Three-mode treatment of a high-gain steady-state free-electron laser

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A description of the steady-state nonlinear behavior of a Compton free-electron-laser (FEL) amplifier is presented, in which the radiation field is written in both linear and nonlinear regimes as the sum of three coupled "normal modes." In the linear regime these modes reduce themselves to the noninteracting modes of the usual linear theory. The two linearly stable modes turn out to be nonlinearly unstable, because their amplitudes grow exponentially as the FEL enters into the nonlinear regime. We derive a self-contained system of three nonlinear dynamical equations for these modes in two different approaches. The system is then reduced to comparatively simple forms in which only cubic nonlinearities appear. In this way, we are able to describe the FEL saturation process and the first peak in the amplitude of the radiation field with a good degree of accuracy. The large-amplitude oscillations in the field intensity, which are the main feature of the FEL nonlinear dynamics, are also reproduced through the nonlinear coupling between the normal modes.

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I. INTRODUCTION

In this paper we investigate the possibility of describing the steady state of a single-pass high-gain Compton free-electron-laser (FEL) amplifier by means of three coupled variables, named "normal modes." Such a description holds both in the linear and in the nonlinear regime, where the radiation-field intensity exhibits a saturation peak and subsequent nearly periodic undamped oscillations.

A general description of a FEL system is achieved by numerical integration of the microscopic dynamical equations for N electrons ($N \gg 1$) and the radiation field [1]. The linearization of such equations is a good approximation for an analytical description of the collective instability in a high-gain FEL, which leads to the exponential growth of the radiation-field amplitude. In fact, this linearization leads to a well-known third-order linear differential equation for the radiation-field amplitude dynamics. Among the three solutions of this equation, one gives an exponentially growing contribution to the field intensity, the two other ones give exponentially decaying or constant contributions. However, this approach does not reproduce the saturation of the signal and the subsequent oscillations in the field intensity. These last features are due to the strong nonlinearity in the coupled electron-radiation system, corresponding to syncrotron oscillations of trapped electrons [2,3] in the "ponderomotive potential," determined by the combined radiation-wiggler field.

The saturation process of the linearly unstable electromagnetic signal was analytically reproduced by several authors [4–9] by reducing the problem to simplified nonlinear differential or integro-differential equations for the radiation field in several cases (small-gain, high-gain Compton-regime single-pass FEL and also oscillator FEL). In some cases [4,8,9] it was found that the saturation in the signal is reproduced by a Landau-Ginzburg equation for the radiation-field amplitude. These analyses predicted the saturation level, but could not reproduce the high-amplitude oscillations in the field intensity.

In order to also model this feature of the FEL nonlinear dynamics, we follow a rather different approach, which appears as a natural extension of the usual linear theory to the nonlinear regime. In fact, we present a "three-coupled-modes" description of the FEL dynamics whose basic assumption is that the radiation field is written both in the linear and in the nonlinear regime as the sum of three interacting "normal modes" or "oscillators." In the linear regime the three modes reduce to the usual independent solutions of the third-order differential equation for the field amplitude derived from the linear theory. Such modes are driven and coupled to each other by the nonlinear contributions to the electron-radiation FEL interaction, thus when the FEL system is in the linear regime these modes are uncoupled, but their linear dynamics is strongly modified as soon as the FEL enters into the nonlinear regime. The model is introduced in Sec. II, after a brief account of the microscopic equations and of the linear theory.

In Sec. III we discuss two different possible ways of reaching a self-contained dynamical description of such "oscillators," i.e., of writing a set of three nonlinear differential equations that describe the FEL dynamics both in the linear and in the nonlinear regimes and which involve only the three modes. After this, we introduce some simplifications of the two schemes, which are sufficient to account for the main features of our "coupled-modes" description. Finally, we obtain a description of the radiation field that is sufficiently correct both in amplitude and phase and is based on a very simple system of three coupled nonlinear equations.

In Sec. IV we conclude with a few remarks. Some details of the calculations are reported in Appendixes A and B.

II. BEHAVIOR OF THE "NORMAL MODES"

In a single-pass FEL a beam of relativistic electrons injected in a suitable magnetostatic field ("wiggler") interacts with a copropagating electromagnetic radiation field. Under proper conditions the system can behave as a high-gain amplifier, generating tunable, coherent, high-peaked power radiation. When propagation effects (due to the different velocities of the electron beam and of radiation) and space-charge effects are neglected, in the steady-state Compton regime, the FEL basic physics is well described by the following one-dimensional microscopic equations, based on the slowly varying envelope approximation for the radiation field [1,10]:

$$\frac{d\theta}{d\bar{z}}=p_j , \tag{1}$$

$$\frac{dp_j}{d\bar{z}}=-(Ae^{i\theta_j}+\text{c.c.})\equiv\frac{d^2\theta_j}{d\bar{z}^2} , \tag{2}$$

$$\frac{dA}{d\bar{z}}=\langle e^{-i\theta}\rangle+i\delta A \tag{3}$$

(where $j=1,\ldots,N$ and c.c. means "complex conjugate"). Note that such equations are written in dimensionless form according to the "universal scaling" adopted in Ref. [1], to which we refer for further details. Here we only recall that $\theta_j=(k+k_w)z_j-wt-2k_w\rho\delta z_j$ and $p_j=(1/\rho)(\gamma_j-\gamma_0)/\gamma_0$ are, respectively, the jth electron phase in the "ponderomotive" field, determined by the combined radiation-wiggler field, and the jth electron relative energy variation ($\gamma_j mc^2$ is the electron energy and $\gamma_0 mc^2$ is the initial energy of the electrons in the monokinetic beam). In the same scaling, $\bar{z}=2k_w\rho(\gamma_r^2/\gamma_0^2)z$ is the longitudinal coordinate, and A is the complex amplitude of the vector potential of the radiation field, from which we obtain the intensity of the electric field as $|E|^2=|A|^2 4\pi\rho n\gamma_0 mc^2$. Finally, $\delta=(1/2\rho)(\gamma_0^2-\gamma_r^2)/\gamma_r^2$ is the detuning parameter that indicates how far the electrons are from the resonance condition (an electron is said to be "resonant" when the frequency of the radiation emitted spontaneously matches that of the radiation to be amplified). In these definitions we introduced the following quantities: the "FEL parameter"

$$\rho=\frac{1}{\gamma_0}\left[\left[\frac{\gamma_0}{\gamma_r}\right]^2\frac{a_w\Omega_p}{4ck_w}\right]^{2/3} ,$$

the wiggler parameter $a_w=e\lambda_w B_w/2\pi mc^2$, the wiggler period $\lambda_w=2\pi/k_w$, the plasma frequency Ω_p, the electron density n, the frequency of the radiation field $\omega=ck=(2\pi/\lambda)c$, and the resonant electron energy $\gamma_r=[\lambda_r(1+a_w^2)/2\lambda]^{1/2}$.

