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Information theory, squeezing, and quantum correlations

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Information theory allows us to make quantitative statements about the strength and nature of the correlations between systems. Application of this theory to the quantized electromagnetic field reveals a special role for the two-mode squeezed states. The nonclassical properties of these states arise from the intermode correlations, and we apply information-theoretic methods to determine the strength of the correlation between specific pairs of observables. This analysis leads to the important *general* result that for any correlated pure state a given pair of single-system observables contains at most only half the information about the correlations. We discuss the implications of this result for the distinction between classical and quantum systems.

I. INTRODUCTION

One of the philosophically most challenging features of quantum theory has been to understand the nature and implications of quantum correlations. The strength of these correlations has led to a debate on the question of completeness and whether quantum theory lacks a hidden element [1–3]. However, experiment has shown that such a hidden element, if it exists, cannot be local [4]. It has been suggested that this nonlocal property violates causality and may be used for superluminal communication. There exists, however, a general proof that the act of observation of one of a pair of correlated quantum systems can in no way affect the outcome of observations on its partner [5].

Quantum correlations, as their name suggests, are responsible for a number of nonclassical effects. This apparently straightforward statement is itself the source of some difficulties associated with the description "nonclassical." At one level a nonclassical effect is one that does not exist in conventional classical theory. Examples of such nonclassical effects are found in the detection of squeezed states of the electromagnetic field [6]. These states have properties that are incompatible with the description of light by the classical Maxwell field [7]. Another, and perhaps more fundamental, criterion for nonclassical behavior is provided by correlated quantum states whose properties are incompatible with the classical notion of local realism. It is consideration of states of this type that has enforced the rejection of local realism and revealed some of the most subtle features of quantum theory [1-4]. Although these nonclassical features coexist in a number of systems [8], they are by no means equivalent. Both of these nonclassical types of behavior will appear in this paper and we will be careful to distinguish between them.

Quantum correlations between two systems have been characterized by the correlations between operators corresponding to observables associated with the individual systems and also with the state describing the two systems. Operator correlations may be quantified by means of the correlation coefficient [9]. However, it is more difficult to quantify the absolute strength of the correlation associated with a quantum state. This problem may be solved by appealing to information theory and defining an index of correlation as the information content of the correlation. This index of correlation has been applied to the discussion of quantum correlations by a number of authors [10–12]. We review the principal properties of the index of correlation and its application to quantum-optical correlations in Sec. II.

The index of correlation is an absolute, quantitative, and observable-independent measure of the strength of the correlations between two systems. It does not, however, tell us which observables we must measure in order to retrieve this information. To answer this question we introduce, in Sec. III, the Shannon index of correlation. This quantity tells us how much of the information residing in the correlations is revealed by measuring a given pair of observables. We derive the important result that, for a pure state, no observation of a pair of single-system observables can provide more than half the information content of the correlations. We have previously highlighted the fundamental significance of the two-mode squeezed states as being the most strongly correlated states of light [12]. In Sec. IV we apply the Shannon index of correlation to these states and find that the most strongly correlated observables are the photon numbers associated with the individual modes. The Shannon index also allows us to ascertain the information contained in the correlations between the single-mode quadratures and phases.

States that maximize the index of correlation (and are therefore optimally correlated) are entangled pure states. Entangled mixtures of states always have a lower index of correlation. Both entanglement—that is, inability to factorize the two-system density matrix—and the superposition principle for probability amplitudes contribute to the strength of the quantum correlations. The significance of the superposition principle in maximizing the index of correlation leads us to speculate, in Sec. V, on the use of information theory to distinguish between classical and quantum correlations.

II. THE INDEX OF CORRELATION

Correlation is a fundamental property of a two-component quantum system and is reflected in our ability, or inability, to factorize the complete density matrix. The information content, Q, of a density matrix, ρ, is defined through its entropy in the following way [13,14]:

$$Q = S^{\max} - S \ , \tag{2.1}$$

where $S^{\max}$ is the maximum possible entropy for the system and S is the actual entropy. Entropy is defined in quantum mechanics as a generalization of the classical Boltzmann entropy [15] so that

$$S = -\mathrm{Tr}\rho \ln\rho \ . \tag{2.2}$$

We have set Boltzmann's constant equal to unity here and shall continue to do so throughout this paper. If the system is in a pure state, the entropy (2.2) is precisely zero and in this case we have as much information about the system as quantum mechanics allows us. If the system is in a mixed state, the entropy is positive and so measures deviations from pure-state behavior. Furthermore, if ρ refers to an isolated system, as we have implicitly assumed above, then the entropy is time independent, as the dynamics of ρ are governed by a unitary transformation.

We shall, in this paper, restrict ourselves to a consideration of correlated quantum systems where the correlation exists between just two component systems. We shall label these component systems by the suffixes a and b. The overall state of the two-component system is described by a density operator ρ and the states of the component systems are described by the reduced density operators ρ_a and ρ_b. The reduced density operators for the $a(b)$ systems are constructed from the full two-component density operator ρ by tracing over one of the component systems. Thus,

$$\rho_{a(b)} = \mathrm{Tr}_{b(a)}\rho \ . \tag{2.3}$$

The operation of tracing over one of the systems is equivalent to neglecting all information about the joint properties of the component systems. The information content of each of the component systems is defined in an analogous way to (2.1) but with the entropies being formed through the reduced density operator so that the entropy for the a system, for example, is given by

$$S_a = -\mathrm{Tr}_a\rho_a \ln\rho_a \ . \tag{2.4}$$

It is the full density operator for the two-component system, ρ, that contains the information about the joint properties of the component systems.

The index of correlation is defined as the information content of the correlation between two systems. Formally, this is written as

$$I_c = Q - (Q_a + Q_b) \ , \tag{2.5}$$

where Q is the information content of the full density operator and $Q_{a(b)}$ is the information content of the reduced density operator for the $a(b)$ system. Failure to measure joint properties of the two systems results in a loss of this quantity of information. Noting that $S^{\max} = S_a^{\max} + S_b^{\max}$, we find that the index of correlation I_c takes the simple and appealing form

$$I_c = S_a + S_b - S \ . \tag{2.6}$$

The index of correlation can be extended to quantum systems with any number of correlated components. If we have N correlated components, then the index of correlation for the N-component quantum system is given by [11, 16]

$$I_c = \sum_{j=1}^{N} S_j - S \ . \tag{2.7}$$

However, as we have previously mentioned, we shall restrict ourselves in this paper to consideration of quantum systems with two components. Examples of such systems in quantum optics are the two-mode squeezed state and the Jaynes-Cummings model of optical resonance. In the former the two component systems are the modes of the electromagnetic field; in the latter the components are the field mode and the two-level atom.

