties, already verified by the unperturbed type-F ladder and factorization functions $K^{(0)}(x,m)$ and $L^{(0)}(m)$, can be easily checked after writing again the factorizability condition (2.9)

$$\frac{d}{dx}K^{(N)}(x,m)=\tfrac{1}{2}[U^{(N)}(x,m)-U^{(N)}(x,m-1)]\ ,$$
$$L^{(N)}(m)=-2K^{(0)}(x,m)K^{(N)}(x,m)$$
$$-\tfrac{1}{2}[U^{(N)}(x,m)+U^{(N)}(x,m-1)]$$
$$-\sum_{\nu=1}^{N-1}K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)\ ,$$

and noting that, as well as $U^{(0)}(x,m)$, the factorizing perturbations $U^{(N)}(x,m)$ will depend on m only via the product $m(m+1)$.

[14] See, for instance, G. W. F. Drake and R. A. Swainson, Phys. Rev. A **42**, 1123 (1990), and references therein.

[15] Note that analytical expressions of the Stark shifts are known to arbitrarily high order. See, for instance, H. J. Silverstone, Phys. Rev. A **18**, 1853 (1978).

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