The driving source of the radiation field is the electron "bunching parameter"

$$b\equiv\langle e^{-i\theta}\rangle\equiv\frac{1}{N}\sum_{j=1}^{N}e^{-i\theta_j} \tag{4}$$

that represents the microscopic average distribution along the coordinate z of a sample of N electrons in the beam.

The general description of the FEL dynamics, based on the numerical integration of Eqs. (1)–(3), shows that the FEL is an unstable system where collective effects occur. The electrons in the beam "self-bunch" on a radiation wavelength scale and the emitted radiation grows exponentially until nonlinear effects limit the conversion of the electron kinetic energy into radiation energy, leading to a saturation peak with subsequent oscillations in the field amplitude.

An analytical description of the exponential growth in the intensity of the electromagnetic signal inside the wiggler is obtained by a linear stability analysis performed on Eqs. (1)–(3) around the equilibrium state defined by $A_0=0$, $\langle e^{-i\theta}\rangle_0=0$, $p_0=0$. From the linearization of such equations we derive the approximate equation

$$A'''(\bar{z})-i\delta A''(\bar{z})-iA(\bar{z})=0 \tag{5}$$

(where the prime denotes $d/d\bar{z}$). Looking for solutions of the form $A(\bar{z})\propto e^{-ik\bar{z}}$, we see that the characteristic equation associated with Eq. (5) is the cubic $k^3+\delta k^2-1=0$. Since it admits two complex roots and one real root [for $\delta<(\frac{27}{4})^{1/3}$], the complex root with positive imaginary part gives the exponential growth in the complex amplitude $A(\bar{z})$. The contributions from the other roots are either merely oscillatory (corresponding to the real root) or exponentially decaying (corresponding to the complex root with negative imaginary part).

Hereafter we will restrict our discussion to the case of exact resonance, i.e., $\delta=0$. In this case the cubic equation reduces to the even simpler form $k^3=1$ and it has two "linearly stable" roots $k_1=1$ and $k_2=-\frac{1}{2}-i\sqrt{3}/2$, together with the "linearly unstable" root $k_3=-\frac{1}{2}+i\sqrt{3}/2$.

Our current purpose is to extend the linear theory to the nonlinear regime in a simple way. From the linear theory we know that the complete solution for the radiation field has the form $A(\bar{z})=\sum_{r=1}^{3}a_r e^{-ik_r\bar{z}}$, where a_r are suitable constants defined by the boundary conditions on system (1)–(3) given at $\bar{z}=0$ and k_r $(r=1,2,3)$ are the roots of the equation $k^3=1$. We assume that such an expression for the field $A(\bar{z})$ is also valid in the nonlinear regime. To do this, we let the amplitudes a_r of the three linearly independent solutions of Eq. (5) vary from their originally constant values. In other words, we define the functions

$$\bar{a}_r(\bar{z})=a_r(\bar{z})e^{-ik_r\bar{z}} \tag{6}$$

and write formally the radiation field, for all values of $\bar{z}$, as

$$\sum_{r=1}^{3}\bar{a}_r(\bar{z})=A(\bar{z}) . \tag{7}$$

The functions of $\bar{a}_r(\bar{z})$, which are obviously not completely determined by Eq. (7), can immediately be found by following a procedure that is similar to the usual method of the variation of the arbitrary constants. In fact, Eq. (7) is closed by taking its first and second derivatives with respect to $\bar{z}$, obtaining the two equations

$$\sum_{r=1}^{3} k_r \bar{a}_r(\bar{z}) = iA'(\bar{z}) \; , \tag{8}$$

$$\sum_{r=1}^{3} k_r^2 \bar{a}_r(\bar{z}) = -A''(\bar{z}) \; , \tag{9}$$

provided that the functions $a_r(\bar{z})$ are chosen in such a way that

$$\sum_{r=1}^{3} a_r'(\bar{z}) e^{-ik_r\bar{z}} = 0 \; , \tag{10}$$

$$\sum_{r=1}^{3} k_r a_r'(\bar{z}) e^{-ik_r\bar{z}} = 0 \; . \tag{11}$$

If we consider the right-hand side (rhs) of Eqs. (7)–(9) as given, such equations yield a linear algebraic nonhomogeneous system of three equations in the three unknown functions $\bar{a}_r(\bar{z})$. The inversion of such a system allows us to write

$$\bar{a}_r(\bar{z}) = \tfrac{1}{3}[A(\bar{z}) + ik_r^2 A'(\bar{z}) - k_r A''(\bar{z})] \; , \tag{12}$$

where we have used the equalities $\sum_{r=1}^{3} k_r = 0$, $k_r^3 = 1$, $k_1 k_2 k_3 = 1$, and $k_1 k_2 + k_2 k_3 + k_3 k_1 = 0$.

The exact behavior of the three functions $\bar{a}_r(\bar{z})$ is obtained substituting the numerical solutions of Eqs. (1)–(3) for $A(\bar{z})$, $A'(\bar{z})$, and $A''(\bar{z})$ and is reported in Figs. 1 and 2. The important point to note is that the amplitudes of the two "linearly stable modes" $\bar{a}_1(\bar{z})$ and $\bar{a}_2(\bar{z})$ start growing exponentially in a region where the total field amplitude is still small with respect to the saturation value.