The entropies for the individual component systems are no longer time independent, in general, as the dynamics of the reduced density operators are not governed by a unitary transformation. In general, these entropies are not monotonically increasing functions of time and can even exhibit periodic behavior [17]. The failure of monotonicity as a property of the reduced entropies is partially compensated by the existence of a remarkable theorem due to Araki and Lieb. This theorem states that for any two-component quantum system the entropies are related by the following triangle inequality [18]:

$$|S_a - S_b| \le S \le S_a + S_b \ . \tag{2.8}$$

If the two-component system is in a pure state so that $S=0$, then the a and the b systems have equal entropies. An interesting example of the use of this relation occurs if we consider a two-level atom interacting with a quantized field mode in a perfect cavity. If the initial state of the atom-field system is pure, then the field can be described at all times by just two quantum states [17]. The Araki-Lieb inequality (2.8) has important implications for the index of correlation. It has been shown [10,12] that the maximum correlation occurs when the two-component system is in a pure state. This is a necessary but not sufficient condition to obtain the maximal correlation. Recalling the fact that for a pure state we have $S_a = S_b = \bar{S}$, it can then be shown [10,12] that the maximum value that the index of correlation can take is just $I_c^{\max} = 2\bar{S}$. The actual value of the index of correlation depends on the nature of the subsystems in question. For example, if the two systems have different numbers of states, the maximal correlation occurs when the total system is in a pure state *and* the system with the smaller number of states has maximum entropy. For the remainder of this paper we shall concern ourselves with pairs of *similar* systems—that is, systems with equal numbers of states. It is not difficult to extend our results to the former case where the systems have unequal numbers of states.

The optimally correlated state of a two-component quantum system is a pure state in which the entropy of one of the component systems has been maximized. This maximization procedure is often subject to physical constraints. Indeed, for unbounded systems the entropy must be constrained by the specification of a mean energy. The state with maximum entropy, but finite mean energy, exhibits thermal fluctuations and is described by a density operator of the general form

$$\rho_{\mathrm{th}}=Z^{-1}(\beta)e^{-\beta H}\ , \tag{2.9}$$

where $Z(\beta)$ is the partition function, H is the system Hamiltonian, and β is the inverse temperature. The maximally correlated state of two similar quantum systems will be a pure state in which each of the component systems are thermal in character. Such states are well known in quantum-statistical mechanics as thermofields [19, 20].

III. THE SHANNON INDEX OF CORRELATION

The index of correlation is an absolute measure of the strength of correlation between two quantum systems. As we have seen, this index can enable us to determine the optimally correlated quantum state of the two systems. However, from an experimental point of view one would be entitled to ask which observables need to be measured to access this information. In other words, we need to determine in exactly which observables this information resides. With this purpose in mind, we introduce a "Shannon" index of correlation which is a direct measure of the information content of the correlation between specific pairs of observables. The use of "Shannon" in this context is for two reasons: Firstly, it serves to distinguish the observable-*dependent* index from the observable-*independent* index, and secondly, this nomenclature has also been applied previously in the study of fluctuations in quantum optics [17]. If $\hat{M}$ and $\hat{N}$ denote two quantum operators representing properties of the a and b systems, respectively, then the Shannon index of correlation between them will be denoted by $I_c(\hat{M},\hat{N})$. Failure to measure these observables jointly will result in a loss of this quantity of information.

We shall, as before, consider two quantum systems which we label with the suffices a and b. The information content of an observable $\hat{M}$ of the a system, when considered independently of the b system, is given by

$$Q_a(\hat{M})=S^{\max}(\hat{M})+\sum_{\mu}\langle\mu|\rho_a|\mu\rangle\ln\langle\mu|\rho_a|\mu\rangle\ , \tag{3.1}$$

where $|\mu\rangle$ is an eigenstate of $\hat{M}$ and $S^{\max}(\hat{M})$ is the maximum entropy associated with the observable. The joint Shannon entropy for two observables $\hat{M}$ and $\hat{N}$ which act in the a and b systems, respectively, is formed in a similar fashion from the diagonal elements of ρ, the full two-system density operator, in the eigenbasis of the two observables. Thus we have

$$S(\hat{M},\hat{N})=-\sum_{\mu}\sum_{\nu}\rho_{\mu\mu,\nu\nu}\ln\rho_{\mu\mu,\nu\nu}\ , \tag{3.2}$$

where $\rho_{\mu\mu,\nu\nu}=\langle\mu|\langle\nu|\rho|\nu\rangle|\mu\rangle$ and $|\nu\rangle$ is an eigenstate of $\hat{N}$. Using the notation that $S_{a(b)}(\hat{T}_{a(b)})$ is the entropy of the observable $\hat{T}_{a(b)}$ in the $a(b)$ system alone and that $S(\hat{M},\hat{N})$ is the joint entropy defined in (3.2), we can define a Shannon index of correlation between the observables $\hat{M}$ and $\hat{N}$ as

$$I_c(\hat{M},\hat{N})=S_a(\hat{M})+S_b(\hat{N})-S(\hat{M},\hat{N})\ . \tag{3.3}$$

This expression now gives the information content of the correlation between the *specific* observables $\hat{M}$ and $\hat{N}$.

Let us suppose that each of these quantum systems is described by a single quantum number. These systems could be, for example, two modes of the electromagnetic field. The state of the two systems is described by a density operator with four summation indices. If we only require the diagonal elements of this density operator, this description simplifies considerably. Denoting the diagonal part of the density operator in the observable bases $|\mu\rangle$ and $|\nu\rangle$ by ρ_{diag}, we obtain for the diagonal part an expression of the form

$$\rho_{\mathrm{diag}}=\sum_{\mu}\sum_{\nu}P_{\mu\nu}|\mu,\nu\rangle\langle\mu,\nu|\ . \tag{3.4}$$

The diagonal part of the reduced density operator for the a system is given by the expression

$$\rho^{a}_{\mathrm{diag}}=\sum_{\mu}\left[\sum_{\nu}P_{\mu\nu}\right]|\mu\rangle\langle\mu|=\sum_{\mu}P_{\mu}|\mu\rangle\langle\mu|\ . \tag{3.5}$$

Remembering the fact that $P_{\mu\nu}\leq P_{\mu}$ with the equality holding only for perfect correlation, we find that the difference between the joint entropy of the two observables and the entropy of the observable $\hat{M}$ obeys the relation

$$S(\hat{M},\hat{N})-S_a(\hat{M})=-\sum_{\mu}\sum_{\nu}P_{\mu\nu}\ln\left[\frac{P_{\mu\nu}}{P_{\mu}}\right]\geq 0\ . \tag{3.6}$$

Thus we arrive at the modified Araki-Lieb inequality for quantum observables, which states that

$$\max(S_a(\hat{M}),S_b(\hat{N}))\leq S(\hat{M},\hat{N})\leq S_a(\hat{M})+S_b(\hat{N})\ . \tag{3.7}$$

This inequality tells us that the observables considered jointly must be at least as disordered as either of the observables alone. If we label the entropies in (3.7) such that $S_a\geq S_b$, we find that the Shannon index of correlation is bounded by $I_c(\hat{M},\hat{N})\leq S_b(\hat{N})$. The maximum possible value for $I_c(\hat{M},\hat{N})$ is just $S_b(\hat{N})$ and the maximum value that this can take is just $S_a(\hat{M})$. Optimally correlated observables must have equal disorder in the individual observables. The above discussion can readily be extended to include systems described by more than one quantum number.

The Shannon index possesses two important properties (the proofs of these properties are given in Appendices A and B). The first of these properties is the physically reasonable result that for any given pair of observables the Shannon index is less than or equal to the full index. Formally, we have

$$I_c(\hat{M},\hat{N})\le I_c \ . \tag{3.8}$$

This expresses the fact that the information contained in the correlation between any two observables cannot exceed the total information content of the correlation between the systems. In other words, a measurement of any given pair of observables will not yield more information than is contained within the two-system state itself. The second, more interesting property, is perhaps not so obvious. In Appendix B we prove the result that for a pure state of the two-component system the Shannon index of correlation for a given pair of observables is less than or equal to half the index of correlation for the state. Formally, this relation is written as

$$I_c(\hat{M},\hat{N})\le \tfrac{1}{2}I_c(\text{pure}) \ . \tag{3.9}$$

This is an important and fundamental result. For pure states of the two-component system any measurement on a single pair of correlated observables can, *at most,* give precisely half the information content about the correlation between the states. This result implies that to test whether a state is optimally correlated, at least three observables must be measured, rather than just the two necessary to give the maximum Shannon index. Any pair of observables which saturate inequality (3.9) are the maximally correlated observables for that state. Furthermore, no other pair of observables will saturate this inequality so that there is a *unique* pair of observables which are maximally correlated. This follows from the result (proved in Appendix B) that any pure-state wave function of two systems may be written in the form

$$|\psi\rangle=\sum_n c_n|\mu_n\rangle\otimes|\nu_n\rangle \ , \tag{3.10}$$

where $|\mu_n\rangle$ and $|\nu_n\rangle$ are observable bases. For any pure state of a two-component system, one can always find a pair of observables which are maximally correlated (this includes the null result where only one of the c_n is nonzero). This result can be stated in another way (see Appendix B): Given a two-component pure state in which one reduced density operator is diagonal in some observable, the procedure of finding the basis in which the other reduced density operator is diagonal *also* gives the perfectly correlated observables of the two systems. This is an important result about correlated quantum systems.

One can also consider the Shannon index as a parameter which simply characterizes the independence of two probability distributions. The modified Araki-Lieb inequality (3.7) therefore holds for any two well-behaved probability distributions. Physically, this inequality states that the uncertainty of a joint distribution must be at least equal to the uncertainty of either of the component systems considered independently. It is then tempting to enlarge upon (3.7) to include classical distributions on phase space. This is fraught with difficulty but an insight into the distinction between classical and quantum correlations can be gained from the Shannon index applied to classical and quantum statistical-mechanical problems. We briefly remark here that at the level of a single pair of observables for a correlated system, there can be no evidence for the quantum nature of the correlation. Thus, all correlations between a given pair of observables can be expressed purely in terms of probability distributions in much the same way as we could describe a classical ensemble. The expressions (3.8) and (3.9) show that to access the region where quantum correlations are manifest, there are two requirements. Firstly, we must measure more than a single pair of observables and, secondly, we must ensure that the quantum state of the two systems is such that the associated index of correlation is more than half of the maximum possible index for the two systems. We shall speculate further on such issues in Sec. V.

In Sec. IV we shall consider a physical example of a correlated quantum system. The example we shall consider is the two-mode squeezed vacuum state of the electromagnetic field [21]. The correlations between the modes are directly responsible for many of the nonclassical features observed in these states [22,23]. The information-theoretic bounds on the correlations that we have derived above and elsewhere [12] reveal new and fundamental insights into the nature of the two-mode squeezed state.

IV. CORRELATIONS AND TWO-MODE SQUEEZING

The two-mode squeezed state belongs to a special class of minimum uncertainty states for which the quantum noise is unevenly distributed between the field quadratures [21]. These states have many unusual properties [6] and potential applications in low-noise communications [24]. One of the most fascinating features of the two-mode squeezed state is the strong correlation that develops between the modes. These correlations have been used to demonstrate that "error-free" communication is possible in the presence of noise [25]. Furthermore, it has been demonstrated, both theoretically [8,26] and experimentally [23,27] that these correlations violate classical bounds and are thus quantum mechanical in origin. Many important experiments and concepts dealing with the correlation in two-mode squeezed states have recently been reviewed [28]. The effect of measurement on the correlation between the modes has also been investigated [29]. In most of what follows we shall be exclusively concerned with the two-mode squeezed vacuum state which, for simplicity, we shall refer to as a "squeezed state." In cases where confusion may arise, however, we refer to the full description.

The two-mode squeezed vacuum state can be expanded in a number state basis as

$$|r,\phi\rangle=(\cosh r)^{-1}\sum_{n=0}^{\infty}(\tanh r)^n e^{in\phi}|n\rangle_a|n\rangle_b \ , \tag{4.1}$$

where a and b refer to the two field modes. It is easy to see from this expression that the photon number in each mode is perfectly correlated with the photon number in the other mode. It is not entirely obvious, however, that the correlation between the modes is responsible for the squeezing properties. We note here that squeezing is a "nonclassical" property in that such a property is incompatible with the description of light by a classical Maxwell field [7]. The strong intermode correlations also give rise to properties which are incompatible with the notion of local realism [8,23]. Thus, both types of nonclassical behavior mentioned in the Introduction coexist in squeezed states and are both caused by the strong correlation between the modes. However, the existence of one of these quantum features does not guarantee the presence of the other. We shall delay any consideration of the violation of classical local realism by squeezed states until Sec. V where we shall speculate on the use of information theory to distinguish between classical and quantum correlations.