Moreover, the growth rate of the functions $|\bar{a}_1(\bar{z})|$ and $|\bar{a}_2(\bar{z})|$ is equal to $3\sqrt{3}/2$, i.e., their growth is three times faster than the growth of the amplitude of the linearly unstable mode $\bar{a}_3(\bar{z})$ in the linear region. The "anomalous" growth of the stable modes is the decisive feature of our description, and it is analytically demonstrated in the following section. It is interpreted in terms of a pumping of the oscillator $\bar{a}_3(\bar{z})$ upon the linearly stable modes. The result is a forced exponential growth of $\bar{a}_1(\bar{z})$ and $\bar{a}_2(\bar{z})$.

Furthermore, note also that when $\bar{a}_1(\bar{z})$ and $\bar{a}_2(\bar{z})$ reach the first peak of saturation their amplitudes are comparable with that of $\bar{a}_3(\bar{z})$. At this point a new regime begins, where there are quasiperiodic energy exchanges between the three oscillators. We could say that the large-amplitude oscillations in the radiation-field intensity are precisely due to these energy exchanges between the three modes.

III. ANALYTICAL MODEL

We now introduce the analytical "coupled-modes" model, where the dynamics of $\bar{a}_r(\bar{z})$ is self-consistently assigned, instead of being written in terms of the numerical solutions of the microscopic system (1)–(3) as in Eq. (12). If we take a further derivative with respect to $\bar{z}$ in Eq. (9)

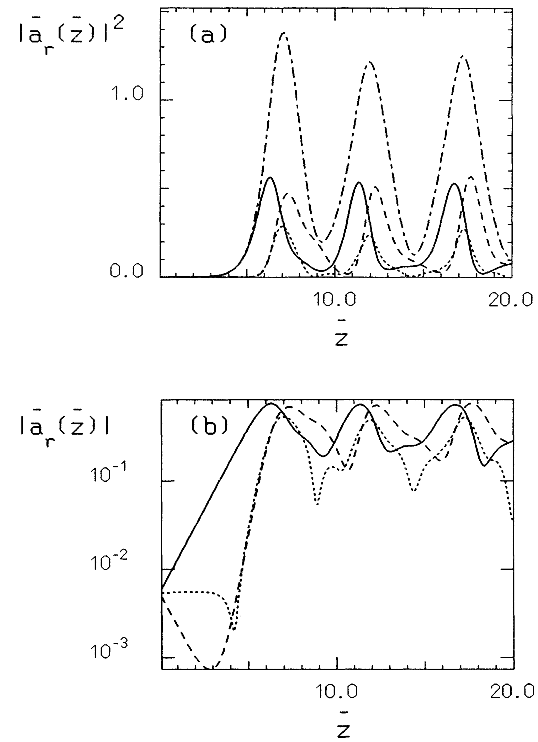

FIG. 1. (a) $|\bar{a}_r(\bar{z})|^2$ as a function of the dimensionless longitudinal coordinate $\bar{z}$ for $r=1$ (dotted line), $r=2$ (dashed line), $r=3$ (solid line). Also shown is the numerical solution of Eqs. (1)–(3) for the field intensity $|A(\bar{z})|^2$ (dot-dashed line). (b) $|\bar{a}_r(\bar{z})|$ vs $\bar{z}$ (log scale) for $r=1$ (dotted line), $r=2$ (dashed line), $r=3$ (solid line).

and recall that $k_r^3 = 1$, by means of Eq. (7) we find

$$\sum_{r=1}^{3} k_r^2 a_r'(\bar{z}) e^{-ik_r\bar{z}} = -A'''(\bar{z}) + iA(\bar{z}) \; . \tag{13}$$

Equations (10), (11), and (13) are a set of equations equivalent to system (7)–(9) and to system (1)–(3), and they provide a linear algebraic nonhomogeneous system of three equations in the three unknown functions $a_r'(\bar{z})$. If we define the function $\Phi(\bar{z})$ such that

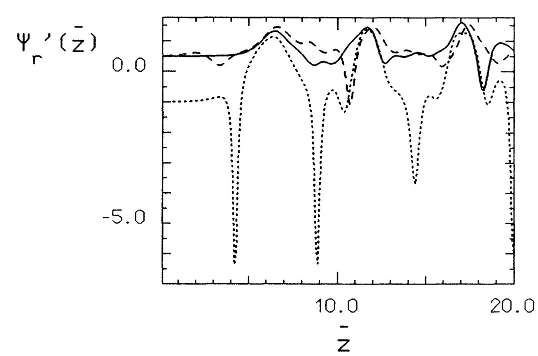

FIG. 2. Derivatives of the phases $\psi_r(\bar{z})$ of the three modes $\bar{a}_r(\bar{z}) \equiv |\bar{a}_r(\bar{z})| e^{i\psi_r(\bar{z})}$ vs $\bar{z}$ for $r=1$ (dotted line), $r=2$ (dashed line), $r=3$ (solid line).

$$\Phi(\bar{z})\equiv -iA'''(\bar{z})-A(\bar{z})\ , \tag{14}$$

the solution of the algebraic system of Eqs. (10), (11), and (13) is the following:

$$a_r'(\bar{z})=-\frac{i}{3}k_r\Phi(\bar{z})e^{ik_r\bar{z}}\ , \tag{15}$$

i.e., from definition (6),

$$\bar{a}_r'(\bar{z})=-ik_r[\bar{a}_r(\bar{z})+\tfrac{1}{3}\Phi(\bar{z})]\ . \tag{16}$$

Thus an analytical description of the FEL dynamics in terms of the functions $\bar{a}_r(\bar{z})$ is given if system (16) is analytically closed on the variables $\bar{a}_r(\bar{z})$.

Actually, note that the functions $a_r(\bar{z})$ are driven and coupled to each other by the quantity $\Phi(\bar{z})\equiv -iA'''(\bar{z})-A(\bar{z})$, which is the nonlinear contribution to the interaction between the N electrons and the radiation in the microscopic system (1)–(3). In fact, recalling that the average value of a dynamic variable $f(\theta_j,p_j)$ is defined as $\langle f(\theta,p)\rangle\equiv(1/N)\sum_{j=1}^{N}f(\theta_j,p_j)$, from system (1)–(3) we obtain the exact equation (involving the collective variables $\langle p^2e^{-i\theta}\rangle$ and $A^*\langle e^{-2i\theta}\rangle$)

$$\begin{aligned}\Phi(\bar{z})&\equiv -iA'''(\bar{z})-A(\bar{z})\\ &\equiv i\langle p^2e^{-i\theta}\rangle+A^*(\bar{z})\langle e^{-2i\theta}\rangle\ .\end{aligned} \tag{17}$$

This is the reason why in the linear regime the functions $a_r(\bar{z})$ are nearly constants [whose values are defined by the boundary conditions at $\bar{z}=0$ on the field $A(\bar{z})$ and its derivatives], while their dynamics are strongly modified as soon as the coupled electron-radiation system enters into the nonlinear regime.