As we have mentioned above, it is the strong correlation between the modes in a two-mode squeezed state which is directly responsible for the squeezing properties. In order to show this, we shall follow an earlier analysis [30], although with the slight modification that our present discussion is recast in terms of the correlation coefficient. Let us consider two general quantum systems, labeled as above by the suffices a and b, and let $\hat{M}$ and $\hat{N}$ be operators in the a and b systems, respectively. The correlation coefficient, which provides a variance-based measure of the degree of correlation between these operators is defined in terms of the covariance

$$\Delta(\hat{M},\hat{N})=\langle\hat{M}\hat{N}\rangle-\langle\hat{M}\rangle\langle\hat{N}\rangle$$

and the variances (denoted by $\Delta^2 M=\langle\hat{M}^2\rangle=\langle\hat{M}\rangle^2$, for example) and is given by

$$\xi(\hat{M},\hat{N})=\frac{\langle\hat{M}\hat{N}\rangle-\langle\hat{M}\rangle\langle\hat{N}\rangle}{[(\langle\hat{M}^2\rangle-\langle\hat{M}\rangle^2)(\langle\hat{N}^2\rangle-\langle\hat{N}\rangle^2)]^{1/2}}$$
$$=\frac{\Delta(\hat{M},\hat{N})}{\Delta M\Delta N}\ . \tag{4.2}$$

The variance in the sum and difference operators is given by

$$\Delta^2(M\pm N)+\Delta^2 M+\Delta^2 N\pm 2\Delta M\Delta N\xi(\hat{M},\hat{N})\ , \tag{4.3}$$

so that if the fluctuations in $\hat{M}+\hat{N}$ are reduced by the correlation, the fluctuations in $\hat{M}-\hat{N}$ are enhanced. This leads to a sort of uncertainty relation between these variances [30]. Let us now consider the case of two field modes, again labeled by the suffices a and b. The X quadrature operators for the modes are defined by

$$\hat{X}_j=\tfrac{1}{2}(\hat{a}_j+\hat{a}_j^\dagger)\ , \tag{4.4}$$

where $j=a,b$ and $\hat{a}_j$ is the annihilation operator for mode j. These quadratures are unsqueezed and in fact display thermal characteristics [31,30]. The joint-mode quadrature operators defined by

$$\hat{X}^{\pm}=\hat{X}_a\pm\hat{X}_b \tag{4.5}$$

can be squeezed, however, if the quadrature correlation coefficient satisfies the following relationship:

$$1\pm\xi(\hat{X}_a,\hat{X}_b)<(2\bar{n}+1)^{-1}\ , \tag{4.6}$$

where $\bar{n}$ is the mean number of photons in each mode. The correlation coefficient is given by

$$\xi(\hat{X}_a,\hat{X}_b)=-\tanh 2r\cos\phi$$

so that for $\phi=0$ and in the limit of large squeezing the single-mode X quadratures become perfectly anticorrelated. Expressing $\bar{n}$ in terms of the squeezing parameter r, we find, as expected, that for $\phi=0$ the inequality is always satisfied for the squeezed state (4.1). We have therefore shown that the squeezing properties of this state are manifest in operators which act in both mode spaces and that it is the correlation between the modes which is directly responsible for these properties.

As we have mentioned above, the most correlated state of two similar quantum systems is a pure state in which each of the component systems is thermal in character. Such states are known as thermofield states [19] and represent the most correlated state possible for two systems subject to a mean energy constraint [12]. The two-mode squeezed state is an example of a bosonic thermofield state [20]. The main utility of the thermofield formalism is that it provides a way of describing thermal ensemble averages as pure-state expectations. This is achieved through the introduction of an identical "fictitious" system which is correlated with the thermal system of interest, or the "real" system. Thus, for many calculations the thermofield formalism is purely a calculational technique and the fictitious system has no physical significance. For the two-mode squeezed state, however, both the real and the fictitious systems are in fact physical field modes.

The pure thermofield vacuum state, denoted by $|0(\beta)\rangle$, is constructed so that the expectation value of any operator $\hat{A}$ on the real system alone reproduces the thermal ensemble average so that

$$\langle 0(\beta)|\hat{A}|0(\beta)\rangle=Z^{-1}(\beta)\sum_n\langle n|\hat{A}|n\rangle e^{-\beta E_n}\ , \tag{4.7}$$

where the state $|n\rangle$ is the "real"-space energy eigenstate with energy E_n. If we label the "fictitious" system by a tilde, then the thermofield vacuum state which reproduces the above expectation value can in general be written as

$$|0(\beta)\rangle=Z^{-1/2}(\beta)\sum_n\exp(i\phi_n-\tfrac{1}{2}\beta E_n)|n,\tilde{n}\rangle\ . \tag{4.8}$$

systems. For the squeezed state (4.1) the phase is given by the relation $\phi_n=n\phi$. The index of correlation between the real and fictitious systems is given by the expression [12]

$$I_c=2[\beta\langle H\rangle+\ln Z(\beta)]\ .$$

This index of correlation is the general expression for any pure state of two quantum systems which can be described by a state of the form (4.8). Consequently, within the thermofield formalism it is possible to construct an unlimited number of two-mode states which are maximally correlated in that they maximize the index of correlation. However, the mere ability to construct such states does not guarantee their physical relevance or the property of squeezing. We shall therefore limit our attention to the specific thermofield state which generates the two-mode squeezed vacuum state. We show in Appendix C that if, in addition, to the mean energy constraint we impose a mean amplitude constraint, then the displaced two-mode squeezed state is the most strongly correlated state of the two field modes.

The index of correlation for the squeezed state (4.1) is given by the expression [12]

$$I_c = 2[\cosh^2 r)\ln(\cosh^2 r) = (\sinh^2 r)\ln(\sinh^2 r)] \ . \tag{4.9}$$

The information is contained in the photon-number correlations and in phase-dependent correlations of, for example, the field quadratures. The simplest way to obtain the number correlation between the modes is to average the density operator over the squeezing angle ϕ. This is the approach previously used [12]. However, we shall, by way of example, calculate the Shannon index of correlation in the manner prescribed in Sec. III. The number-state matrix elements of the squeezed-state density operator are, from (4.1), given by

$$\begin{aligned} {}_b\langle n|\,{}_a\langle n'|\rho|n'\rangle_a\,|n\rangle_b &= (\cosh r)^{-2}\tanh^{2n} r\,\delta_{n,n'} \ , \\ {}_a\langle n|\rho_a|n\rangle_a = {}_b\langle n|\rho_b|n\rangle_b &= (\cosh r)^{-2}\tanh^{2n} r \ . \end{aligned} \tag{4.10}$$

If we denote the photon-number operator for a field mode by $\hat{N}$, then the Shannon index of correlation for the photon number is, from (3.1)–(3.3), given by

$$I_c(\hat{N}_a,\hat{N}_b) = -\sum_{n=0}^{\infty}\left[\frac{\tanh^{2n} r}{\cosh^2 r}\right]\ln\left[\frac{\tanh^{2n} r}{\cosh^2 r}\right] = \tfrac{1}{2}I_c \ . \tag{4.11}$$

The photon numbers are perfectly correlated and, as such, carry precisely half the information about the correlation between the modes. As we have previously shown [12], the two-mode squeezed state is an optimally correlated state of the two field modes and the above result for the photon-number correlation gives an explicit example of the general relation (3.8). The photon numbers are the optimally correlated observables for the squeezed state (4.1). Other observables will also be correlated, but no other pair of observables will be as strongly correlated as the photon numbers for the modes. This is reflected in the variance of the photon-number difference between the modes which is given by [30]

$$\Delta^2(N_a - N_b) = 0 \ . \tag{4.12}$$

This result is a nonclassical prediction in the sense that it cannot be reproduced by classical Maxwell theory. If we consider two phase-locked coherent states, the fluctuation in the photon-number difference is Poissonian. It has been recently shown that the sum of the phases of the two field modes in a two-mode squeezed vacuum state locks to the phase of the squeezing parameter [32]. This reflects the strong phase correlation between the modes. To determine this phase correlation, we require the elements of the relevant density operators in the phase basis.