The "coupled modes" model can be analytically closed on the variables $\bar{a}_r(\bar{z})$ when the explicit dependence on the electron dynamics can be eliminated from Eq. (17). This is done by means of two different approaches, both outlined in the following subsections. Details of the calculations are reported also in the Appendixes A and B.

A. Collective scheme

In the first scheme the driving function (17) is expressed in terms of the variables $A(\bar{z})$, $A'(\bar{z})$, and $A''(\bar{z})$ by neglecting the "second-harmonic" contribution $A^*(\bar{z})\langle e^{-2i\theta}\rangle$, and by writing the following approximated expression for the quantity $\langle p^2e^{-i\theta}\rangle$:

$$\langle p^2e^{-i\theta}\rangle=b\langle p^2\rangle-2b\langle p\rangle^2+2\langle p\rangle\langle pe^{-i\theta}\rangle\ .$$

This last equality was first introduced in the collective variables model of an FEL [11] and is based on the factorization ansatz

$$\langle(p-\langle p\rangle)^2e^{-i\theta}\rangle\approx\langle(p-\langle p\rangle)^2\rangle\langle e^{-i\theta}\rangle\ .$$

The calculation is explained in Appendix A. The final result is the "collective" nonlinear differential equation for the radiation field

$$\begin{aligned}-iA'''(\bar{z})-A(\bar{z})=&-2A'(\bar{z})[A(\bar{z})A'^*(\bar{z})-\text{c.c.}]\\ &+2A''(\bar{z})|A(\bar{z})|^2\\ &-2iA'(\bar{z})|A(\bar{z})|^4\ ,\end{aligned} \tag{18}$$

where we supposed for simplicity $|A|_0=0$. Note that the nonlinearities that are present in the driving term (18) are of third and fifth order in $A(\bar{z})$ and its derivatives. From Eq. (18) we calculate the nonlinear driving function $\Phi(\bar{z})\equiv -iA'''(\bar{z})-A(\bar{z})$ in terms of the "oscillators" $\bar{a}_r(\bar{z})$ using Eqs. (7)–(9). The result is

$$\begin{aligned}\Phi_c(\bar{z})\equiv&\sum_{r,s,t=1}^{3}\Gamma_{rst}\bar{a}_r(\bar{z})\bar{a}_s^*(\bar{z})\bar{a}_t(\bar{z})\\ &-2\sum_{r,s,t,u,v=1}^{3}k_r\bar{a}_r(\bar{z})\bar{a}_s^*(\bar{z})\bar{a}_t(\bar{z})\bar{a}_u^*(\bar{z})\bar{a}_v(\bar{z})\ ,\end{aligned} \tag{19}$$

where

$$\Gamma_{rst}=-(k_r+k_t)(k_r+k_s^*+k_t)\ . \tag{20}$$

When the driving function (19) is considered, the solutions of Eq. (16) give the total radiation field $A(\bar{z})=\sum_{r=1}^{3}\bar{a}_r(\bar{z})$ shown in Fig. 3.

B. Iterative scheme

In the second scheme the electron dynamics is eliminated from the nonlinear driving function (17) by solving Eqs. (1)–(3) through an iterative calculation that is similar to an iteration procedure used in a different context by Kimel and Elias [6]. The calculation is given in detail in Appendix B. Here we only say that it is performed preserving the full pendulum nonlinearity both in the phase equation (2) and in the field equation (3). This fact leads to a nonlinear integro-differential equation for the radiation field with nonlinear terms of all orders in $A(\bar{z})$. The "series" that is generated in this way is truncated by taking into account only nonlinear terms of the third order in the field $A(\bar{z})$. Although we limit ourselves to a self-consistent third-order calculation, the scheme can, in principle, yield all higher-order coupling terms.

The integro-differential nonlinear equation for the radiation field is the following:

$$\begin{aligned}-iA'''(\bar{z})-A(\bar{z})=2\Bigg[&\Omega'(\bar{z})\int_0^{\bar{z}}d\bar{z}'A(\bar{z}')\Omega^*(\bar{z}')-\Omega'(\bar{z})\int_0^{\bar{z}}d\bar{z}'A^*(\bar{z}')\Omega(\bar{z}')\\ &+\Omega'^*(\bar{z})\int_0^{\bar{z}}d\bar{z}'A(\bar{z}')\Omega(\bar{z}')\Bigg]+2A^*(\bar{z})\int_0^{\bar{z}}d\bar{z}'\int_0^{\bar{z}'}d\bar{z}''A(\bar{z}'')\Omega(\bar{z}'')\\ &-\Omega'^2(\bar{z})\Omega^*(\bar{z})-2\Omega(\bar{z})|\Omega'(\bar{z})|^2-2\Omega^2(\bar{z})A^*(\bar{z})\ ,\end{aligned} \tag{21}$$

where we have defined

$$\Omega(\bar{z})=\int_0^{\bar{z}}d\bar{z}'\int_0^{\bar{z}'}d\bar{z}''A(\bar{z}'') . \quad (22)$$

At last, we have to close the dynamical equations (16) for the variables $\bar{a}_r(\bar{z})$ by expressing the function $\Phi(\bar{z}) \equiv -iA'''(\bar{z})-A(\bar{z})$ through Eq. (21) in terms of the "modes" $\bar{a}_r(\bar{z})$.

If we again take into account only those nonlinear terms that are cubic in the three unknown variables $\bar{a}_r(\bar{z})$, the calculation can be carried out as the amplitudes $a_r(\bar{z})$ were actually independent if $\bar{z}$. In fact, in the explicit evaluation of integrals of the form

$$\sum_{r=1}^{3}\int_0^{\bar{z}}d\bar{z}'a_r(\bar{z}')e^{-ik_r\bar{z}'} ,$$

we can first make an integration by parts writing

$$\sum_{r=1}^{3}\int_0^{\bar{z}}d\bar{z}'a_r(\bar{z}')e^{-ik_r\bar{z}'} = \sum_{r=1}^{3}\left[\left[\frac{i}{k_r}a_r(\bar{z}')e^{-ik_r\bar{z}'}\right]_0^{\bar{z}} - \frac{i}{k_r}\int_0^{\bar{z}}d\bar{z}'a_r'(\bar{z}')e^{-ik_r\bar{z}'}\right] . \quad (23)$$

Then we may neglect the integral containing the first derivatives of the amplitudes $a_r(\bar{z})$ since, according to Eq. (15), it would eventually lead to nonlinear terms of the fifth order.

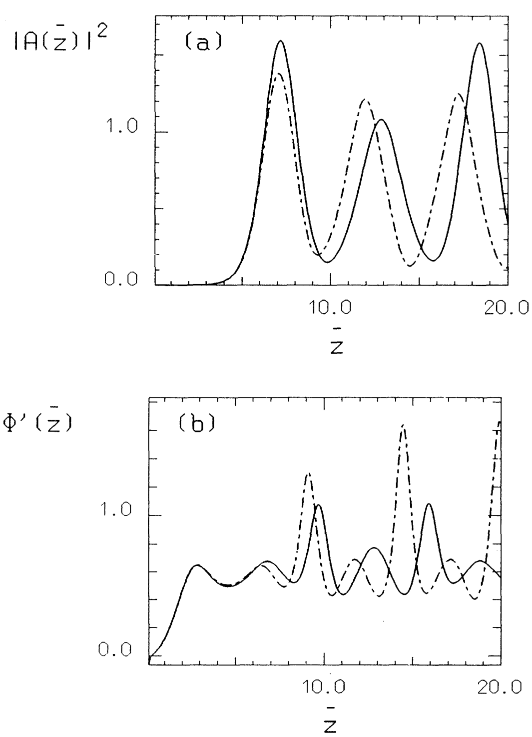

FIG. 3. (a) $|\bar{a}_1(\bar{z})+\bar{a}_2(\bar{z})+\bar{a}_3(\bar{z})|^2$ vs $\bar{z}$ as obtained from Eqs. (16) and (19) (solid line) and $|A(\bar{z})|^2$ vs $\bar{z}$ as obtained from Eqs. (1)–(3) (dot-dashed line). (b) Derivatives of the phases $\phi(\bar{z})$ of the previous two fields.

Finally, with a little more algebra, we are led to the driving function

$$\Phi_i(\bar{z})\equiv \sum_{r,s,t=1}^{3}\Lambda_{rst}\bar{a}_r(\bar{z})\bar{a}_s^*(\bar{z})\bar{a}_t(\bar{z}) , \quad (24)$$

where

$$\Lambda_{rst}=\frac{2(k_t+k_s^*)}{k_rk_s^{*2}k_t^2}(1-\delta_{k_t,k_s^*})-\frac{2}{k_t^2k_s^*(k_r+k_t)} +\frac{2}{k_t^2(k_r+k_t)^2}-\frac{1}{k_rk_s^{*2}k_t}-\frac{2}{k_r^2k_t^2}+\frac{2}{k_r^2k_s^*k_t} \quad (25)$$

and δ_{k_t,k_s^*} is the usual Kronecker symbol.

C. Simplified collective and iterative schemes

In the previous two schemes we found that the nonlinear driving function $\Phi(\bar{z})$ for the dynamical evolution of $\bar{a}_r(\bar{z})$ is expressed as the sum of third-order and (in the case of the "collective scheme") also fifth-order nonlinear terms in the three variables $\bar{a}_r(\bar{z})$, as given by Eqs. (19) and (24).

It is possible to simplify further the previous two schemes by neglecting some terms among the several forcing terms that are contained in the coupling functions $\Phi_c(\bar{z})$ or $\Phi_i(\bar{z})$. It must be noted at this point that if we approximate the dynamical evolution of $a_3(\bar{z})$ by means of any reduced coupling function $\tilde{\Phi}(\bar{z})$, namely, if we write

$$a_3'(\bar{z})=-\frac{i}{3}k_3\tilde{\Phi}(\bar{z})e^{ik_3\bar{z}} , \quad (26)$$

the same choice must be also done for consistency in the other two equations for $a_1(\bar{z})$ and $a_2(\bar{z})$. In fact from Eq. (15) it follows for $r=1,2$

$$a_r'(\bar{z})=k_{r+1}e^{-i(k_3-k_r)\bar{z}}a_3'=-\frac{i}{3}k_r\tilde{\Phi}(\bar{z})e^{ik_r\bar{z}} . \quad (27)$$

To understand the main features of the model, it is enough to consider as a driving term the third-order term containing $\bar{a}_3|\bar{a}_3|^2$, and write, therefore, system (16) as

$$\bar{a}_r'(\bar{z})=-ik_r[\bar{a}_r(\bar{z})+\tfrac{1}{3}\lambda\bar{a}_3(\bar{z})|\bar{a}_3(\bar{z})|^2] \quad (28)$$

(where the coupling factor is $\lambda=\Gamma_{333}$ in the collective scheme or $\lambda=\Lambda_{333}$ in the iterative scheme). If we do this, we obtain the analytical demonstration of the "anomalous" exponential growth of $\bar{a}_1(\bar{z})$ and $\bar{a}_2(\bar{z})$ in the first nonlinear region. The choice of this driving term is understandable if we recall that $\bar{a}_3(\bar{z})$ is the only mode whose amplitude grows exponentially in the linear regime, so that the term $\bar{a}_3(\bar{z})|\bar{a}_3(\bar{z})|^2$ is the fastest-growing third-order nonlinear term. Furthermore, we neglect higher-order nonlinear coupling terms, since their amplitudes are negligible with respect to the third-order ones.