The properties of the Hermitian optical phase operator have been described in detail elsewhere [33]. The operator exists in an $(s+1)$-dimensional state space Ψ spanned either by $(s+1)$ number states, $|n\rangle$, or $(s+1)$ orthonormal phase states. The phase states can be expanded in terms of the number states as

$$|\theta_m\rangle = (s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle \qquad (m=0,1,2,\ldots,s) \ . \tag{4.13}$$

The $(s+1)$ phase values θ_m are equally spaced and lie in the range $[\theta_0,\theta_0+2\pi)$. Thus, we have

$$\theta_m = \theta_0 + \frac{2\pi m}{(s+1)} \ . \tag{4.14}$$

The value of θ_0 is arbitrary, although care must be taken in the choice of this quantity in order to avoid spurious "window" effects [33]. We emphasize that the limit $s\to\infty$ is taken only after c numbers, such as expectation values, have been calculated. Physical results are obtained in this limit.

The matrix elements required for a calculation of the Shannon index for the phase are given, in the limit of large s, by

$$\begin{aligned} {}_a\langle\theta_m|\rho_a|\theta_m\rangle_a &= {}_b\langle\theta_{m'}|\rho_b|\theta_{m'}\rangle_b = \frac{1}{(s+1)} \ , \\ {}_b\langle\theta_{m'}|\,{}_a\langle\theta_m|\rho|\theta_m\rangle_a\,|\theta_{m'}\rangle_b &= \frac{[(s+1)\cosh r]^{-2}}{1+\tanh^2 r - 2\tanh r\cos(\phi-\theta_m-\theta_{m'})} \ . \end{aligned} \tag{4.15}$$

To avoid confusion with the squeezing angle ϕ, we denote the phase operator by $\hat{\Phi}$ [34] so that the Shannon entropies in phase for the individual modes become

$$S_a(\hat{\Phi}_a) = S_b(\hat{\Phi}_b) = \ln(s+1) \ . \tag{4.16}$$

This is the maximum possible phase entropy and results from a completely random phase distribution. This is consistent with the fact that each of the modes are thermal in character and individually possess no phase information. The calculation of the joint phase entropy is more difficult. However, if we take the continuum limit so that θ_m and $\theta_{m'}$ become continuous variables and we use the integral relationships [35]

$$\begin{aligned} &\int_{2\pi} d\theta\,\frac{1}{1+\tanh^2 r - 2\tanh r\cos(A-\theta)} = 2\pi\cosh^2 r \ , \\ &\int_{2\pi} d\theta\,\frac{\ln[1+\tanh^2 r - 2\tanh r\cos(A-\theta)]}{1+\tanh^2 r - 2\tanh r\cos(A-\theta)} = -4\pi\cosh^2 r\,\ln(\cosh^2 r) \ , \end{aligned} \tag{4.17}$$

then the joint phase entropy becomes

$$S(\hat{\Phi}_a,\hat{\Phi}_b) = 2\ln(s+1) - \ln(\cosh^2 r) \ . \tag{4.18}$$

The Shannon index of correlation for the mode phases then becomes

$$I_c(\hat{\Phi}_a,\hat{\Phi}_b)=\ln(\cosh^2 r) \ . \tag{4.19}$$

The density-of-states function cancels in the Shannon index but not in the individual entropies. This is a particularly nice feature of this measure of correlation.

We note that

$$I_c(\hat{N}_a,\hat{N}_b)>I_c(\hat{\Phi}_a,\hat{\Phi}_b)$$

so that the photon-number operators are more strongly correlated than the phase operators for the modes. This is expected and arises because a measurement of $\hat{N}_a$ immediately and precisely predicts the result of a measurement of $\hat{N}_b$. The difference between these Shannon indices is given by

$$I_c(\hat{N}_a,\hat{N}_b)-I_c(\hat{\Phi}_a,\hat{\Phi}_b)=\sinh^2 r\,\ln(\coth^2 r) \tag{4.20}$$

and in the limit $r\to\infty$ this difference tends to unity. This reflects the fact that in this limit the two phases become strongly locked [32]. The large r forms of these Shannon indices are given by

$$I_c(\hat{N}_a,\hat{N}_b)=I_c(\hat{\Phi}_a,\hat{\Phi}_b)=2r \ , \tag{4.21}$$

so that while the two indices have the same limiting form for large r, the difference between them tends to unity.

We have seen that the photon numbers between the modes are perfectly correlated and that the phase correlation between the modes is essentially perfect at large r. It is important to determine the correlation between the phase of one mode and the photon number of the other. The required single-mode matrix elements are given above but the joint entropy requires the matrix element of the full two-mode density operator and is given by

$${}_a\langle n|{}_b\langle\theta_m|\rho|\theta_m\rangle_b|n\rangle_a=\frac{1}{(s+1)}\left[\frac{\tanh^n r}{\cosh r}\right]^2 . \tag{4.22}$$

The joint entropy for the number and phase is therefore given by

$$S(\hat{N}_a,\hat{\Phi}_b)=S_a(\hat{N}_a)+S_b(\hat{\Phi}_b) \ . \tag{4.23}$$

The Shannon index of correlation is, therefore, zero—that is,

$$I_c(\hat{N}_a,\hat{\Phi}_b)=0 \ . \tag{4.24}$$

This result is easily understood: Measuring the number (phase) of one mode yields *precisely zero* information about the phase (number) of the other mode. For example, if mode a is found to have n photons, then mode b is in a number state with n photons—that is, a state of random phase. It can be shown that $\hat{N}$ and $\hat{\Phi}$ are uncorrelated directly from the density operator. We have the factorization property

$${}_a\langle n|{}_b\langle\theta_m|\rho|\theta_m\rangle_b|n\rangle_a=P_a(n)P_b(\theta_m)$$

so that the photon-number statistics for one mode are independent of the phase statistics for the other.

The Shannon index for the quadratures can also be calculated in a straightforward way from the diagonal elements of ρ in the X quadrature basis. For the state (4.1) with $\phi=0$, these diagonal elements are given by

$$\langle X_a,X_b|\rho|X_a,X_b\rangle =\pi^{-1}\exp[\cosh2r(X_a^2+X_b^2)+2\sinh2r X_a X_b] \ . \tag{4.25}$$

Instead of summations we now have integrals over the continuous variables X_a and X_b. As before, we shall find that any density-of-states factors cancel in the index of correlation. The entropies for the single-mode quadratures are easily calculated to be

$$S_a(\hat{X}_a)=S_b(\hat{X}_b)=\tfrac{1}{2}[\ln(\pi\cosh2r)+1] \tag{4.26}$$

and a slightly more involved calculation gives the joint entropy for the single-mode quadratures as

$$S(\hat{X}_a,\hat{X}_b)=\ln(\pi)+1 \ . \tag{4.27}$$

The Shannon index for the X quadratures then becomes

$$I_c(\hat{X}_a,\hat{X}_b)=\ln(\cosh2r) \tag{4.28}$$

and in the large-r limit this has the value $2r$. The above Shannon index does not depend on whether we measure the squeezed or unsqueezed quadrature; the Y quadratures are as strongly correlated as the X quadratures. Furthermore, a similar calculation shows that

$$I_c(\hat{X}_a,\hat{Y}_b)=I_c(\hat{Y}_a,\hat{X}_b)=0 \ . \tag{4.29}$$

The quadratures are therefore not cross correlated. The quadrature correlation is stronger than the phase correlation between the modes and we find that

$$I_c(\hat{N}_a,\hat{N}_b)=\tfrac{1}{2}I_c>I_c(\hat{X}_a,\hat{X}_b)>I_c(\hat{\Phi}_a,\hat{\Phi}_b) \ . \tag{4.30}$$

In the limit of large squeezing the difference between the Shannon indices of the above correlated observables becomes negligible and the number, phase, and quadrature correlations become equally strong.

The two-mode squeezed vacuum state is an important state of the two-mode field. Subject to a mean energy constraint, there are *no other* quantum states of the two-mode system which yield a stronger correlation. This places the two-mode squeezed state in a very special position in quantum optics. We shall speculate in Sec. V that the two-mode squeezed state belongs to a special class of two-system states which, to some extent, display maximal violations of local realism. Setting speculation aside, we have shown that the two-mode squeezed state is a maximally correlated quantum state (subject to the energy constraint) and that the photon-number operators for the modes form the most correlated observables. We have also explicitly calculated the correlations between the mode phases and between the mode quadratures. We find, as expected, that these observables are not as strongly correlated as the photon numbers.

V. DISCUSSION

In the preceding sections we have developed an information-theoretic approach to correlated quantum systems. We have applied this to correlations of quantum states and to correlations of specific observables. Throughout, we have purposely restricted our treatment to quantum systems comprised of two-component systems. It is apparent that the index of correlation allows us to determine some fundamental and general relations between the correlations of observables and the correlation of states. One of the most philosophically challenging features of correlated quantum systems is their apparent violation of local realism. For such correlated systems, observation of one component system appears to cause an instantaneous response in its partner. These correlations are not fully characterized by sets of observation probabilities between pairs of observables [36]. In this section we suggest that quantum mechanics enables systems to be *twice* as strongly correlated as classical physics would allow and that the source of this excess correlation is the superposition principle for probability amplitudes. The index of correlation allows us to speculate on the boundary between quantum and classical correlations, but the bounds derived therefrom are not as strong as those of the Bell inequalities. We should remark here that an information-theoretic treatment of the Bell inequalities has been derived [37].

In order to discuss the distinction between classical and quantum correlations, it is important to have a measure of correlation between two classical systems. By "classical" physics we mean, in this context, a local realistic theory that is complete without recourse to hidden variables. Classical ensembles are defined in terms of distributions on phase space and an entropy can be associated with such distributions. We note, however, that there can be considerable difficulties in defining entropies of continuous variables in classical mechanics and some of these are addressed in the review by Wehrl [38]. Guided by the fact that, for both continuous and discrete variables in quantum mechanics, the density-of-states factors cancel in the index of correlation, we shall assume a similar property in a classical description. With these remarks in mind, we are in a position to describe a "classical" index of correlation I_c^{cl}, which gives the information content of the correlation between two classical systems. This index is given by Eq. (2.6) where the entropies are now defined through classical distributions on phase space.

In order to compare classical and quantum regimes, we must first introduce a figure of merit for characterizing the strength of correlation in both classical and quantum systems. We define the quantum and classical figures of merit, respectively, by

$$\eta^{\mathrm{qu}}=\frac{I_c}{S^{\max}}=\frac{I_c}{S_a^{\max}+S_b^{\max}}\ ,$$
$$\eta^{\mathrm{cl}}=\frac{I_c}{\tilde{S}^{\max}}=\frac{I_c^{\mathrm{cl}}}{\tilde{S}_a^{\max}+\tilde{S}_b^{\max}}\ . \tag{5.1}$$

We have used the notation that superscript max refers to the maximum entropy of the system subject to any physical constraints. $S^{\max}$ is therefore the maximum possible entropy of the two-component system subject to any physical constraints. The tilde is used to remind us that we are dealing with a classically derived entropy and the comments of the previous paragraphs apply. This figure of merit suffers from the potentially serious problem that any density-of-states factors will no longer cancel as they did for the index of correlation. This should be borne in mind in the following discussion, especially where one may be faced with classical entropies of continuous variables. We remark that this problem does not occur in the quantum-mechanical definition of entropy so that the figure of merit defined above is a sensible measure and any general conclusions we may come to concerning correlated quantum systems will be valid.

The figures of merit η are bounded in the quantum and classical cases by

$$\eta^{\mathrm{qu}}\le 1\ ,$$
$$\eta^{\mathrm{cl}}\le \tfrac{1}{2}\ . \tag{5.2}$$

The crucial difference here is that while a classically correlated system, defined in terms of probability distributions, must display a disorder at least equal to either of its components, a quantum system can have a *zero* total entropy. It is the superposition principle for probability amplitudes that allows us to make this distinction in quantum mechanics. The measurement of an observable of an ensemble of identically prepared systems will, in general, lead to a spread of different results. In classical physics the entropy of the system will be greater than or equal to the entropy associated with this distribution of results. However, a quantum system may be in a pure-state superposition of the eigenstates of the measured observable. The entropy of the ensemble is zero but measurement of the chosen observable produces a distribution of different results. Once again, we emphasize that it is the appearance of probability amplitudes in quantum mechanics which implies that quantum-mechanical systems obey the quantum Araki-Lieb inequality (2.8) rather than the "classical" Araki-Lieb inequality (3.7). This leads us to the speculative conclusion that quantum mechanics allows systems to be *twice as strongly correlated as classical mechanics.*