It is clear that for $r=3$ we have a Landau-Ginzburg equation. Within both schemes the amplitude of the analytical solution of this equation saturates to the same value (independent of the initial conditions) that is

$$\lim_{\bar{z}\gg 1} |\bar{a}_3(\bar{z})|^2 = -3\frac{\mathrm{Im}(k_3)}{\mathrm{Im}(\lambda k_3)} . \tag{29}$$

Moreover, the mode $\bar{a}_3(\bar{z})$ has clearly the role of pumping mode. This pump effect leads to an exponential growth of the linearly stable oscillators $\bar{a}_1(\bar{z})$ and $\bar{a}_2(\bar{z})$. The resulting fields are reported in Fig. 4, while in Fig. 5 the field $A(\bar{z})=\sum_{r=1}^{3}\bar{a}_r(\bar{z})$ is compared with the numerical solution of the $2N+2$ microscopic equations (1)–(3). Note that, as we said in the previous section, the nonlinear growths of $\bar{a}_1(\bar{z})$ and $\bar{a}_2(\bar{z})$ are three times faster than the linear growth of $\bar{a}_3(\bar{z})$ even in this simplest model. This behavior is due to the presence of the coupling term $\bar{a}_3(\bar{z})|\bar{a}_3(\bar{z})|^2$. In fact, to obtain the correct growth rate of $\bar{a}_1(\bar{z}),\bar{a}_2(\bar{z})$ it is sufficient to substitute in Eq. (28) the linear expression for the linearly unstable mode $\bar{a}_3(\bar{z})=a_3 e^{-ik_3\bar{z}}$.

If we extrapolate the validity of Eq. (28) up to the full nonlinear region, we can see that the oscillations in the intensity of the radiation field $A(\bar{z})$ (even if with an incorrect period) are an intrinsic feature of this model. In fact, the main difference of this simplified model with respect to the previous Landau-Ginzburg models [4,8,9] is that now the Landau-Ginzburg equation holds for the oscillator $\bar{a}_3(\bar{z})$ and not for the total field $A(\bar{z})$. Thus even this very simplified model is also able to reproduce, beyond the saturation, the oscillations in the amplitude of the radiation field through the coupling of $\bar{a}_1(\bar{z})$ and $\bar{a}_2(\bar{z})$ with $\bar{a}_3(\bar{z})$.

A more correct description of the $A(\bar{z})$ dynamics is given if we consider the following step in the choice of the coupling function $\Phi(\bar{z})$. A straightforward extension of the preceding Landau-Ginzburg model is to consider as forcing terms the "diagonal" ones, so that each oscillator has its own Landau-Ginzburg equation and is symmetrically driven by the other two. Equations (16) become

$$\begin{aligned}\bar{a}_r'(\bar{z}) = -ik_r\{\bar{a}_r(\bar{z})+\tfrac{1}{3}[&\Lambda_{111}\bar{a}_1(\bar{z})|\bar{a}_1(\bar{z})|^2\\ &+\Lambda_{222}\bar{a}_2(\bar{z})|\bar{a}_2(\bar{z})|^2\\ &+\Lambda_{333}\bar{a}_3(\bar{z})|\bar{a}_3(\bar{z})|^2]\}\end{aligned} \tag{30}$$

(in the iterative scheme). The radiation field obtained by solving numerically Eqs. (30) is shown in Fig. 6. We observe that with the forcing term given in these equations

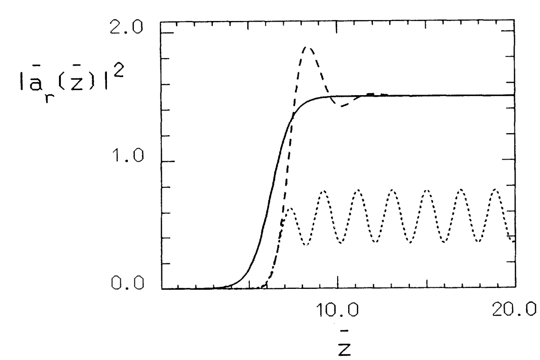

FIG. 4. $|\bar{a}_r(\bar{z})|^2$ vs $\bar{z}$ for $r=1$ (dotted line), $r=2$ (dashed line), $r=3$ (solid line) as obtained from Eq. (28) (iterative scheme).

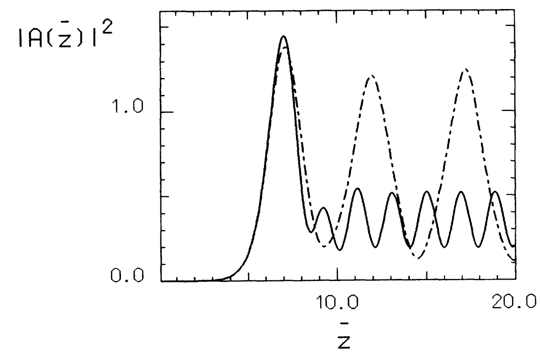

FIG. 5. $|\bar{a}_1(\bar{z})+\bar{a}_2(\bar{z})+\bar{a}_3(\bar{z})|^2$ vs $\bar{z}$ as obtained from Eq. (28) (solid line) in the iterative scheme and $|A(\bar{z})|^2$ vs $\bar{z}$ as obtained from Eqs. (1)–(3) (dot-dashed line).

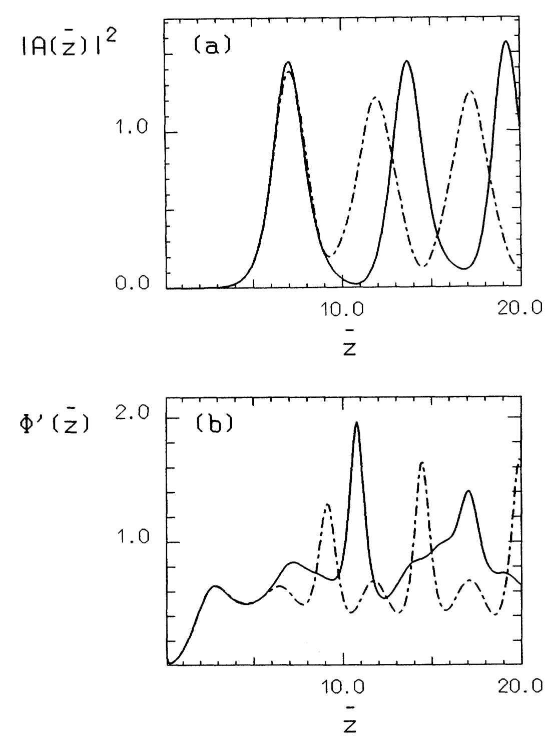

FIG. 6. (a) $|\bar{a}_1(\bar{z})+\bar{a}_2(\bar{z})+\bar{a}_3(\bar{z})|^2$ vs $\bar{z}$ as obtained from Eq. (30) (solid line) in the iterative scheme and $|A(\bar{z})|^2$ vs $\bar{z}$ as obtained from Eqs. (1)–(3) (dot-dashed line). (b) Derivatives of the phases $\phi(\bar{z})$ of the previous two fields.

the agreement with the numerical solutions of the microscopic $2N+2$ equations appears to be qualitatively correct, supporting the validity of the description in terms of three "oscillators."