Unfortunately, this information-theoretic distinction, while providing valuable insights, is of little immediate use in assessing the possible violation of local realism. It is clear from the above inequalities (5.2) that conventional classical mechanics cannot access the region $\frac{1}{2}<\eta\le 1$ which is precisely the region in which we expect to find violations of Bell's inequalities. However, the strength of correlation can be enhanced in classical mechanics at the expense of the introduction of hidden variables. In order, then, to determine the index of correlation and the figure of merit for the system, we would need to include this extra information associated with the hidden variables. Naturally, we could not obtain this additional information by direct observation of the systems. Thus the quantum-classical bounds we have described above are of little operational use in distinguishing between quantum mechanics and a local hidden-variable version of classical mechanics. In other words, any experiment we might consider to test local realism cannot access the information about the correlation associated with the hidden variables; the only information about correlations accessible to experiment is that contained within the real, physical variables. It is therefore possible to construct a hidden-variable theory for a correlated system which will, experimentally, yield the same index of correlation as that of the quantized system [16]. For this reason the above quantum-classical bound is not as strong as those of the Bell inequalities, or indeed the information-theoretical bounds derived by Braunstein and Caves [37]. However, the bounds we have suggested give a clear insight into the origin of the distinction between conventional classical mechanics and quantum mechanics. It is precisely the extra information inherent in the superposition of probability amplitudes in quantum mechanics which allows a quantum ensemble to have zero entropy.

An optimally correlated quantum system must be in a pure state, but measurement of a specific pair of individual system observables cannot reveal the existence of the pure state. This means that measurement of *one* property of each of a correlated pair of systems will not distinguish between the classical and quantum regimes. It is only by also measuring other noncommuting observables that the quantum nature of the correlation can be established. As we have remarked above, one can supplement classical physics with hidden variables which may then admit the existence of correlations in the nonclassical regime. Tests to distinguish between quantum mechanics and rival classical theories must therefore probe the region $\frac{1}{2} < \eta \leq 1$. In order to test quantum mechanics in this regime, we need to measure more than a single pair of observables. An example of such a test is provided by the Bell inequalities which distinguish between quantum mechanics and local hidden-variable theories by considering more than one observable for at least one of the correlated systems [3]. The index of correlation has highlighted the fact that it is the pure-state entanglement of probability amplitudes, for which there is no classical analog, which admits this extra correlation in quantum mechanics.

Let us now consider the quantum figure of merit for correlated systems, leaving aside the question of any classical or quantum bounds. As we have already indicated, the quantum definitions of entropy do not suffer from the limitations of classical entropies. The quantum figure of merit for the correlation between two systems is, therefore, a general parameter for quantum-mechanical correlations *irrespective* of its comparison with any classical measures. A correlated quantum system prepared in a state which saturates the inequality (5.2) so that $\eta = 1$ belongs to a special class of quantum states: these states are the maximally correlated states that quantum mechanics will allow. In other words, given a state of general two-component system with $\eta = 1$, we cannot find another state of *any two quantum systems* which is more strongly correlated. The two-mode squeezed state (4.1) saturates this inequality so that $\eta(\text{squeezed}) = 1$. This places the squeezed states among a very special class of two-component quantum states.

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APPENDIX A

In this appendix we prove the inequality

$$I_c \geq I_c(\hat{A}, \hat{B}) \ . \tag{A1}$$

The proof of this relation depends on three theorems:

(1) The first theorem is a property of the relative entropy which we quote without proof. The theorem is contained in the review article by Wehrl [38]. This theorem states that for density operators ρ and σ in a product space $H_1 \otimes H_2$, the relative entropy satisfies the relation

$$S(\sigma|\rho) \geq S(\sigma_1|\rho_1) \ , \tag{A2}$$

where we have written

$$S(\sigma|\rho) = \mathrm{Tr}\rho(\ln\rho - \ln\sigma)$$

and the subscript 1 implies that a trace has been taken over the Hilbert space H_2.

(2) The second theorem states that if the Hilbert space H_1 is doubled to make $H_1 \otimes \tilde{H}_1$ and the density matrix elements ρ_{mn} are replaced with ρ_{mn} so that $|m\rangle \to |m, \tilde{m}\rangle$, then the entropy remains unchanged. This follows quite simply from the fact that making the transformation from the single space to the doubled space in the way prescribed *leaves the density matrix elements unchanged.* Thus, $\rho(H_1)$ and $\rho(H_1 \otimes \tilde{H}_1)$ have the same eigenvalue spectrum [39].

(3) The third theorem simply states that given a density matrix in the double space, that is,

$$\rho' = \sum_n \sum_n \rho_{mn} |m, \tilde{m}\rangle\langle n, \tilde{n}| \ , \tag{A3}$$

then the reduced density operator in the space H_1 is the diagonal in the $|m\rangle$ basis. This is trivial to prove from (A3).

Let us now consider a Hilbert space $H_a \otimes H_b$ with a state on this space described by a density operator ρ. Let us also construct a density operator σ from the reduced density operators so that $\sigma = \rho_a \otimes \rho_b$. The relative entropy $S(\sigma|\rho)$ is then identically equal to the index of correlation. We now write the density operators ρ and σ in a particular basis so that, for example,

$$\rho = \sum_A \sum_{A'} \sum_B \sum_{B'} \rho_{A,A';B,B'} |A, B\rangle\langle A', B'| \ . \tag{A4}$$

Now we double the basis so that the new density operators are in the space $H_a \otimes \tilde{H}_a \otimes H_b \otimes \tilde{H}_b$. We have the new density operator ρ' in this space given by

$$\rho' = \sum_A \sum_{A'} \sum_B \sum_{B'} \rho_{A,A';B,B'} |A, \tilde{A}\rangle \otimes |B, \tilde{B}\rangle\langle A', \tilde{A}'| \otimes \langle B', \tilde{B}'| \ . \tag{A5}$$

Now we have from theorem (2) above that $S(\rho)=S(\rho')$ and from theorem (3) we have $S(\rho')=S(\hat{A},\hat{B})$. Similar results hold for σ. Tracing over the tilde bases, we find that

$$\begin{aligned} &\mathrm{Tr}_t\rho'=\rho(\hat{A},\hat{B})\ ,\\ &\mathrm{Tr}_t\sigma'=\rho_a(\hat{A})\otimes\rho_b(\hat{B})\ , \end{aligned} \tag{A6}$$

and we have used the notation that subscript t refers to the tilde system and that $\rho(\hat{A},\hat{B})$ is the density operator expressed in the basis defined by these operators. We then find the relation, from theorem (1), that

$$S(\sigma|\rho)\geq S(\rho_a(\hat{A})\otimes\rho_b(\hat{B})|\rho(\hat{A},\hat{B}))\ , \tag{A7}$$

which is equivalent to the inequality (A1). The theorem is thus proven.