Finally, we remark about the possibility of obtaining better agreements with different choices of the coupling terms. For instance, if we consider as driving terms in Eq. (15) the terms that are "nonresonant" with $a_1(\bar{z})$, i.e., all the terms in $\Phi(\bar{z})$ such that $\mathrm{Re}(k_r-k_s^*+k_t)\neq\mathrm{Re}(k_1)$, the radiation field $A(\bar{z})$ is better reproduced both in amplitude and phase so that the agreement is qualitatively and even quantitatively remarkable. The resulting radiation field, obtained in the collective scheme, i.e., with the coupling coefficients as given by Eq. (20), is shown in Fig. 7.

IV. CONCLUSIONS

Summarizing, we have shown that the steady-state behavior of a high-gain Compton FEL amplifier can be described in terms of the nonlinear interaction between three "modes" that reduce to the usual linear modes in the region where the lethargy of the radiation field is still the dominating feature of the FEL process.

One of the basic findings is that the two modes that are stable from the linear point of view actually are nonlinearly unstable, and they grow exponentially to appreciable levels in the first nonlinear region. The final evolution of the system is therefore dominated by strong energy exchanges between the three modes.

We have also shown how it is possible to build up a simplified description of the interaction among these three oscillators in terms of only cubic nonlinearities. The results that are obtained by solving system (30), although only qualitatively correct, lead to the conviction that much insight into the physical aspects of the FEL process can be gained along the line indicated in this paper.

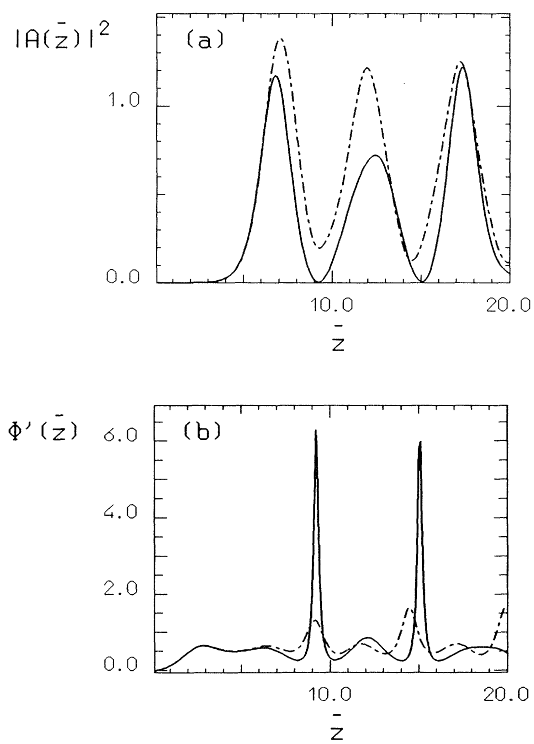

FIG. 7. (a) $|\bar{a}_1(\bar{z})+\bar{a}_2(\bar{z})+\bar{a}_3(\bar{z})|^2$ vs $\bar{z}$ as obtained from Eq. (16) with the "nonresonant terms" for $a_1(\bar{z})$ as driving terms (solid line) in the collective scheme, and $|A(\bar{z})|^2$ vs $\bar{z}$ as obtained from Eqs. (1)–(3) (dot-dashed line). (b) Derivatives of the phases $\phi(\bar{z})$ of the previous two fields.

APPENDIX A

In this appendix we calculate the driving function $\Phi(\bar{z})\equiv -iA'''(\bar{z})-A(\bar{z})$ in the collective scheme. We recall that the following equation holds:

$$-iA'''-A=i\langle p^2e^{-i\theta}\rangle+A^*\langle e^{-2i\theta}\rangle \ . \tag{A1}$$

We neglect the "second-harmonic" term $A^*\langle e^{-2i\theta}\rangle$ and we use the following factorization ansatz [11]:

$$\langle (p-\langle p\rangle)^2e^{-i\theta}\rangle\approx\langle (p-\langle p\rangle)^2\rangle\langle e^{-i\theta}\rangle \ , \tag{A2}$$

from which follows

$$-iA'''-A=ib\langle p^2\rangle-2ib\langle p\rangle^2+2i\langle p\rangle\langle pe^{-i\theta}\rangle \ , \tag{A3}$$

where $b\equiv\langle e^{-i\theta}\rangle$. The approximations written above have been checked numerically [11], moreover they are certainly suitable for the electron distributions corresponding to the boundary conditions usually considered (almost unbunched and monokinetic electron beams).

As the microscopic system (1)–(3) admits the following constants of motion:

$$\left\langle\frac{p^2}{2}\right\rangle-i(Ab^*-A^*b)-\delta|A|^2=\left\langle\frac{p^2}{2}\right\rangle_0-i(Ab^*-A^*b)_0-\delta|A|_0^2 \ ,$$

$$\langle p\rangle+|A|^2=|A|_0^2 \ ,$$

Eq. (A3) is rewritten as

$$\begin{aligned}-iA'''-A=&-2b[(Ab^*-A^*b)-(Ab^*-A^*b)_0]\\&-2ib(|A|^2-|A|_0^2)^2\\&+2i\langle pe^{-i\theta}\rangle(|A|_0^2-|A|^2)\end{aligned} \tag{A4}$$

(where we supposed $\delta=0$, $\langle p\rangle_0=0$, $\langle p^2\rangle_0=0$). By recalling the system (1)–(3), this last equation can be written as

$$\begin{aligned}-iA'''-A=&-2A'[(AA'^*-A^*A')\\&-(AA'^*-A^*A')_0]\\&-2iA'(|A|^2-|A|_0^2)^2\\&-2A''(|A|_0^2-|A|^2) \ .\end{aligned} \tag{A5}$$

As a last observation, we note that the above approximations lead to the truncation of the hierarchy that would be generated writing from Eqs. (1)–(3) the exact evolution equations for the macroscopic collective variables A, b, and $P\equiv\langle pe^{-i\theta}\rangle$. In this way, the closed system for the variables A, b, and P reads

$$A'=b\ , \tag{A6}$$

$$b'=-iP\ , \tag{A7}$$

$$P'=-A+2b(Ab^*-A^*b)-2b(Ab^*-A^*b)_0 +2i(|A|^2-|A|_0^2)[P+b(|A|^2-|A|_0^2)]\ , \tag{A8}$$

and is equivalent to system (16) when the complete coupling function $\Phi_c(\bar z)$ as given in Eq. (19) is considered.

APPENDIX B

In this appendix the driving function (17) is rewritten as a function of the radiation field by an iterative approach on the microscopic equations

$$\theta_j''(\bar z)=-[A(\bar z)e^{i\theta_j(\bar z)}+\text{c.c.}]\ , \tag{B1}$$

$$A'(\bar z)=\langle e^{-i\theta(\bar z)}\rangle\ . \tag{B2}$$

We consider the solution of the pendulum equation (B1):

$$\theta_j(\bar z)=\theta_{j,0}-I[Ae^{i\theta}](\bar z)\ , \tag{B3}$$

where we have defined the function

$$I[Ae^{i\theta}](\bar z)\equiv\int_0^{\bar z}d\bar z'\int_0^{\bar z'}d\bar z''[A(\bar z'')e^{i\theta_j(z'')}+\text{c.c.}] \tag{B4}$$

and have considered a cold electron beam (i.e., $p_{j,0}=0$). Note that the function $I[Ae^{i\theta}](\bar z)$ takes into account the pendulum nonlinearity.

Now we substitute the formal solution (B3) for $\theta_j(\bar z)$ in the field equation

$$A'(\bar z)=\langle e^{-i\theta(\bar z)}\rangle=\langle e^{-i\theta_0}e^{iI[Ae^{i\theta}](\bar z)}\rangle \tag{B5}$$

and we expand the bunching parameter in Eq. (B5) as a power series in $I[Ae^{i\theta}](\bar z)$ up to the third order:

$$A'(\bar z)\approx\left\langle e^{-i\theta_0}\left[1+iI[Ae^{i\theta}](\bar z)+\frac{i^2}{2!}I^2[Ae^{i\theta}](\bar z)+\frac{i^3}{3!}I^3[Ae^{i\theta}](\bar z)\right]\right\rangle\ . \tag{B6}$$

If we substitute for $\theta(\bar z)$ the formal solution (B3) of the nonlinear pendulum equation (B1) we obtain

$$A'(\bar z)\approx\left\langle e^{-i\theta_0}\left[1+iI[Ae^{i\theta_0}e^{-iI[Ae^{i\theta}](\bar z)}](\bar z)+\frac{i^2}{2!}I^2[Ae^{i\theta_0}e^{-iI[Ae^{i\theta}](\bar z)}](\bar z)+\frac{i^3}{3!}I^3[Ae^{i\theta_0}e^{-iI[Ae^{i\theta}](\bar z)}](\bar z)\right]\right\rangle\ . \tag{B7}$$

We see that in order to keep the nonlinear terms up to third order in the radiation field $A(\bar z)$, each exponential $e^{-iI[Ae^{i\theta}](\bar z)}$ has to be further expanded as follows:

$$A'(\bar z)\approx\left\langle e^{-i\theta_0}\left\{1+iI\left[Ae^{i\theta_0}\left[1-iI[Ae^{i\theta_0}](\bar z)+\frac{(-i)^2}{2!}I^2[Ae^{i\theta_0}](\bar z)\right]\right](\bar z)+\frac{i^2}{2!}I^2[Ae^{i\theta_0}\{1-iI[Ae^{i\theta_0}](\bar z)\}](\bar z)+\frac{i^3}{3!}I^3[Ae^{i\theta_0}](\bar z)\right\}\right\rangle\ . \tag{B8}$$

Performing explicitly the averages and taking into account that $\langle e^{-i\theta_0}\rangle=0$, by a twofold differentiation of the final result we obtain

$$\begin{aligned}-iA'''(\bar z)-A(\bar z)=2&\left[\Omega'(\bar z)\int_0^{\bar z}d\bar z'A(\bar z')\Omega^*(\bar z')-\Omega'(\bar z)\int_0^{\bar z}d\bar z'A^*(\bar z')\Omega(\bar z')+\Omega'^*(\bar z)\int_0^{\bar z}d\bar z'A(\bar z')\Omega(\bar z')\right]\\&+2A^*(\bar z)\int_0^{\bar z}d\bar z'\int_0^{\bar z'}d\bar z''A(\bar z'')\Omega(\bar z'')-\Omega'^2(\bar z)\Omega^*(\bar z)-2\Omega(\bar z)|\Omega'(\bar z)|^2-2\Omega^2(\bar z)A^*(\bar z)\ ,\end{aligned} \tag{B9}$$

where we have defined

$$\Omega(\bar z)=\int_0^{\bar z}d\bar z'\int_0^{\bar z'}d\bar z''A(\bar z'')\ . \tag{B10}$$

We observe that our calculation differs from that of Gallardo *et al.* performed in Ref. [7] since they neglect the pendulum nonlinearity, replacing the function $\exp\{iI[Ae^{i\theta}](\bar z)\}$ with the function $\exp\{iI[Ae^{i\theta_0}](\bar z)\}$ in the field equation (B5). On the contrary, since we keep the complete solution for $\theta_j(\bar z)$ both in the phase equation (B1) and in the rhs of the field equation (B5), our scheme is actually based on a self-consistent third-order expansion in the field equation.

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