APPENDIX B

In this appendix we prove the result that measurement of a single pair of correlated observables, for a pure state of a two-component system, can, at most, give only half the information about the correlation between the states. In order to prove this, we first need to prove the result that any two-component pure state may be written in the form

$$|\psi\rangle=\sum_n \pi_n|\mu_n\rangle\otimes|\nu_n\rangle\ . \tag{B1}$$

This tells us that for any pure state of a two-component system one can always find a pair of observables which are maximally correlated.

Consider the general expression for a pure state of two quantum systems:

$$|\psi\rangle=\sum_i\sum_j c_{ij}|\mu_i\rangle_a|\Omega_j\rangle_b\ , \tag{B2}$$

where we assume that the above expansion is in an observable basis derived from Hermitian operators so that $\hat{\mu}|\mu_i\rangle=\mu_i|\mu_i\rangle$ and $\hat{\Omega}|\Omega_j\rangle=\Omega_j|\Omega_j\rangle$ with these states being orthonormal. Let us now define new, normalized states in the b system by

$$\begin{aligned} &|\nu_i\rangle_b=\frac{1}{\pi_i}\sum_j c_{ij}|\Omega_j\rangle_b\ ,\\ &\pi_i=\left[\sum_j|c_{ij}|^2\right]^{1/2}\ . \end{aligned} \tag{B3}$$

In general, the $|\nu_i\rangle_b$ do not describe an orthogonal basis and are thus not eigenstates of an observable. The overlap between these states is given by

$${}_b\langle\nu_k|\nu_i\rangle_b=\frac{1}{\pi_i\pi_k}\sum_j c_{kj}^*c_{ij}\ . \tag{B4}$$

These states are only orthonormal if the coefficients satisfy the relation

$$\sum_j c_{kj}^*c_{ij}=\delta_{ik}\pi_i^2\ . \tag{B5}$$

The reduced density operator for the a system is given by the expression

$$\rho_a=\sum_i\sum_j\sum_k c_{ij}c_{kj}^*|\mu_i\rangle\langle\mu_k|\ , \tag{B6}$$

so that if we now assume that the $|\mu_k\rangle$ form the basis in which ρ_a is diagonal, then Eq. (B5) must be satisfied. Thus the observables represented by the operators $\hat{\mu}$ and $\hat{\nu}$ are the maximally correlated observables for the two-component system. The reduced density operators are diagonal in the bases of these observables. The assertion of Eq. (B1) is now proven.

Let us now consider two quantum systems labeled by the suffixes a and b. Using the property that the Shannon entropy for an observable in one of these systems is greater than or equal to the total entropy, we find that the Shannon index of correlation obeys the inequality

$$I_c(\hat{A},\hat{B})\geq S_a+S_b-S(\hat{A},\hat{B})\ , \tag{B7}$$

where $\hat{A}$ and $\hat{B}$ represent observables in the a and b systems, respectively. Using the modified Araki-Lieb inequality (3.7), we find that the maximum Shannon index is just given by

$$I_c^{\max}(\hat{A},\hat{B})=S_a(\hat{A})=S_b(\hat{B})=S(\hat{A},\hat{B})\ , \tag{B8}$$

so that the left-hand side of inequality (B7) can be replaced by $I_c^{\max}(\hat{A},\hat{B})$ and is saturated precisely when $S_a=S_a(\hat{A})$ and $S_b=S_b(\hat{B})$. However, we see from (B1) that this occurs when $\hat{A}$ and $\hat{B}$ determine the bases in which ρ_a and ρ_b are diagonal. This is sufficient to determine that the maximal Shannon entropy for these observables is precisely half that of the index of correlation for the pure state. The inequality (3.9) is therefore proven.

APPENDIX C

In order to determine the most strongly correlated state of the two-mode field subject to both a mean energy and a mean amplitude constraint, we must maximize the single-mode entropy subject to the following constraints:

$$\begin{aligned} &\mathrm{Tr}(\rho)=1\ ,\\ &\mathrm{Tr}(\hat{a}^\dagger\hat{a}\rho)=\bar{n}\ ,\\ &\mathrm{Tr}(\hat{a}\rho)=\alpha\ , \end{aligned} \tag{C1}$$

where $\bar{n}=n_{\mathrm{th}}+|\alpha|^2$ and n_{th} is the number of thermal photons. Using the method of Lagrange multipliers, we find the variational equation

$$1+\ln\rho+\lambda+\beta\left[\hat{a}^\dagger\hat{a}+\frac{\gamma}{\beta}\hat{a}^\dagger+\frac{\gamma^*}{\beta}\hat{a}\right]=0\ , \tag{C2}$$

where the constants λ, β, and γ are to be determined. This condition can also be written as

$$1+\ln\rho+\lambda'+\beta\hat{A}^\dagger\hat{A}=0\ , \tag{C3}$$

where

$$\begin{aligned} &\lambda'=\lambda-\frac{|\lambda|^2}{\beta}\ ,\\ &\hat{A}=\hat{D}\left[\frac{\gamma}{\beta}\right]\hat{a}\hat{D}^{-1}\left[\frac{\gamma}{\beta}\right]\ , \end{aligned} \tag{C4}$$

and $\hat{D}$ is the Glauber displacement operator [40]. The new condition (C3) is precisely the equation we would obtain if we were required to maximize the entropy of a density operator $\tilde{\rho}$ subject to the constraints

$$\begin{aligned} &\mathrm{Tr}(\tilde{\rho})=1 \ , \\ &\mathrm{Tr}(\hat{A}^{\dagger}\hat{A}\tilde{\rho})=n_{\mathrm{th}} \ . \end{aligned} \tag{C5}$$

The constraint on the average photon number in (C5) is chosen to be consistent with the earlier constraints (C1). The solution of this is just the Boltzmann distribution given by

$$\tilde{\rho}=\frac{e^{-\beta\hat{A}^{\dagger}\hat{A}}}{\mathrm{Tr}[\exp(-\beta\hat{A}^{\dagger}\hat{A})]} \ . \tag{C6}$$

As the trace of an operator is invariant under a unitary transformation, we find that the density operator $\tilde{\rho}$ is given by

$$\tilde{\rho}=\hat{D}\left(\frac{\gamma}{\beta}\right)\rho_{\mathrm{th}}\hat{D}^{-1}\left(\frac{\gamma}{\beta}\right) , \tag{C7}$$

and we have written the thermal density operator as ρ_{th}. Two-mode states such that each of the single modes are described by density operators of the form (C7) are coherent thermofield states [41] and are formally identical to the displaced two-mode squeezed vacuum state. The most correlated two-mode state subject to a mean energy and amplitude constraint is the displaced two-mode squeezed vacuum state.

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